Week 8:

Verification of a fitted ARMA model

Stochastic modelling of trend

Last week

Setting: data $Y_1, \ldots, Y_n$ from a stationary series $\{Y_t\}$ $\rightsquigarrow$ fit a feasible ARMA model

$\hookrightarrow$ determine the model order

$\hookrightarrow$ estimate the model parameters

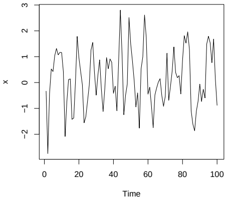
- point estimates
- it is possible to derive formulas for std. deviations of the estimators $\rightsquigarrow$ testing of significance

Next step

$\hookrightarrow$ model verification

Example

Data: $Y_1, \ldots, Y_{100}$



1. Based on some criteria $\rightsquigarrow$ choose AR(2) model

$$Y_t = \varphi_1 Y_{t-1} + \varphi_2 Y_{t-2} + \varepsilon_t$$

2. Estimation (e.g. MLE) $\rightsquigarrow$ $\widehat{\varphi}_1 = 0.6634$, $\widehat{\varphi}_2 = -0.3137$.
   Estimated model:

$$Y_t = 0.6634\, Y_{t-1} - 0.3137\, Y_{t-2} + \widehat{\varepsilon}_t$$

Function arima

```
>arima(x,order=c(2,0,0),include.mean=FALSE)

Coefficients:
                 ar1              ar2
      0.663439742961  -0.313670847370
s.e.  0.095764265201   0.098148294295

sigma^2 estimated as 0.83124222026:  log likelihood = -132.9,  aic = 271.81
```

Function arma (tseries):

```
> library(tseries)
> summary(arma(x,order=c(2,0),include.intercept=FALSE))

Model: ARMA(2,0)


Coefficient(s):
            Estimate       Std. Error  t value   Pr(>|t|)
ar1  0.6531591632695  0.0921202981258  7.09028 1.3383e-12 ***
ar2 -0.2967312994312  0.0920865474614 -3.22231  0.0012716 **
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Fit:
sigma^2 estimated as 0.779002581682,  Conditional Sum-of-Squares = 77.03,
 AIC = 262.81
```

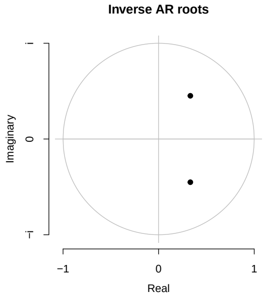
Verification of a fitted model

Consider a fitted ARMA model

$$\widehat{\varphi}(B)Y_t = \widehat{\theta}(B)\widehat{\varepsilon}_t$$

Checking stationarity

- roots of $\widehat{\varphi}(z)$, or their inverses



(not necessary if we use MLE with stationarity constraints)

- impulse response function

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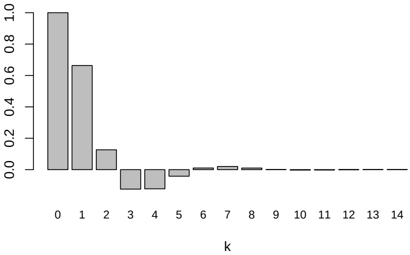
- Compute and plot the corresponding effect on Y_{s+k} for $k \geq 0$
- If the model is stationary $\rightsquigarrow$ the impulse fades away to 0



Examination of residuals

Consider a fitted ARMA model

$$\widehat{\varphi}(B)Y_t = \widehat{\theta}(B)\widehat{\varepsilon}_t$$

The residuals $\{\widehat{\varepsilon}_t\}$ should behave like a white noise

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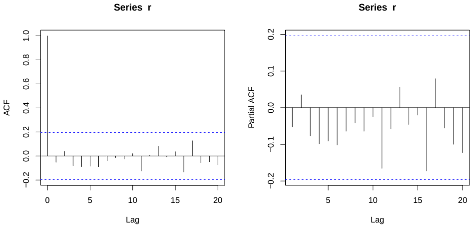

