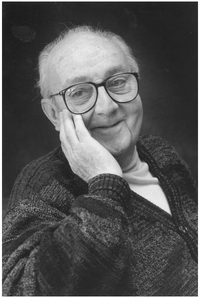
Week 6:

Box-Jenkins methodology

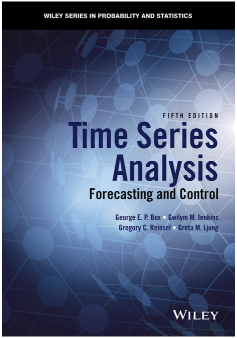
Box-Jenkins methodology

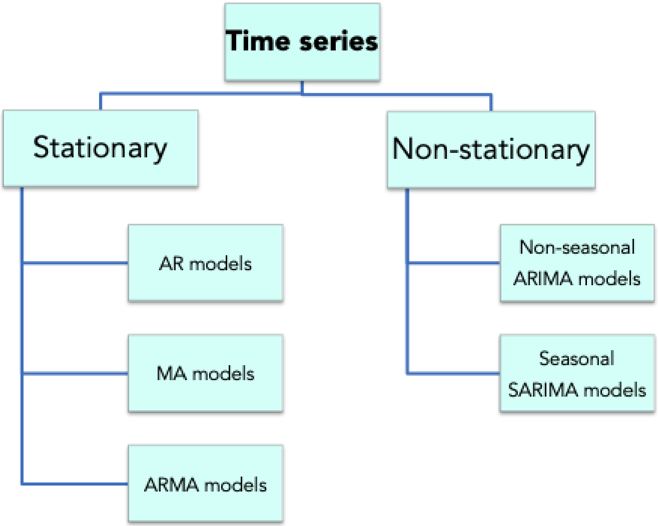
- AutoRegressive Integrated Moving Average (ARIMA) models
- 1970s, popularized by Box and Jenkins
- rely on autocorrelation patterns in the data



George E. P. Box
1919 – 2013

Gwilym M. Jenkins
1932 – 1982




Time series
Stationary
Non-stationary
AR models
MA models
ARMA models
Non-seasonal
ARIMA models
Seasonal
SARIMA models

Notions and definitions from Stochastic Processes II

Time series $\{Y_t\}$

- strict stationarity
- (weak) stationarity
- white noise WN
- autocovariance function $\{\gamma_k\}$
- autocorrelation function (ACF) $\{\rho_k\}$
- partial autocorrelation function (PACF) $\{\rho_{kk}\}$

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Technical tools:

- Hilbert space $L_2 = \{X: \text{ random variable } \mathrm{E}X^2 < \infty\}$ with inner product
$$\langle X, Y \rangle = \mathrm{E}XY$$

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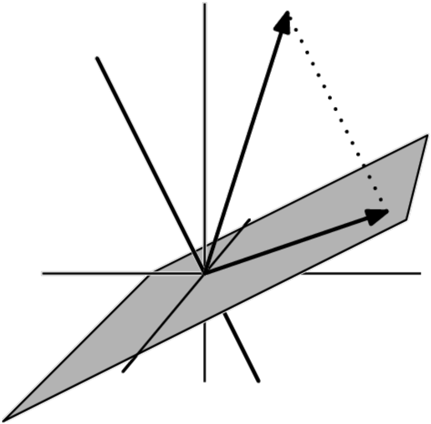
$$\langle X, Y \rangle = \mathsf{E}XY$$
- orthogonality: $X \perp Y$ if $\langle X, Y \rangle = 0$
- projection on subspace $\mathcal{H}$:
$$X = \underbrace{P_{\mathcal{H}}X}_{\in \mathcal{H}} + \underbrace{(X - P_{\mathcal{H}}X)}_{\perp \mathcal{H}}$$
and
$$\|X - P_{\mathcal{H}}X\|^2 = \min_{Y \in \mathcal{H}} \|X - Y\|^2$$

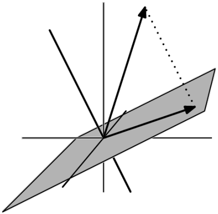
Projection principal



$$X - P_{\mathcal{H}} X \perp V \quad \text{for all } V \in \mathcal{H}$$

Projection principal II.

Let $\{Y_t\}$ be a stochastic process and $\mathcal{H} = \mathcal{H}_1^k = \{\sum_{i=1}^k c_i Y_i\}$.



