Nečas Center for Mathematical Modeling

Numerical solution of continuum physics problems with FEniCS or how to use FEM and not vary (too much) about programming ...

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Numerical solution of continuum physics problems with FEniCS or how to use FEM and not vary (too much) about programming ...

✓	**lecture 1**	introduction to FEniCS/python, FEM and how to solve laplace equation
✓	**lecture 2**	boundary conditions, time discretization, convection-diffusion equation, (stabilization)
✓	**lecture 3**	Stokes, incompressible Navier-Stokes equations
☞	**lecture 4**	linear and non-linear elasticity
	lecture 5	ALE-method, level-set method

Elasticity

Find displacement vector $\mathbf{u}$ such that:

$$
\begin{aligned}
\varrho_0 \frac{\partial \mathbf{v}}{\partial t} - \operatorname{div} \mathbf{P} &= \varrho_0 \mathbf{f} && \text{in } \Omega \\
\mathbf{u} &= \mathbf{u}_D && \text{on } \Gamma_D \\
\mathbf{Pn} &= \mathbf{f}_N && \text{on } \Gamma_N
\end{aligned}
$$

- $\mathbf{v}$ - the vector of velocity ($\frac{\partial \mathbf{u}}{\partial t} = \mathbf{v}$)
- $\mathbf{u}$ - the vector of displacement
- $\mathbf{P}(\mathbf{u})$ - the first Piola-Kirchoff stress tensor
- $\mathbf{f}$ - body force per unit mass
- $\mathbf{u}_D$ - given boundary displacement
- $\mathbf{f}_N$ - given boundary force/traction

- weak form: given $(\mathbf{u}^n, \mathbf{v}^n)$ find $(\mathbf{u}, \mathbf{v}) \in V \times V$ such that

$$
\begin{aligned}
(\frac{\mathbf{u} - \mathbf{u}^n}{dt}, \bar{\mathbf{u}}) &= (\mathbf{v}, \bar{\mathbf{u}}) \\
(\frac{\mathbf{v} - \mathbf{v}^n}{dt}, \bar{\mathbf{v}}) + a(\mathbf{u}; \bar{\mathbf{v}}) &= L(\bar{\mathbf{v}})
\end{aligned}
$$

  holds for all $(\bar{\mathbf{u}}, \bar{\mathbf{v}}) \in V \times V$
- where $(f, g) = \int_\Omega f \cdot g \,\mathrm{d}x$, $a(\mathbf{u}, \mathbf{w}) = \int_\Omega \mathbf{P}(\mathbf{u}) \cdot \nabla \mathbf{w} \,\mathrm{d}x$ and $L(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}x + \int_{\Gamma_N} \mathbf{f}_N \cdot \mathbf{v} \,\mathrm{d}s$

Elasticity - material laws

- deformation gradient $\mathbf{F} = \mathbf{I} + \nabla u$, $\quad J = \det \mathbf{F}$, $\quad \mathbf{B} = \mathbf{F}\mathbf{F}^T$, $\mathbf{E} = \frac{1}{2}(\mathbf{B} - \mathbf{I})$
- Cauchy stress tensor $\boldsymbol{\sigma} = \frac{1}{J}\mathbf{P}\mathbf{F}^T$
- elastic $\Rightarrow$ no dissipation, "stored energy" potential Ψ,

$$\mathbf{P} = \varrho_0 \frac{\partial \Psi}{\partial \mathbf{F}}$$

- Neo-Hookean model: $\Psi = \frac{\mu}{2}(\operatorname{tr} \mathbf{E}) + \frac{\lambda}{2}(J-1)^2$

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T} = \mathbf{F}(2\mu\mathbf{E} + \lambda J(J-1))\mathbf{F}^{-T}$$

- St. Venant- Kirchhoff model:

$$\mathbf{P} = 2\mu\mathbf{E} + \lambda(\operatorname{tr} \mathbf{E})\mathbf{I}$$