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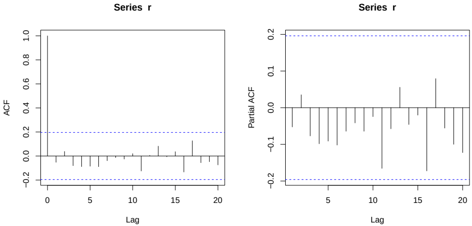

- use portmanteau tests

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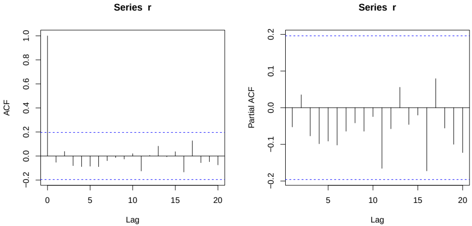

- use portmanteau tests
  $\rightsquigarrow$ recall week 5 tests of randomness

Portmanteau tests for fitted ARMA diagnostics

Let $\{\widehat{\varepsilon}_t\}$ be residuals of a fitted ARMA(p, q) and $\{r_k\}$ its sample ACF

Test statistics (Box–Pierce)

$$Q = n \sum_{k=1}^{K} r_k^2$$

or (Ljung–Box)

$$Q^* = n(n+2) \sum_{k=1}^{K} \frac{r_k^2}{n-k}$$

should be asymptotically χ^2_{K-p-q}

(Notice the change in degrees of freedom.)

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Testing procedure:

- fix $K > 1$
- if $Q^* > \chi^2_{K-p-q}(1-\alpha) \rightsquigarrow$ the considered model is not suitable

Example

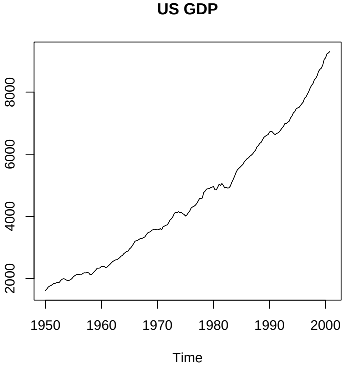
```
> a=arima(x,order=c(2,0,0),include.mean=FALSE)
> r=resid(a)

> Box.test(r,lag=5,fitdf=2)

Box-Pierce test

data:  r
X-squared = 2.576705486928, df = 3, p-value = 0.46158801884

> Box.test(r,lag=5,fitdf=2,type="Ljung-Box")

Box-Ljung test

data:  r
X-squared = 2.726753122961, df = 3, p-value = 0.435700010104
```

Stochastic Modeling of Trend

Nonstationarity

So far: data $Y_1, \ldots, Y_n$ from a stationary series $\{Y_t\}$

In economy and finance: majority of time series are nonstationary



Consequences:

- ARMA models not suitable
- in regression: spurious regression

Different types of non-stationarity

Let $\{\varepsilon_t\}$ be a sequence of iid variables $\sim (0, \sigma^2)$

Consider two simple models:

1. **Linear trend model:**

$$Y_t = \alpha_0 + \alpha t + \varepsilon_t$$

2. **Random walk with a drift:**

$$Y_t = \alpha + Y_{t-1} + \varepsilon_t = \alpha t + \sum_{i=1}^{t} \varepsilon_i + Y_0,$$

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   $\rightsquigarrow$ deterministic nonstationarity
   if a deterministic trend is eliminated $Y_t - \alpha_0 - \alpha t \rightsquigarrow$ stationary series

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Different ways to achieve stationarity

Comparison

For model 1 compute:

1. $\mathrm{E}\, Y_t$
2. $\mathrm{Var}\, Y_t$
3. $\mathrm{Cov}\,(Y_t, Y_s)$
4. What happens if we use ΔY_t.

For model 2 and $Y_0 = 0$ compute:

1. $\mathrm{E}\, Y_t$
2. $\mathrm{Var}\, Y_t$
3. $\mathrm{Cov}\,(Y_t, Y_s)$
4. What happens if we use $Y_t - \alpha t$.

Random walk with a drift vs. AR(1) model

Model

$$Y_t = \alpha + Y_{t-1} + \varepsilon_t$$

is AR(1) with an intercept

$$Y_t = \alpha + \phi_1 Y_{t-1} + \varepsilon_t$$

for $\phi_1 = 1$

- Recall that AR(1) is stationary iff $|\phi_1| < 1$.
- If $\phi_1 = 1 \rightsquigarrow 1 - \phi_1 z = 0$ has a root $z = 1$, i.e. a unit root.

