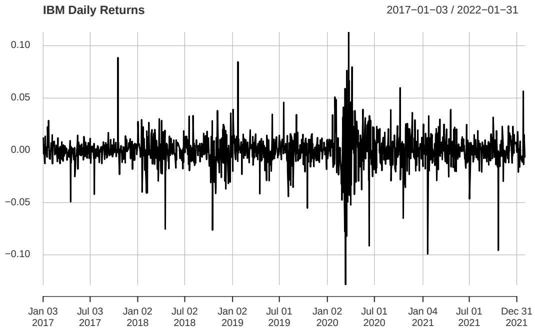
Week 11: ARCH and GARCH models

Volatility

= conditional variance (of e.g. underlying asset return) ⇝ not directly observable



Last week: General time series model

Let $\mathcal{F}_{t-1} = \sigma\{Y_s, s \leq t-1\}$ be information known up to time $t-1$

$$Y_t = \mu(\mathcal{F}_{t-1}) + \sigma(\mathcal{F}_{t-1})\varepsilon_t$$

with ε_t iid $(0, 1)$

$\hookrightarrow$ $\mu(\mathcal{F}_{t-1})$ conditional mean $\mathsf{E}[Y_t|\mathcal{F}_{t-1}]$
Example of model for μ: AR(2) $\mu(\mathcal{F}_{t-1}) = \varphi_1 Y_{t-1} + \varphi_2 Y_{t-2}$

$\hookrightarrow$ $\sigma(\mathcal{F}_{t-1})^2$ volatility $\mathsf{Var}\,[Y_t|\mathcal{F}_{t-1}]$

Last week: General time series model

Let $\mathcal{F}_{t-1} = \sigma\{Y_s, s \leq t-1\}$ be information known up to time $t-1$

$$Y_t = \mu(\mathcal{F}_{t-1}) + \sigma(\mathcal{F}_{t-1})\varepsilon_t$$

with ε_t iid $(0, 1)$

$\hookrightarrow$ $\mu(\mathcal{F}_{t-1})$ conditional mean $\mathrm{E}[Y_t|\mathcal{F}_{t-1}]$
Example of model for μ: AR(2) $\mu(\mathcal{F}_{t-1}) = \varphi_1 Y_{t-1} + \varphi_2 Y_{t-2}$

$\hookrightarrow$ $\sigma(\mathcal{F}_{t-1})^2$ volatility $\mathrm{Var}\,[Y_t|\mathcal{F}_{t-1}]$

Now focus on modelling $\sigma(\mathcal{F}_{t-1})^2$, so we consider $e_t = Y_t - \mu(\mathcal{F}_{t-1})$

$\hookrightarrow$ then

$$\mathrm{E}[e_t|\mathcal{F}_{t-1}] = 0$$

and for $s < t$

$$\mathrm{Cov}\,(e_t, e_s) = \mathrm{E}[e_t e_s] = \mathrm{E}[\mathrm{E}[e_t e_s|\mathcal{F}_{t-1}] = \mathrm{E}[e_s \mathrm{E}[e_t|\mathcal{F}_{t-1}] = 0$$

so $\{e_t\}$ uncorrelated, but with possibly non-constant conditional variance

$$\mathrm{Var}\,[e_t|\mathcal{F}_{t-1}] = \mathrm{Var}\,[Y_t|\mathcal{F}_{t-1}] = \sigma^2(\mathcal{F}_{t-1})$$

ARCH model by Engle (1982)

ARCH (autoregressive conditional heteroscedasticity)

ARCH(r) model:

$$e_t = \sigma_t \varepsilon_t,$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_r e_{t-r}^2$$

where ε_t are iid

$$\mathrm{E}\varepsilon_t = 0, \quad \mathrm{Var}\,\varepsilon_t = 1$$

and

$$\alpha_0 > 0, \quad \alpha_1, \ldots, \alpha_r \in [0, 1), \quad \sum_{i=1}^{r} \alpha_i < 1 \qquad \text{(A)}$$

ARCH model by Engle (1982)

ARCH (autoregressive conditional heteroscedasticity)

ARCH(r) model:

$$e_t = \sigma_t \varepsilon_t,$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_r e_{t-r}^2$$

where ε_t are iid

$$\mathrm{E}\varepsilon_t = 0, \quad \mathrm{Var}\,\varepsilon_t = 1$$

and

$$\alpha_0 > 0, \quad \alpha_1, \ldots, \alpha_r \in [0, 1), \quad \sum_{i=1}^{r} \alpha_i < 1 \tag{A}$$

Example: ARCH(1):

$$e_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2$$

large $e_{t-1}^2 \rightsquigarrow$ large conditional volatility of $e_t \rightsquigarrow$ larger uncertainty