- linear ealsticity: assume $|\nabla \mathbf{u}|$ is small, then $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - I) \approx \boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$

$$\mathbf{P} = \boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + (\lambda \operatorname{tr} \boldsymbol{\varepsilon})\mathbf{I}$$

the Lamme constants $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$, $\mu = \frac{E}{2(1+\nu)}$ can be given in terms of Young modulus E and Poisson ratio ν

Elasticity

- weak form: given $(\mathbf{u}^n, \mathbf{v}^n)$ find $(\mathbf{u}, \mathbf{v}) \in V \times V$ such that

$$(\frac{\mathbf{u} - \mathbf{u}^n}{dt}, \bar{\mathbf{u}}) = (\mathbf{v}, \bar{\mathbf{u}})$$
$$(\frac{\mathbf{v} - \mathbf{v}^n}{dt}, \bar{\mathbf{v}}) + a(\mathbf{u}; \bar{\mathbf{v}}) = L(\bar{\mathbf{v}})$$

  holds for all $(\bar{\mathbf{u}}, \bar{\mathbf{v}}) \in V \times V$
- where

$$a(\mathbf{u}, \mathbf{w}) = \int_\Omega \mathbf{P}(\nabla \mathbf{u}) \cdot \nabla \mathbf{w} \, \mathrm{d}x$$

- Choice of mixed FE spaces $V_h \times V_h$ - as long as the material is compressible any continuous FE is ok
- FEniCS code for mixed spaces:

```
V = VectorFunctionSpace(mesh, "CG", 2)
W = MixedFunctionSpace([V, V])

w = Function(W)
(u, v) = split(w)

(_u, _v) = TestFunctions(W)
```

Elasticity

- weak form: given $(\mathbf{u}^n, \mathbf{v}^n)$ find $(\mathbf{u}, \mathbf{v}) \in V \times V$ such that

$$(\frac{\mathbf{u} - \mathbf{u}^n}{dt}, \bar{\mathbf{u}}) = (\mathbf{v}, \bar{\mathbf{u}})$$
$$(\frac{\mathbf{v} - \mathbf{v}^n}{dt}, \bar{\mathbf{v}}) + a(\mathbf{u}; \bar{\mathbf{v}}) = L(\bar{\mathbf{v}})$$

  holds for all $(\bar{\mathbf{u}}, \bar{\mathbf{v}}) \in V \times V$
- FEniCS form definitions:

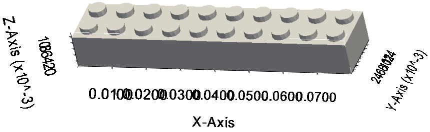
```
I = Identity(v.geometric_dimension())
F=I+\grad(u)
J=det(F)
B = F * F.T
e = 0.5*(grad(u) + grad(u).T)
P = 2*mu*e + lamb*tr(e)*I

EL = ..... + inner(P,grad(_v))*dx + ...

J = derivative(EL, w)
problem=NonlinearVariationalProblem(EL, w, bcs, J)
solver=NonlinearVariationalSolver(problem)
solver.solve()
```

Task no.4 - Elasticity

1. cleanup the file to you preferred style
2. compare the time dependent `elast.py` problem for 1st order metod (implicit Euler, $\theta = 1$) and 2nd order method (Crank-Nicolson, $\theta = \frac{1}{2}$)
3. extend the `elast.py` to 3D, you can use the mesh `lego_beam.xml`

   the outer dimensions are: $[0.1, 79.9] \times [0.1, 15.9] \times [0.0, 11.3]$ (in milimeters, e.g. they need to be multiplied by $\times 10^{-3}$).
4. Demonstrate the "self-penetration" problem, see `http://www.karlin.mff.cuni.cz/~madlik/fstrin/results/` for example

- How to log in to the cluster ....

```
ssh -Y -C IName@10.4.8.14
```

- Copy files or folders from/to a cluster - to be run on your local machine
  - copy local folder to your home directory on the cluster

    ```
    scp -r folder IName@10.4.8.14:~/
    ```

  - copy a folder from cluster to your local current directory

    ```
    scp -r IName@10.4.8.14:~/path/folder ./
    ```

- Lectures and example downloadable at `http://www.karlin.mff.cuni.cz/~hron/warsaw_2014/`