Then

$$P_{\mathcal{H}} X = \sum_{i=1}^k \beta_i Y_i,$$

where $\beta_1, \ldots, \beta_k$ minimize $\mathrm{E}(X - \sum_{i=1}^k \alpha_i Y_i)^2$ among all linear combinations. Computation of β's:

$$\langle X - \sum_{i=1}^k \beta_i Y_i, Y_j \rangle = 0, \quad \text{for all } j = 1, \ldots, k,$$

$$\mathrm{E}(X - \sum_{i=1}^k \beta_i Y_i) Y_j = 0$$

Partial ACF

$\{Y_t\}$ weakly stationary, centered:

$$\rho_{kk} = \mathrm{cor}(Y_{k+1} - \widehat{Y}_{k+1}, Y_1 - \widehat{Y}_1),$$

where

$$\widehat{Y}_{k+1} = P_{\mathcal{H}_2^k} Y_{k+1},$$
$$\widehat{Y}_1 = P_{\mathcal{H}_2^k} Y_1,$$

with

$$\ldots, Y_1, \underbrace{Y_2, Y_3, \ldots, Y_k}_{\mathcal{H}_2^k}, Y_{k+1}, \ldots$$

Recall $P_{\mathcal{H}_2^k} Y_{k+1} = \sum_{i=2}^k \beta_i Y_i$, such that $\beta_2, \ldots, \beta_k$ minimize $\mathrm{E}(Y_{k+1} - \sum_{i=2}^k \alpha_i Y_i)^2$ among all linear combinations.

Alternative definition of partial ACF

If

$$P_{\mathcal{H}_1^k} Y_{k+1} = \phi_1 Y_k + \phi_2 Y_{k-1} + \cdots + \phi_k Y_1,$$

then

$$\rho_{kk} = \phi_k.$$

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then

$$\rho_{kk} = \phi_k.$$

Computation: The projection implies

$$\langle Y_{k+1} - P_{\mathcal{H}_1^k} Y_{k+1}, Y_j \rangle = 0, \quad j = 1, \ldots, k,$$

which gives

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_k \end{pmatrix} = \begin{pmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & \rho_0 & \rho_1 & \cdots & \rho_{k-2} \\ \vdots & & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_k \end{pmatrix} \tag{1}$$

$\rightsquigarrow$ solve to get $\rho_{kk} = \phi_k$

Notions and definitions

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- autocovariance function $\{\gamma_k\}$
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Sample estimates

- sample mean
- sample autocovariance function $\{c_k\}$
- sample ACF $\{r_k\}$
- sample PACF $\{r_{kk}\}$

Practical recommendation: $n > 50$, $k < n/4$

Sample PACF

PACF

$$\begin{pmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & \rho_0 & \rho_1 & \cdots & \rho_{k-2} \\ \vdots & & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_0 \end{pmatrix}^{-1} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_k \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \rho_{kk} \end{pmatrix}$$

$\rightsquigarrow$ Sample PACF: replace ρ_k by r_k,

Sample PACF

PACF

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$\rightsquigarrow$ Sample PACF: replace ρ_k by r_k, i.e.

$$\begin{pmatrix} 1 & r_1 & r_2 & \cdots & r_{k-1} \\ r_1 & 1 & r_1 & \cdots & r_{k-2} \\ \vdots & & & & \\ r_{k-1} & r_{k-2} & r_{k-3} & \cdots & 1 \end{pmatrix}^{-1} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \end{pmatrix} = \begin{pmatrix} \widehat{\phi}_1 \\ \widehat{\phi}_2 \\ \vdots \\ r_{kk} \end{pmatrix}$$

Sample PACF

PACF

$$\begin{pmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & \rho_0 & \rho_1 & \cdots & \rho_{k-2} \\ \vdots & & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_0 \end{pmatrix}^{-1} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_k \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \rho_{kk} \end{pmatrix}$$

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$\rightsquigarrow$ recursive Durbin–Levinson Algorithm

Asymptotics for sample ACF for iid

Proposition. If $\{Y_t\}$ are iid and let K be fixed, then

$$\sqrt{n}\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_K \end{pmatrix} \xrightarrow{D} \mathrm{N}_K(\mathbf{0}, \boldsymbol{I}_K), \quad n \to \infty$$

Application

$\hookrightarrow$ for n large $r_1, \ldots, r_K$ approximately iid $\mathrm{N}(0, 1/n)$

$\hookrightarrow$ r_k lies outside $\left(-\frac{u_{0.975}}{\sqrt{n}}, \frac{u_{0.975}}{\sqrt{n}}\right)$ with asymptotic probability 5% for each $1 \leq k \leq K$, independently