Properties

Let $\mathcal{F}_t = \sigma\{e_s, s \leq t\}$

- If (A) holds, then $\{e_t\}$ is weakly stationary

Properties

Let $\mathcal{F}_t = \sigma\{e_s, s \leq t\}$

- If (A) holds, then $\{e_t\}$ is weakly stationary
- Mean:

$$\mathrm{E}[e_t|\mathcal{F}_{t-1}] = \mathrm{E}[\sigma_t \varepsilon_t|\mathcal{F}_{t-1}] = \sigma_t \mathrm{E}\varepsilon_t = 0$$

  and so also

$$\mathrm{E}e_t = \mathrm{E}\mathrm{E}[e_t|\mathcal{F}_{t-1}] = 0$$

Properties

Let $\mathcal{F}_t = \sigma\{e_s, s \leq t\}$

- If (A) holds, then $\{e_t\}$ is weakly stationary
- Mean:

$$\mathrm{E}[e_t|\mathcal{F}_{t-1}] = \mathrm{E}[\sigma_t \varepsilon_t|\mathcal{F}_{t-1}] = \sigma_t \mathrm{E}\varepsilon_t = 0$$

  and so also

$$\mathrm{E}e_t = \mathrm{E}\mathrm{E}[e_t|\mathcal{F}_{t-1}] = 0$$

- Conditional variance

$$\mathrm{Var}\,[e_t|\mathcal{F}_{t-1}] = \mathrm{Var}\,[\sigma_t \varepsilon_t|\mathcal{F}_{t-1}] = \sigma_t^2$$

  and unconditional variance

$$\mathrm{Var}\; e_t = \mathrm{E}(\sigma_t^2) \to \mathrm{Var}\; e_t = \frac{\alpha_0}{1 - \alpha_1 - \ldots - \alpha_r}$$

Properties

Let $\mathcal{F}_t = \sigma\{e_s, s \leq t\}$

- If (A) holds, then $\{e_t\}$ is weakly stationary
- Mean:

$$E[e_t|\mathcal{F}_{t-1}] = E[\sigma_t \varepsilon_t|\mathcal{F}_{t-1}] = \sigma_t E\varepsilon_t = 0$$

  and so also

$$Ee_t = EE[e_t|\mathcal{F}_{t-1}] = 0$$

- Conditional variance

$$\text{Var}\,[e_t|\mathcal{F}_{t-1}] = \text{Var}\,[\sigma_t \varepsilon_t|\mathcal{F}_{t-1}] = \sigma_t^2$$

  and unconditional variance

$$\text{Var}\,e_t = E(\sigma_t^2) \to \text{Var}\,e_t = \frac{\alpha_0}{1 - \alpha_1 - \ldots - \alpha_r}$$

- Covariance for $s < t$

$$\text{Cov}\,(e_t, e_s) = Ee_t e_s = E[E[e_t e_s|\mathcal{F}_{t-1}]] = E[e_s E[e_t|\mathcal{F}_{t-1}]] = 0$$

Properties

Let $\mathcal{F}_t = \sigma\{e_s, s \leq t\}$

- If (A) holds, then $\{e_t\}$ is weakly stationary
- Mean:

$$\mathrm{E}[e_t|\mathcal{F}_{t-1}] = \mathrm{E}[\sigma_t \varepsilon_t|\mathcal{F}_{t-1}] = \sigma_t \mathrm{E}\varepsilon_t = 0$$

  and so also

$$\mathrm{E}e_t = \mathrm{E}\mathrm{E}[e_t|\mathcal{F}_{t-1}] = 0$$

- Conditional variance

$$\mathrm{Var}\,[e_t|\mathcal{F}_{t-1}] = \mathrm{Var}\,[\sigma_t \varepsilon_t|\mathcal{F}_{t-1}] = \sigma_t^2$$

  and unconditional variance

$$\mathrm{Var}\, e_t = \mathrm{E}(\sigma_t^2) \rightarrow \mathrm{Var}\, e_t = \frac{\alpha_0}{1 - \alpha_1 - \ldots - \alpha_r}$$

- Covariance for $s < t$

$$\mathrm{Cov}\,(e_t, e_s) = \mathrm{E}e_t e_s = \mathrm{E}[\mathrm{E}[e_t e_s|\mathcal{F}_{t-1}]] = \mathrm{E}[e_s \mathrm{E}[e_t|\mathcal{F}_{t-1}]] = 0$$