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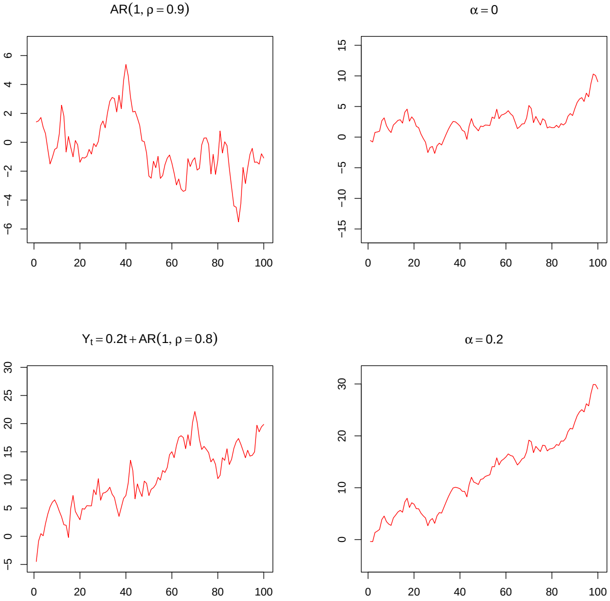
- If $\phi_1 = 1 \rightsquigarrow 1 - \phi_1 z = 0$ has a root $z = 1$, i.e. a unit root.

- it is not easy to distinguish a stationary AR(1) with ϕ_1 close to 1 and a random walk from a single trajectory
- statistical tests for unit root (will be described later today)

Trend stationarity vs. unit root



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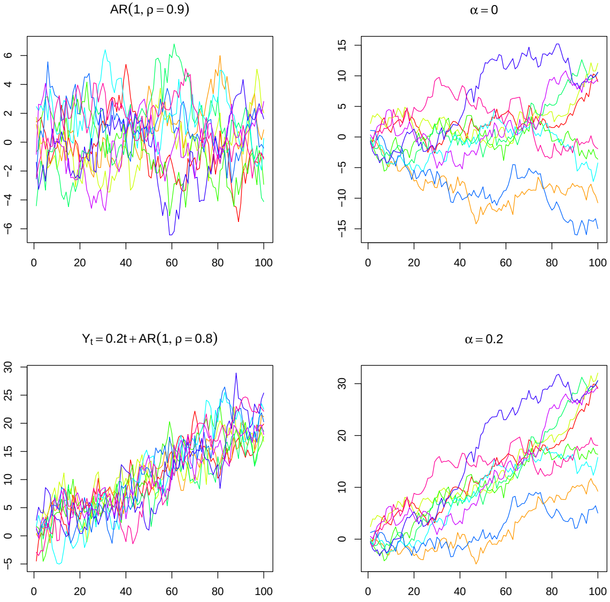

Differencing operator

$$\Delta Y_t = Y_t - Y_{t-1} = (1 - B) Y_t$$

Δ^d defined recursively

$$\Delta^d (Y_t) = \Delta(\Delta^{d-1} Y_t)$$

so

$$\begin{aligned}
\Delta^2 Y_t &= \Delta(Y_t - Y_{t-1}) = \Delta(Y_t) - \Delta(Y_{t-1}) = Y_t - 2Y_{t-1} + Y_{t-2}, \\
\Delta^3 Y_t &= \Delta(Y_t - 2Y_{t-1} + Y_{t-2}) = Y_t - 3Y_{t-1} + 3Y_{t-2} - Y_{t-3} \\
&\vdots
\end{aligned}$$

or see that

$$\Delta^d (Y_t) = (1 - B)^d Y_t = \left(\sum_{k=0}^{d} \binom{d}{k} (-1)^k B^k \right) Y_t = \sum_{k=0}^{d} \binom{d}{k} (-1)^k Y_{t-k}$$

Modelling of trend

1. Deterministic stationarity:

$$Y_t = Tr_t + u_t,$$

   where

   $\hookrightarrow$ Tr_t is a deterministic time trend

   $\hookrightarrow$ $\{u_t\}$ is a centred stationary process

   Modelling:

   - use known techniques for estimation of trend
   - be careful with testing
   - estimation can be improved if the correlation structure of $\{u_t\}$ is taken into account (see Financial Econometrics course)

2. Stochastic stationarity:

$$\Delta^d Y_t$$

   is a (generally non-centred) stationary process $\rightsquigarrow$ ARIMA models (I stands for *integrated*)

ARIMA model

ARIMA(p, d, q):

$$\varphi(B)\left(\Delta^d Y_t\right) = \alpha + \theta(B)\varepsilon_t$$

where

- $\{\varepsilon_t\}$ is WN
- $$\varphi(B) = 1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p,$$
  $$\theta(B) = 1 + \theta_1 z + \cdots + \theta_q z^q,$$

  such that the roots of $\varphi(z)$ lie outside the unit circle
- $\varphi(B)\Delta^d = \varphi(B)(1-B)^d$ generalized autoregressive operator $\rightsquigarrow$ polynomial $\varphi(z)(1-z)^d$: d times the unit root

Principle of ARIMA

1. find suitable smallest d such that $\Delta^d Y_t$ stationary
2. model $\Delta^d Y_t$ using a suitable ARMA

Choice of d

Typically $d \in \{0, 1, 2\}$

- Explore plots of Y_t, ΔY_t, $\Delta^2 Y_t$ ... and their sample ACF and PACF
- Use statistical tests for unit roots (see later)
- Some software: information criteria AIC, BIC

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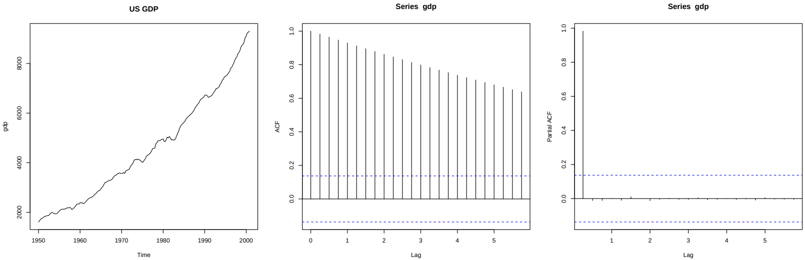
Be careful with overdifferencing.

Example: If $\{\varepsilon_t\}$ is a white noise (i.e. stationary), then $\Delta\varepsilon_t$ is a stationary MA(1) with $\theta_1 = -1$

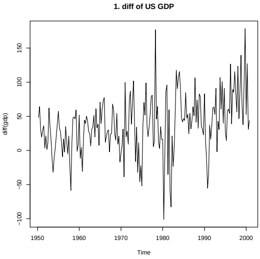
$$\Delta\varepsilon_t = \varepsilon_t - \varepsilon_{t-1}$$

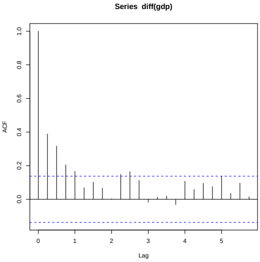
which is non-invertible and has a larger variance.