$\hookrightarrow$ bounds in plots

$\hookrightarrow$ more general situation: Bartlett's formula (later)

Asymptotics for sample PACF for iid

If $\{Y_t\}$ are iid, then

$$\sqrt{n}\begin{pmatrix}\widehat{\phi}_1\\ \widehat{\phi}_2\\ \vdots\\ r_{kk}\end{pmatrix} = \underbrace{\begin{pmatrix}1 & r_1 & r_2 & \dots & r_{k-1}\\ r_1 & 1 & r_1 & \dots & r_{k-2}\\ \vdots & & & & \\ r_{k-1} & r_{k-2} & r_{k-3} & \dots & 1\end{pmatrix}^{-1}}_{\stackrel{P}{\to}\boldsymbol{I}_K} \underbrace{\cdot\sqrt{n}\cdot\begin{pmatrix}r_1\\ r_2\\ \vdots\\ r_k\end{pmatrix}}_{\stackrel{D}{\to}\mathrm{N}_K(\mathbf{0},\boldsymbol{I}_K)},$$

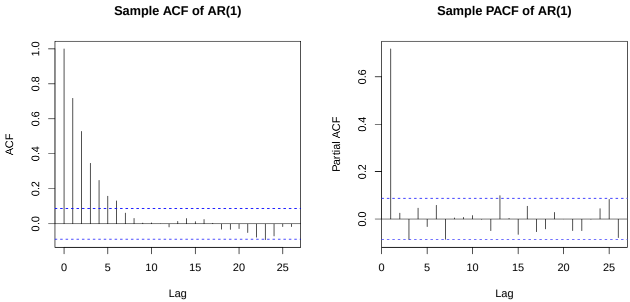
so

$$\sqrt{n}r_{kk} \stackrel{D}{\to} \mathrm{N}(0,1), \quad n\to\infty.$$

$\hookrightarrow$ more general situation: Quenouille's approximation (later)

Correlograms

- plot of $\{r_k\}$ or $\{r_{kk}\}$



- bounds:

$$\left(-\frac{u_{0.975}}{\sqrt{n}}, \frac{u_{0.975}}{\sqrt{n}}\right)$$

- try to determine the model from truncation point k_0
- now: basic TS models and how their $\{\rho_k\}$ and $\{\rho_{kk}\}$ look like

Basic models

Motivation: Wold decomposition: $\{Y_t\}$ is weakly stationary purely nondeterministic (i.e. it does not contain a component process whose future values can be perfectly predicted from the past values), then

$$Y_t = \sum_{i=1}^{\infty} \psi_i \varepsilon_{t-i} + \varepsilon_t,$$

where $\{\varepsilon_t\}$ is a WN, $\sum_{i=1}^{\infty} \psi_i^2 < \infty$.

Linear process: MA(∞) $Y_t = \sum_{i=1}^{\infty} \psi_i \varepsilon_{t-i} + \varepsilon_t$

$\hookrightarrow$ if $\sum \psi_i^2 < \infty \rightsquigarrow$ convergence in mean square

$\hookrightarrow$ if $\sum |\psi_i| < \infty \rightsquigarrow$ convergence almost surely

Linear process

Let $\sum_{i=1}^{\infty} |\psi_i| < \infty$, $\{\varepsilon_t\}$ WN

$$Y_t = \sum_{i=1}^{\infty} \psi_i \varepsilon_{t-i} + \varepsilon_t = \psi(B)\varepsilon_t$$

$\hookrightarrow$ B lag operator: $BX_t = X_{t-1}$, $B^k X_t = X_{t-k}$

$\hookrightarrow$

$$\psi(z) = \sum_{i=0}^{\infty} \psi_i z^i \quad \text{with } \psi_0 = 1$$

converges absolutely for all $|z| \leq 1$

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  converges absolutely for all $|z| \leq 1$
- invertible if

  $$\varepsilon_t = \sum_{i=0}^{\infty} \pi_i Y_{t-i} = \pi(B) Y_t$$

  with $\sum_{i=1}^{\infty} |\pi_i| < \infty$. If it holds then $\pi(z) = \psi(z)^{-1}$.