$\{e_t\}$ is a white noise process of dependent variables with volatility σ_t^2

AR representation

See that

$$e_t^2 = \sigma_t^2 \varepsilon_t^2 = \sigma_t^2 + \underbrace{\sigma_t^2(\varepsilon_t^2 - 1)}_{u_t} = \alpha_0 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_r e_{t-r}^2 + u_t$$

where

$$\mathrm{E} u_t = \mathrm{E} \sigma_t^2(\varepsilon_t^2 - 1) = 0$$

and they are uncorrelated

$\rightsquigarrow \{e_t^2\}$ follows an AR(r) model

AR representation

See that

$$e_t^2 = \sigma_t^2 \varepsilon_t^2 = \sigma_t^2 + \underbrace{\sigma_t^2(\varepsilon_t^2 - 1)}_{u_t} = \alpha_0 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_r e_{t-r}^2 + u_t$$

where

$$\mathrm{E} u_t = \mathrm{E} \sigma_t^2(\varepsilon_t^2 - 1) = 0$$

and they are uncorrelated

$\rightsquigarrow \{e_t^2\}$ follows an AR(r) model

Practical consequence: Look at ACF and PACF of $\{e_t^2\}$ if ARCH(r) model suitable.

Distribution of ε_t

- $\hookrightarrow$ recall that $\mathrm{E}\varepsilon_t = 0$, $\mathrm{Var}\,\varepsilon_t = 1$
- $\hookrightarrow$ distribution typically assumed to be
  - normal $\mathrm{N}(0, 1)$
  - standardized t_ν with $\nu > 2$:

    if $Z \sim t_\nu \rightsquigarrow \mathrm{E}Z = 0$, $\mathrm{Var}\,Z = \frac{\nu}{\nu-2} \rightsquigarrow \varepsilon = \sqrt{\frac{(\nu-2)}{\nu}} \cdot Z$ satisfies $\mathrm{E}\varepsilon = 0$ and $\mathrm{Var}\,\varepsilon = 1$

Distribution of ε_t

- $\hookrightarrow$ recall that $\mathrm{E}\varepsilon_t = 0$, $\mathrm{Var}\,\varepsilon_t = 1$
- $\hookrightarrow$ distribution typically assumed to be
  - normal $\mathrm{N}(0, 1)$
  - standardized t_ν with $\nu > 2$:

    if $Z \sim t_\nu \rightsquigarrow \mathrm{E}Z = 0$, $\mathrm{Var}\,Z = \frac{\nu}{\nu-2} \rightsquigarrow \varepsilon = \sqrt{\frac{(\nu-2)}{\nu}} \cdot Z$ satisfies $\mathrm{E}\varepsilon = 0$ and $\mathrm{Var}\,\varepsilon = 1$
- $\hookrightarrow$ even if $\varepsilon_t \sim \mathrm{N}(0, 1) \rightsquigarrow e_t$ NOT normal

  kurtosis for ARCH(1)

$$\frac{\mathrm{E}e_t^4}{(\mathrm{Var}\,e_t)^2} = \frac{3(1-\alpha_1^2)}{1-3\alpha_1^2} > 3$$

  and it is finite only if $\alpha_1^2 < 1/3 \rightsquigarrow$ leptokurtic (heavy-tailed) distribution $\rightsquigarrow$ more ”outliers”

Example: Simulated data

ARCH(1): $e_t = \sigma_t \varepsilon_t$, $\sigma_t^2 = 0.1 + 0.3 e_{t-1}^2$, $n = 1\,000$, $\varepsilon_t \sim \mathrm{N}(0, 1)$

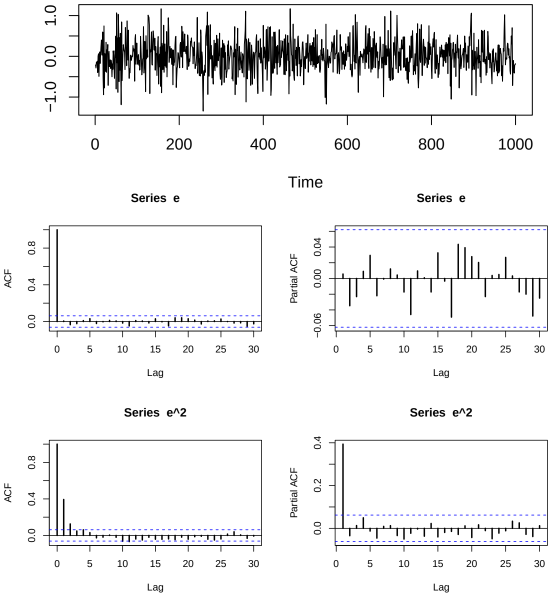
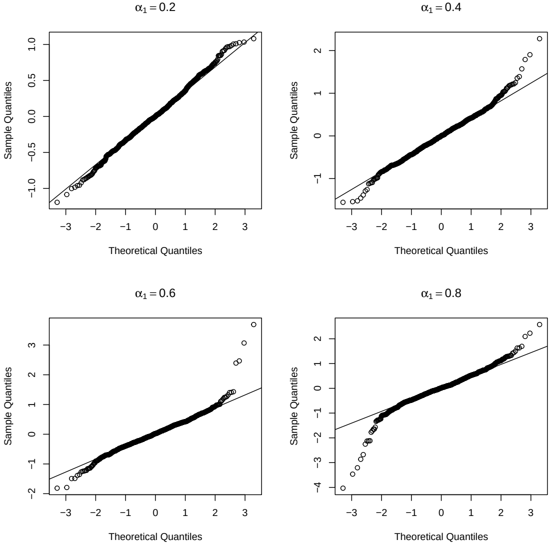