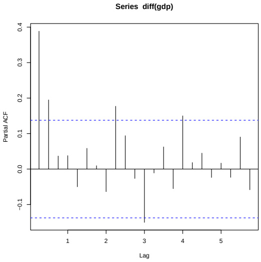
US GDP



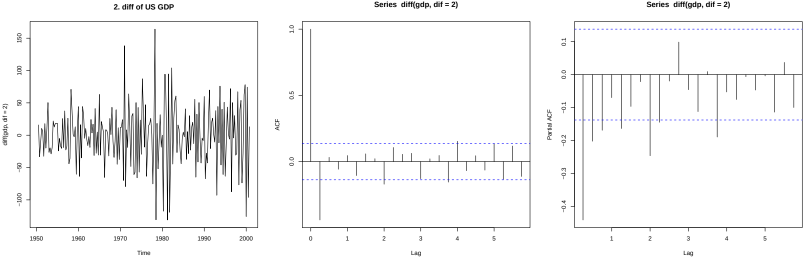
US GDP: ΔY_t







US GDP: $\Delta^2 Y_t$



Note: Intercept in ARIMA models

$$\varphi(B)\left(\Delta^d Y_t\right) = \alpha + \theta(B)\varepsilon_t$$

- $d = 0 \rightsquigarrow$ ARMA(p, q) with an intercept $\rightsquigarrow$

$$\mathrm{E}Y_t = \frac{\alpha}{1 - \varphi_1 - \ldots - \varphi_p}$$

so α determines the level of the series

- $d = 1$: series $\Delta Y_t = Y_t - Y_{t-1}$ satisfies

$$\mathrm{E}\Delta Y_t = \frac{\alpha}{1 - \varphi_1 - \ldots - \varphi_p} =: \mu,$$

so

$$\mathrm{E}Y_t = \mathrm{E}Y_{t-1} + \mathrm{E}\Delta Y_t = \mathrm{E}Y_{t-1} + \mu = \mu \cdot t + \mathrm{E}Y_0,$$

so α determines the slope

Note: Log returns

Let P_t be a price of some financial asset (e.g. a stock)

- return
$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$
- log-return
$$r_t = \log\left(\frac{P_t}{P_{t-1}}\right) = \log P_t - \log P_{t-1}$$
i.e. r_t corresponds to $\Delta \log P_t$

- very often $\{r_t\}$ is a (shifted) white noise

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i.e. r_t corresponds to $\Delta \log P_t$
- see that if x is small, then
$$\log(1 + x) \approx 1 + x$$
so
$$r_t = \log\left(\frac{P_t}{P_{t-1}}\right) = \log\left(\frac{P_t - P_{t-1}}{P_{t-1}} + 1\right) = \log(R_t + 1) \approx R_t$$
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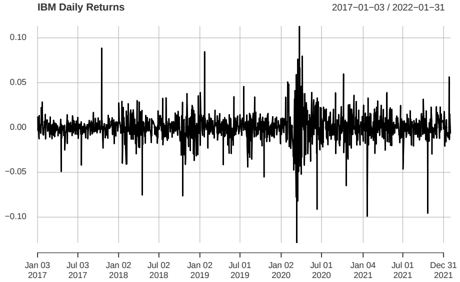
Example: Log returns



Tests of Unit Root

Simplest situation:

$$Y_t = \rho Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{WN}(0, \sigma^2)$$

Test

$$H_0 : \rho = 1$$

against

$$H_1 : \rho < 1.$$

Note: In practice H_1 often means $\rho \in (0, 1)$.

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Transformation: Subtract Y_{t-1} from both sides $\rightsquigarrow$

$$\Delta Y_t = \underbrace{(\rho - 1)}_{\theta} Y_{t-1} + \varepsilon_t$$

then

$$H_0 : \theta = 0 \quad \text{and} \quad H_1 : \theta < 0$$

Dickey–Fuller Test

$$\Delta Y_t = \theta Y_{t-1} + \varepsilon_t$$

Idea: regress ΔY_t on Y_{t-1} and test $\theta = 0$ using a standard t-test

$$T = \frac{\widehat{\theta}}{sd(\widehat{\theta})}$$

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Problem: under H_0 the standard asymptotics does not apply

- $\hookrightarrow$ T is not asymptotically N(0, 1)
- $\hookrightarrow$ asymptotic distribution of T more complicated $\rightsquigarrow$ Dickey-Fuller distribution $\rightsquigarrow$ critical values c_α tabulated

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Reject H_0 if

$$T < c_\alpha$$

if $\alpha = 0.05 \rightsquigarrow c_\alpha = -2.86$ (compare: normal quantile $u_{0.05} = -1.65$)

Trend variants of DF test

- DF test: under $H_1 \rightsquigarrow \{Y_t\}$ is a stationary centered AR(1)

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More general model:

$$Y_t = \alpha + \delta t + \rho Y_{t-1} + \varepsilon_t,$$

the same transformation $\rightsquigarrow$

$$\Delta Y_t = \alpha + \delta t + \theta Y_{t-1} + \varepsilon_t$$

and $H_0 : \theta = 0$ against $H_1 : \theta < 0$

Case I. $\delta = 0$ and $\delta = 0$ considered

Case II. $\delta = 0 \rightsquigarrow$ under H_0 RW with a drift, under H_1 stationary non-centred process