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with $\sum_{i=1}^{\infty} |\pi_i| < \infty$. If it holds then $\pi(z) = \psi(z)^{-1}$.
- ↪ useless model for practice: infinite number of unknown ψ_i

Innovations

$$Y_t = \sum_{i=1}^{\infty} \psi_i \varepsilon_{t-i} + \varepsilon_t = \psi(B)\varepsilon_t$$

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ε_t is called innovation:

- Let Y_t be invertible. Then

$$\varepsilon_t = Y_t - \sum_{i=1}^{\infty} \psi_i \varepsilon_{t-i} = Y_t - P_{\mathcal{H}_{-\infty}^{t-1}} Y_t$$

Innovations

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- If $\{Y_t\}$ normal and $\mathcal{F}_{t-1} = \sigma\{Y_s, s \leq t-1\}$, then

$$P_{\mathcal{H}_{-\infty}^{t-1}} Y_t = \mathrm{E}[Y_t | \mathcal{F}_{t-1}]$$

  and

$$\varepsilon_t = Y_t - \mathrm{E}[Y_t | \mathcal{F}_{t-1}]$$

  $\rightsquigarrow$ ε_t corresponds to unpredictable movements in values of Y_t

Moving Average Process MA(q)

Let $\{\varepsilon_t\}$ WN$(0, \sigma^2)$:

$$Y_t = \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} + \varepsilon_t = \theta(B)\varepsilon_t$$

for

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$$

Properties:

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Properties:

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  analytic for $|z| \leq 1$, i.e. if the roots of $\theta(z)$ lie outside the unit circle

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  $\rightsquigarrow$ truncation point $k_0 = q$
- PACF: $\{\rho_{kk}\}$ no truncation point

Important Exercises

1. Consider MA(q) process $\{Y_t\}$.
   - Compute $\mathrm{E}\, Y_t$, $\mathrm{Var}\, Y_t$.
   - Compute ACF ρ_k.

2. Consider MA(1) process $Y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1}$
   - Express the ACF.
   - Express the invertibility condition in terms of θ_1.

3. Consider MA(2):

$$Y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$$

   - Express the ACF.
   - Express the invertibility condition in terms of the set for (θ_1, θ_2).

PACF of MA

MA(1): $Y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1}$

$\hookrightarrow$ then

$$\rho_{kk} = \frac{(-1)^{k+1}\theta_1^k(1-\theta_1^2)}{1-\theta_1^{2(k+1)}} = \frac{(-1)^{k+1}\theta_1^k}{1+\theta_1^2+\theta_1^4+\cdots+\theta_1^{2k}}$$

(not easy to show)

$\hookrightarrow$ no truncation point k_0

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MA(q):

$\hookrightarrow$ ρ_{kk} has no truncation point k_0

$\hookrightarrow$ $\rho_{kk} \to 0$ as $k \to \infty$

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MA(q):

$\hookrightarrow$ ρ_{kk} has no truncation point k_0

$\hookrightarrow$ $\rho_{kk} \to 0$ as $k \to \infty$

$\hookrightarrow$ ρ_{kk} *bounded by a linear combination of geometrically decreasing sequences and sinusoids with geometrically decreasing amplitudes*
(see shape of ACF for AR(p))

Autoregressive Process AR(p)

$$Y_t = \varphi_1 Y_{t-1} + \cdots + \varphi_p Y_{t-p} + \varepsilon_t,$$

or

$$\varphi(B) Y_t = \varepsilon_t$$

for $\varphi(z) = 1 - \varphi_1 z - \ldots - \varphi_p z^p$

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Properties:

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Properties:

- stationary if roots of $\varphi(z) = 0$ lie outside the unit circle
- in that case centered and

$$\mathrm{Var}\,(Y_t) = \frac{\sigma^2}{1 - \varphi_1^2 - \ldots - \varphi_p^2}$$

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- ACF ρ_k is a solution of Yule Walker equations
  $\rightsquigarrow$ no truncation point
- PACF

$$\rho_{kk} = 0 \quad \text{for } k > p$$

  $\rightsquigarrow$ truncation point $k_0 = p$

Important Exercises

1. Consider a stationary AR(p) process
   - Compute $\text{Var}(Y_t)$.
   - Derive the Yule Walker equations.
   - Discuss the form of $\{\rho_k\}$, i.e. the form of a solution of a general system of difference equations.

2. Consider an AR(1) model $Y_t = \varphi_1 Y_{t-1} + \varepsilon_t$.
   - Express the stationarity condition in terms of φ_1.
   - Compute ACF.
   - Show that $\rho_{kk} = 0$ for $k > 1$.

3. Consider an AR(2) model $Y_t = \varphi_1 Y_{t-1} + \varphi_2 Y_{t-2} + \varepsilon_t$.
   - Express the stationarity condition in terms of φ_1 and φ_2.
   - Compute ACF.
   - Show that $\rho_{kk} = 0$ for $k > 2$.