Illustration: heavy tails



Example: Simulated data II.

ARCH(1): $e_t = \sigma_t \varepsilon_t$, $\sigma_t^2 = 0.1 + 0.3 e_{t-1}^2$, $n = 1\,000$, ε_t standardized t_5

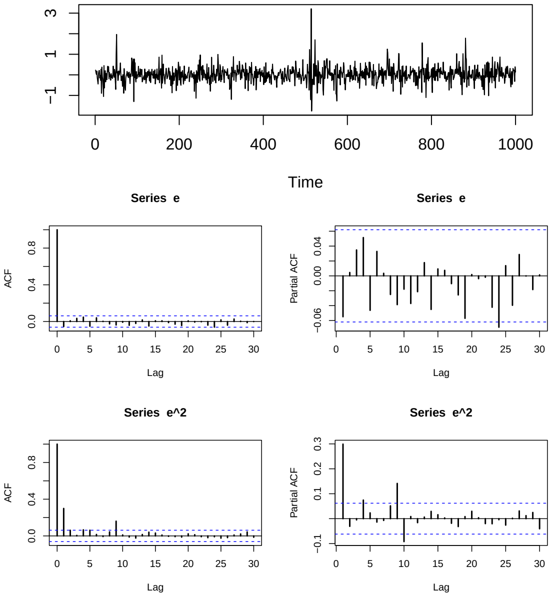
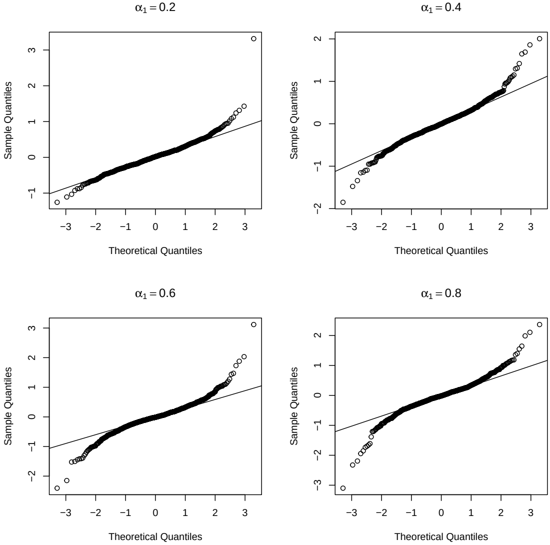


Illustration: heavy tails for *t* innovations



Building an ARCH model

ARCH model is suitable for series which

- $\hookrightarrow$ are uncorrelated
- $\hookrightarrow$ their squares e_t^2 exhibit correlation as an AR series

Setting: Consider data $e_1, \ldots, e_n$ from a series $\{e_t\}$

0. Check that $\{e_t\}$ is uncorrelated: ACF and PACF

   If not $\rightsquigarrow$ fit an ARMA model first and then continue with residuals

Building an ARCH model

ARCH model is suitable for series which

- $\hookrightarrow$ are uncorrelated
- $\hookrightarrow$ their squares e_t^2 exhibit correlation as an AR series

Setting: Consider data $e_1, \dots, e_n$ from a series $\{e_t\}$

0. Check that $\{e_t\}$ is uncorrelated: ACF and PACF
   If not $\rightsquigarrow$ fit an ARMA model first and then continue with residuals
1. Choose the model order: correlogram for e_t^2 (it follows AR(r))

Building an ARCH model

ARCH model is suitable for series which

- ↪ are uncorrelated
- ↪ their squares e_t^2 exhibit correlation as an AR series

Setting: Consider data $e_1, \dots, e_n$ from a series $\{e_t\}$

0. Check that $\{e_t\}$ is uncorrelated: ACF and PACF
   If not ⇝ fit an ARMA model first and then continue with residuals
1. Choose the model order: correlogram for e_t^2 (it follows AR(r))
2. Estimate the model parameters
   - ↪ maximum likelihood
     requires assumption on distribution of ε_t
   - ↪ Gaussian quasi-maximum likelihood method

Building an ARCH model

ARCH model is suitable for series which

- ↪ are uncorrelated
- ↪ their squares e_t^2 exhibit correlation as an AR series

Setting: Consider data $e_1, \ldots, e_n$ from a series $\{e_t\}$

0. Check that $\{e_t\}$ is uncorrelated: ACF and PACF
   If not $\rightsquigarrow$ fit an ARMA model first and then continue with residuals
1. Choose the model order: correlogram for e_t^2 (it follows AR(r))
2. Estimate the model parameters
   - ↪ maximum likelihood
     requires assumption on distribution of ε_t
   - ↪ Gaussian quasi-maximum likelihood method
3. Model verification.