Case III. under H_1: deterministic time trend

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Case III. under H_1: deterministic time trend

Testing procedure:

- fit the model and compute the t-statistic for H_0
- different DF critical values for cases I., II. and III. $\rightsquigarrow$ tabulated

Augmented Dickey Fuller test

- DF test: under $H_0 \rightsquigarrow \Delta Y_t$ is an uncorrelated sequence
- ADF test $\rightsquigarrow$ allows ΔY_t to follow an AR model under H_0

Example: AR(1)

$$\Delta Y_t = \alpha + \theta Y_{t-1} + \varphi_1 \Delta Y_{t-1} + \varepsilon_t$$

with $|\varphi_1| < 1$ and test

$$H_0 : \theta = 0 \quad \text{against} \quad H_1 : \theta < 0$$

Then

$\hookrightarrow$ under $H_0 \rightsquigarrow \{\Delta Y_t\}$ stationary AR(1), so $\{Y_t\}$ ARIMA(1,1,0)

$\hookrightarrow$ under $H_1 \rightsquigarrow \{Y_t\}$ follows a non-centred stationary AR(2) model

Augmented Dickey Fuller test

Procedure for AR(p):

- Regress ΔY_t on $Y_{t-1}, \Delta Y_{t-1}, \ldots \Delta Y_{t-p}$
- Compute the t statistics for coefficient standing next to Y_{t-1}
- Use the same DF critical values as Case II

Choice of p:

- if p too large $\rightsquigarrow$ smaller power
- if p too small $\rightsquigarrow$ incorrect size of the test
- book recommendations: take the frequency of the data into account
- R: formula
$$k = \left\lfloor (n-1)^{1/3} \right\rfloor$$

US GDP

```
> adf.test(gdp,k=0)

Augmented Dickey-Fuller Test

data:  gdp
Dickey-Fuller = 1.618931877674, Lag order = 0, p-value = 0.99
alternative hypothesis: stationary

Warning message:
In adf.test(gdp, k = 0) : p-value greater than printed p-value

> adf.test(gdp)

Augmented Dickey-Fuller Test

data:  gdp
Dickey-Fuller = 0.2835363509014, Lag order = 5, p-value = 0.99
alternative hypothesis: stationary

Warning message:
In adf.test(gdp) : p-value greater than printed p-value
```

US GDP (cont.)

```
> adf.test(diff(gdp))

Augmented Dickey-Fuller Test

data:  diff(gdp)
Dickey-Fuller = -5.427919342951, Lag order = 5, p-value = 0.01
alternative hypothesis: stationary

Warning message:
In adf.test(diff(gdp)) : p-value smaller than printed p-value
```

Other tests

Phillips–Perron Test:

- uses robust (HAC) standard errors for the standard DF test statistics

KPSS Test:

- recall that DF: H_0 : non-stationarity and H_1 : stationarity
  if DF does not reject $H_0 \rightsquigarrow$ either H_0 holds or not enough power
- Kwiatkowski–Phillips–Schmidt–Shin:

$$H_0 : \text{stationarity} \quad H_1 : \text{non-stationarity}$$

- Model

$$Y_t = \alpha + \delta t + r_t + \varepsilon_t, \quad r_t = r_{t-1} + u_t,$$

  where u_t are iid $N(0, \sigma_u^2)$
  $\rightsquigarrow$ LM test (score test) for $H_0 : \sigma_u^2 = 0$
- combine DF and KPSS test. If conclusions differ $\rightsquigarrow$ inconclusive verdict

US GDP

```
> kpss.test(gdp)

KPSS Test for Level Stationarity

data:  gdp
KPSS Level = 4.062992006614, Truncation lag parameter = 4, p-value = 0.01

Warning message:
In kpss.test(gdp) : p-value smaller than printed p-value


> kpss.test(diff(gdp))

KPSS Test for Level Stationarity

data:  diff(gdp)
KPSS Level = 1.194908945277, Truncation lag parameter = 4, p-value = 0.01

Warning message:
In kpss.test(diff(gdp)) : p-value smaller than printed p-value
```