Gaussian (normal) MLE

Let $e_1, \ldots, e_n$ be observed data and assume $\varepsilon_t \sim \mathrm{N}(0, 1)$

Then for $t \geq r + 1$

$$e_t | \mathcal{F}_{t-1} \sim \mathrm{N}(0, \sigma_t^2)$$

and the joint density of $e_{r+1}, \ldots, e_n$ given $e_1, \ldots, e_r$ is

$$\prod_{t=r+1}^{n} \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left\{-\frac{e_t^2}{2\sigma_t^2}\right\}$$

(use the same derivation as for an ARMA)

Gaussian (normal) MLE

Let $e_1, \ldots, e_n$ be observed data and assume $\varepsilon_t \sim \mathrm{N}(0,1)$

Then for $t \geq r+1$

$$e_t | \mathcal{F}_{t-1} \sim \mathrm{N}(0, \sigma_t^2)$$

and the joint density of $e_{r+1}, \ldots, e_n$ given $e_1, \ldots, e_r$ is

$$\prod_{t=r+1}^{n} \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left\{-\frac{e_t^2}{2\sigma_t^2}\right\}$$

(use the same derivation as for an ARMA)

The log-likelihood is

$$\ell(\boldsymbol{\alpha}) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=r+1}^{n}\left[\log(\sigma_t^2) + \frac{e_t^2}{\sigma_t^2}\right]$$

where

$$\sigma_t^2 = \sigma_t^2(\boldsymbol{\alpha}) = \alpha_0 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_r e_{t-r}^2.$$

Gaussian (normal) MLE

Let $e_1, \ldots, e_n$ be observed data and assume $\varepsilon_t \sim \mathrm{N}(0,1)$

Then for $t \geq r+1$

$$e_t | \mathcal{F}_{t-1} \sim \mathrm{N}(0, \sigma_t^2)$$

and the joint density of $e_{r+1}, \ldots, e_n$ given $e_1, \ldots, e_r$ is

$$\prod_{t=r+1}^{n} \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left\{ -\frac{e_t^2}{2\sigma_t^2} \right\}$$

(use the same derivation as for an ARMA)

The log-likelihood is

$$\ell(\boldsymbol{\alpha}) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=r+1}^{n}\left[\log(\sigma_t^2) + \frac{e_t^2}{\sigma_t^2}\right]$$

where

$$\sigma_t^2 = \sigma_t^2(\boldsymbol{\alpha}) = \alpha_0 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_r e_{t-r}^2.$$

Then

$$\widehat{\boldsymbol{\alpha}}_n = \mathrm{argmax}_{\boldsymbol{\alpha}}\, \ell(\boldsymbol{\alpha}) = \mathrm{argmin}_{\boldsymbol{\alpha}} \sum_{t=r+1}^{n}\left[\log(\sigma_t^2) + \frac{e_t^2}{\sigma_t^2}\right]$$

Other estimations

- MLE with different distributional assumption for ε_t
  - $\varepsilon_t \sim$ standardized t_ν,
  - possibility to estimate ν together with $\boldsymbol{\alpha}$
- Gaussian quasilikelihood estimation (QML):
  - take
    $$\widehat{\boldsymbol{\alpha}}_n = \text{argmin}_{\boldsymbol{\alpha}} \sum_{t=r+1}^{n} \left[\log(\sigma_t^2) + \frac{e_t^2}{\sigma_t^2}\right]$$
    even though we know that the normality assumption might not hold
  - such QML estimator is consistent and asymptotically normal under very general conditions ($\mathrm{E}\varepsilon_t^4 < \infty$)
  - valid standard errors and possibility for testing

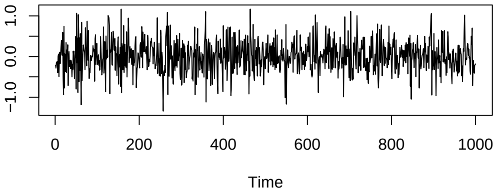
Model verification and predictions

For ARCH(r) fitted to $e_1, \dots, e_n$

1. compute $\widehat{\sigma}_t^2$ sequentially using the estimated parameters
2. compute
$$\widetilde{e}_t = \frac{e_t}{\widehat{\sigma}_t}$$
3. check ACF and PACF for $\{\widetilde{e}_t^2\}$, possibly apply Q-test (portmanteau test of Ljung-Box) for ACF of $\widetilde{e}_t^2$
4. distributional assumptions can be checked by histograms, QQ-plots

Example

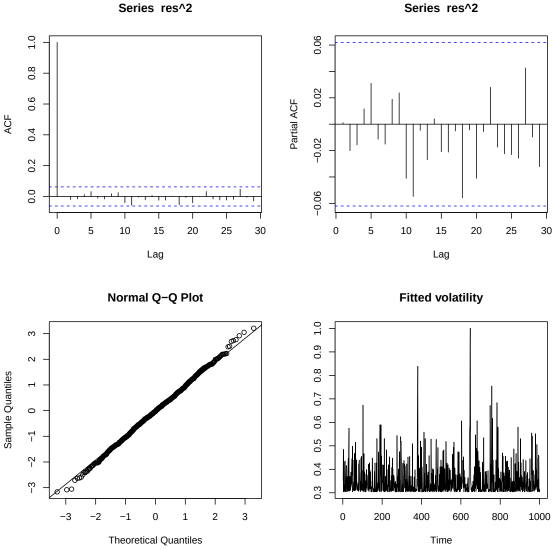
Continue with the simulated data $\{e_t\}$



$\rightsquigarrow$ fitted ARCH(1) model

$$\sigma_t^2 = 0.092 + 0.294 e_{t-1}^2$$

Example: Verification



GARCH model

GARCH(r, s)

$$e_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{r} \alpha_i e_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2$$

where ε_t are iid with $\mathrm{E}\varepsilon_t = 0$ and $\mathrm{Var}\,\varepsilon_t = 1$ and

$$\alpha_0 > 0, \quad \alpha_i \geq 0, \quad \beta_j \geq 0, \quad \sum_{i=1}^{r} \alpha_i + \sum_{j=1}^{s} \beta_j < 1 \qquad \text{(G)}$$

- if (G) holds then $\{e_t\}$ is weakly stationary
- model GARCH(1,1)

$$e_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

the most popular: only 3 parameters, but capable of modelling general volatility

Properties of GARCH(r,s)

- Mean:

$$\mathrm{E}(e_t|\mathcal{F}_{t-1}) = \sigma_t \mathrm{E}(\varepsilon_t|\mathcal{F}_{t-1}) = 0$$

and

$$\mathrm{E}(e_t) = \mathrm{E}[\mathrm{E}(e_t|\mathcal{F}_{t-1})] = \mathrm{E}[\sigma_t \mathrm{E}(\varepsilon_t|\mathcal{F}_{t-1})] = 0$$

Properties of GARCH(r,s)

- Mean:

$$\mathrm{E}(e_t|\mathcal{F}_{t-1}) = \sigma_t \mathrm{E}(\varepsilon_t|\mathcal{F}_{t-1}) = 0$$

  and

$$\mathrm{E}(e_t) = \mathrm{E}[\mathrm{E}(e_t|\mathcal{F}_{t-1})] = \mathrm{E}[\sigma_t \mathrm{E}(\varepsilon_t|\mathcal{F}_{t-1})] = 0$$

- Variance:

$$\mathrm{Var}\,[e_t|\mathcal{F}_{t-1}] = \sigma_t^2$$

  stationarity $\rightsquigarrow$

$$\mathrm{Var}\, e_t = \mathrm{E}\sigma_t^2 = \mathrm{E}\left(\alpha_0 + \sum_{i=1}^{r} \alpha_i e_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2\right)$$

  and so

$$\mathrm{Var}\, e_t = \frac{\alpha_0}{1 - \sum_{i=1}^{r} \alpha_i - \sum_{j=1}^{s} \beta_j}.$$

Properties of GARCH(r,s)

- Mean:

$$E(e_t|\mathcal{F}_{t-1}) = \sigma_t E(\varepsilon_t|\mathcal{F}_{t-1}) = 0$$

and

$$E(e_t) = E[E(e_t|\mathcal{F}_{t-1})] = E[\sigma_t E(\varepsilon_t|\mathcal{F}_{t-1})] = 0$$

- Variance:

$$\text{Var}\,[e_t|\mathcal{F}_{t-1}] = \sigma_t^2$$

stationarity $\rightsquigarrow$

$$\text{Var}\, e_t = E\sigma_t^2 = E\left(\alpha_0 + \sum_{i=1}^{r} \alpha_i e_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2\right)$$

and so

$$\text{Var}\, e_t = \frac{\alpha_0}{1 - \sum_{i=1}^{r} \alpha_i - \sum_{j=1}^{s} \beta_j}.$$

- covariance

$$E\varepsilon_t\varepsilon_s = 0$$

$\rightsquigarrow$ $\{e_t\}$ is a white noise

ARMA representation for $\{e_t^2\}$

$$e_t^2 = \sigma_t^2 \varepsilon_t^2 = \sigma_t^2 + \underbrace{\sigma_t^2(\varepsilon_t^2 - 1)}_{u_t}$$

and so $\sigma_{t-j}^2 = e_{t-j}^2 - u_{t-j}$ and

$$\begin{aligned} e_t^2 &= \alpha_0 + \sum_{i=1}^{r} \alpha_i e_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2 + u_t \\ &= \alpha_0 + \sum_{i=1}^{r} \alpha_i e_{t-i}^2 + \sum_{j=1}^{s} \beta_j (e_{t-j}^2 - u_{t-j}) + u_t \\ &= \alpha_0 + \sum_{i=1}^{\max\{r,s\}} (\alpha_i + \beta_i) e_{t-i}^2 - \sum_{j=1}^{s} \beta_j u_{t-j} + u_t \end{aligned}$$

$\rightsquigarrow$ ARMA($\max\{r, s\}, s$) with noise $\{u_t\}$

Construction

1. Choose model orders
   typically try/use GARCH(1,1)
2. Estimate parameters as

$$(\widehat{\boldsymbol{\alpha}}_n, \widehat{\boldsymbol{\beta}}_n) = \text{argmin}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \sum_{t=r+1}^{n} \left[\log(\sigma_t^2) + \frac{e_t^2}{\sigma_t^2} \right],$$

   where $\sigma_t^2 = \sigma_t^2(\boldsymbol{\alpha}, \boldsymbol{\beta})$ are computed recursively with some initial setting (e.g. $\sigma_1 = \cdots = \sigma_s = 0$)
3. Model verification = same as for ARCH

Predictions of volatility

Consider data $e_1, \ldots, e_n$ from GARCH(1,1)

$$e_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

1. estimate the model parameters $\widehat{\alpha}_0, \widehat{\alpha}_1, \widehat{\beta}_1$ and computed sequentially

$$\widehat{\sigma}_t^2 = \widehat{\alpha}_0 + \widehat{\alpha}_1 e_{t-1}^2 + \widehat{\beta}_1 \widehat{\sigma}_{t-1}^2$$

for $t = 2, \ldots, n$ and some initial $\widehat{\sigma}_1^2$

Predictions of volatility

Consider data $e_1, \dots, e_n$ from GARCH(1,1)

$$e_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

1. estimate the model parameters $\widehat{\alpha}_0, \widehat{\alpha}_1, \widehat{\beta}_1$ and computed sequentially

$$\widehat{\sigma}_t^2 = \widehat{\alpha}_0 + \widehat{\alpha}_1 e_{t-1}^2 + \widehat{\beta}_1 \widehat{\sigma}_{t-1}^2$$

   for $t = 2, \dots, n$ and some initial $\widehat{\sigma}_1^2$
2. 1 step ahead volatility prediction

$$\widehat{\sigma}_{n+1}^2 = \widehat{\alpha}_0 + \widehat{\alpha}_1 e_n^2 + \widehat{\beta}_1 \widehat{\sigma}_n^2$$

Predictions of volatility

Consider data $e_1, \ldots, e_n$ from GARCH(1,1)

$$e_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

1. estimate the model parameters $\widehat{\alpha}_0, \widehat{\alpha}_1, \widehat{\beta}_1$ and computed sequentially
$$\widehat{\sigma}_t^2 = \widehat{\alpha}_0 + \widehat{\alpha}_1 e_{t-1}^2 + \widehat{\beta}_1 \widehat{\sigma}_{t-1}^2$$
for $t = 2, \ldots, n$ and some initial $\widehat{\sigma}_1^2$
2. 1 step ahead volatility prediction
$$\widehat{\sigma}_{n+1}^2 = \widehat{\alpha}_0 + \widehat{\alpha}_1 e_n^2 + \widehat{\beta}_1 \widehat{\sigma}_n^2$$
3. for $k > 1$: $\sigma_{n+k}^2 = \alpha_0 + \alpha_1 e_{n+k-1}^2 + \beta_1 \sigma_{n+k-1}^2$ and
$$e_{n+k-1}^2 = \sigma_{n+k-1}^2 \varepsilon_{n+k-1}^2 = \sigma_{n+k-1}^2 + \sigma_{n+k-1}^2 (\varepsilon_{n+k-1}^2 - 1)$$

Predictions of volatility

Consider data $e_1, \dots, e_n$ from GARCH(1,1)

$$e_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

1. estimate the model parameters $\widehat{\alpha}_0, \widehat{\alpha}_1, \widehat{\beta}_1$ and computed sequentially
$$\widehat{\sigma}_t^2 = \widehat{\alpha}_0 + \widehat{\alpha}_1 e_{t-1}^2 + \widehat{\beta}_1 \widehat{\sigma}_{t-1}^2$$
for $t = 2, \dots, n$ and some initial $\widehat{\sigma}_1^2$
2. 1 step ahead volatility prediction
$$\widehat{\sigma}_{n+1}^2 = \widehat{\alpha}_0 + \widehat{\alpha}_1 e_n^2 + \widehat{\beta}_1 \widehat{\sigma}_n^2$$
3. for $k > 1$: $\sigma_{n+k}^2 = \alpha_0 + \alpha_1 e_{n+k-1}^2 + \beta_1 \sigma_{n+k-1}^2$ and
$$e_{n+k-1}^2 = \sigma_{n+k-1}^2 \varepsilon_{n+k-1}^2 = \sigma_{n+k-1}^2 + \sigma_{n+k-1}^2 (\varepsilon_{n+k-1}^2 - 1)$$
so $\mathrm{E}[e_{n+k-1}^2 | \mathcal{F}_n] = \sigma_{n+k-1}^2$ and
$$\widehat{\sigma}_{n+k}^2 = \widehat{\alpha}_0 + \widehat{\alpha}_1 \underbrace{\widehat{e}_{n+k-1}^2}_{\widehat{\sigma}_{n+k-1}^2} + \widehat{\beta}_1 \widehat{\sigma}_{n+k-1}^2 = \widehat{\alpha}_0 + (\widehat{\alpha}_1 + \widehat{\beta}_1) \widehat{\sigma}_{n+k-1}^2$$

ARMA GARCH model: Summary

Consider data $Y_1, \ldots, Y_n$ from a stationary series $\{Y_t\}$

1. Fit an ARMA(p, q) model to $Y_1, \ldots, Y_n$.

ARMA GARCH model: Summary

Consider data $Y_1, \ldots, Y_n$ from a stationary series $\{Y_t\}$

1. Fit an ARMA(p, q) model to $Y_1, \ldots, Y_n$.
2. Compute residuals $\{e_t\}$ and verify white noise assumptions.

ARMA GARCH model: Summary

Consider data $Y_1, \ldots, Y_n$ from a stationary series $\{Y_t\}$

1. Fit an ARMA(p, q) model to $Y_1, \ldots, Y_n$.
2. Compute residuals $\{e_t\}$ and verify white noise assumptions.
3. Check ACF and PACF of $\{e_t^2\}$. If some important correlations $\rightsquigarrow$ model volatility

ARMA GARCH model: Summary

Consider data $Y_1, \ldots, Y_n$ from a stationary series $\{Y_t\}$

1. Fit an ARMA(p, q) model to $Y_1, \ldots, Y_n$.
2. Compute residuals $\{e_t\}$ and verify white noise assumptions.
3. Check ACF and PACF of $\{e_t^2\}$. If some important correlations $\rightsquigarrow$ model volatility
4. Fit a suitable ARCH or GARCH model to $\{e_t\}$.

ARMA GARCH model: Summary

Consider data $Y_1, \ldots, Y_n$ from a stationary series $\{Y_t\}$

1. Fit an ARMA(p, q) model to $Y_1, \ldots, Y_n$.
2. Compute residuals $\{e_t\}$ and verify white noise assumptions.
3. Check ACF and PACF of $\{e_t^2\}$. If some important correlations $\rightsquigarrow$ model volatility
4. Fit a suitable ARCH or GARCH model to $\{e_t\}$.
5. Predictions:
   - Use the fitted ARMA model for mean predictions of Y_{n+k}.
   - Use the fitted GARCH model for volatility predictions of Y_{n+k}.

ARMA GARCH model: Summary

Consider data $Y_1, \ldots, Y_n$ from a stationary series $\{Y_t\}$

1. Fit an ARMA(p, q) model to $Y_1, \ldots, Y_n$.
2. Compute residuals $\{e_t\}$ and verify white noise assumptions.
3. Check ACF and PACF of $\{e_t^2\}$. If some important correlations $\rightsquigarrow$ model volatility
4. Fit a suitable ARCH or GARCH model to $\{e_t\}$.
5. Predictions:
   - Use the fitted ARMA model for mean predictions of Y_{n+k}.
   - Use the fitted GARCH model for volatility predictions of Y_{n+k}.

Note: ARMA and GARCH part can be estimated also simultaneously

Further reading

Book:

- 8.3.6 Various Modifications of GARCH Models
- 8.3.1 Historical Volatility and EWMA Models