Contents

Chapter 1. Introduction 3
1. Present status 3
2. Objectives 4
3. Procedure 5

Chapter 2. Mixture theory 7
1. Preliminary definitions 7
2. Balance laws of general mixture 9
3. Binary mixture of a solid and a fluid 12

Chapter 3. Applications 19
1. One dimensional problem 19
2. Two dimensional problem 29

Chapter 4. Conclusion 41

Bibliography 43

CHAPTER 1

Introduction

The ability to model and predict the mechanical behavior of biological tissues is very important in several areas of bio-engineering and medicine. For example, a good mathematical model for biological tissue could be used in such areas as early recognition or prediction of heart muscle failure, advanced design of new treatments and operative procedures, and the understanding of atherosclerosis and associated problems. Other possible applications include development of virtual reality programs for training new surgeons or designing new operative procedures (see [**1**, **2**]), and last but not least the design of medical instruments or artificial replacements with mechanical and other properties as close as possible to the original parts (see [**3**]). These are some of the areas where a good mathematical model of soft tissue is essential for success.

1. Present status

1.1. Soft tissue modeling. Muscle tissue consists of muscle fibers and capillary blood vessels aligned in one direction and amorphous extracellular matrix and interstitial fluid which fills the voids and is limited in its movement. The size of the capillaries is of the order of the diameter of the red blood cell and their distribution in the muscle tissue is regular. This fact allows us to use some kind of homogenization to describe the material. The uni-directional orientation of the muscle fibers also suggests that transversely isotropic symmetry is appropriate. An overview of models for soft tissues including models of active muscle contraction is presented in [**4**] and [**5**].

Since a mathematical description of such materials on a microscopical level would be too complicated, other means of description are used to describe mathematically the mechanics of such materials. One possible description is based on the mixture theory. The biphasic mixture model (see [**6**, **7**]), where the tissue is represented by mixture of solid and fluid components, has been successfully used to model compression of cartilage (see [**8**, **9**]) and shear of skin (see [**10**]). In [**11**, **12**] a hierarchical mixture model is developed to analyze blood flow through biological tissues accounting for different levels of the arterial tree. In [**1**, **2**] a real time prediction of soft tissue deformation during surgery is investigated using mixture theory and in [**3**] mixture theory is applied to model milk expression during breast feeding.

Another important part of the description is the constitutive relations for the materials. In [**13**] and [**14**] a constitutive equation for passive myocardium is suggested and the material parameters are determined from experiments. Several simple constitutive relations for soft tissues are compared in [**15**] and [**6**]. In the framework of mixture theory, the constitutive relation for the interaction between the constituents has to be specified.

1.2. Continuum theory of mixtures. The problem of fluid saturated porous media was studied first by Fick [**16**] and then by Darcy [**17**]. These works gave the basis for the development of mixture theory based on modern continuum mechanics in the works of Truesdell [**18, 19**] and [**20**]. A good overview of the historical development is given in [**21, 22**], and other important works on the development of mixture theories include [**23, 24, 25, 26, 27, 28, 29**] and [**30**]. The important issue of constraints within the frame-work of mixture theory is discussed in [**31, 32**], whereas in [**33**] and [**34**] thermodynamic ideas are applied to mixtures and the question of suitable boundary conditions is discussed in [**35**].

The basic idea of mixture theory is the assumption of co-occupancy, i.e., at each spatial point there is certain fraction of each constituent (with associated fields) and there are prescribed balance equations for each constituent of the mixture as is usual for a single continuum, with additional terms representing the interaction between constituents within the mixture.

Mixture theory was first applied to swelling and diffusion in rubber materials [**36, 22**], mechanics of skin [**10**], compression of cartilage [**8, 9**] and blood perfusion through biological tissues in [**11, 12**].

1.3. Numerical solution. A numerical solution of the resulting equations of mixture theory poses great challenge since it includes the features of nonlinear elasticity and fluid mechanics coupled together and the volume additivity constraint corresponding to incompressibility constraint in a single continuum.

There have been several numerical studies of mixture models. One dimensional diffusion of fluid through an isotropic material is solved in [**37**], for transversely isotropic materials in [**36**] and in [**38**] a one dimensional diffusion through isotropic stretched slab is solved using a velocity boundary condition.

Finite element solutions of mixture models for the small deformation, linear elastic case are presented in [**9, 12**] and for nonlinear large deformation description of various soft tissues in [**8, 39, 40, 41, 42, 43, 44**].

2. Objectives

The main objective of this work is to develop and investigate a mathematical model which would make suitable basis for the description of the mechanical behavior of soft, hydrated tissues with significant perfusion. A typical example of interest is the case of muscle with significant blood perfusion through the capillary blood vessels. Such perfusion can significantly alter the overall mechanical response of the tissue.

The specific aims of this investigation are:

- To employ the continuum mixture theory to derive the appropriate balance relations.
- To investigate requirements on constitutive relations and appropriate boundary conditions in the context of the mixture theory.
- To solve illustrative steady, one and two dimensional problems of diffusion through a finite slab, and to develop and implement appropriate numerical schemes. The later will rely on finite element discretization for finding approximate solution to the continuous, nonlinear, steady, two dimensional problem.

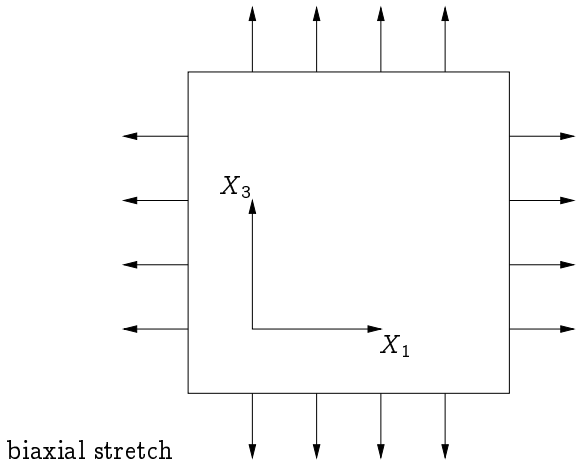

FIGURE 1. Schematic view of biaxial stretching experiment.

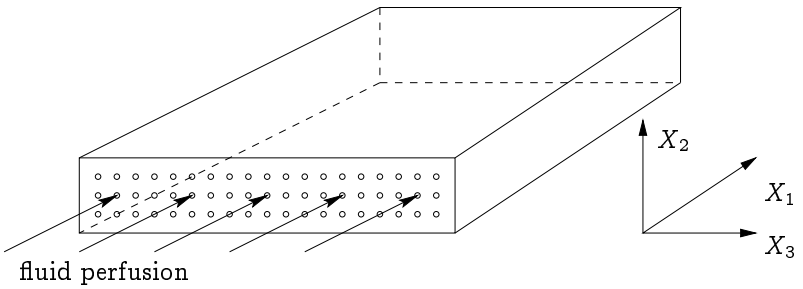

FIGURE 2. Schematic view of muscle perfusion.

3. Procedure

A typical experiment we would like to model is the biaxial stretching of a muscle specimen, see figure 1, with simultaneous perfusion through the capillaries as schematically depicted in figure 2. In such a problem we consider following characteristics

- The tissue specimen in the experiment, and even in its natural environment, is subjected to large deformation.
- Muscle fibers in the specimen are oriented in one direction.
- The perfusion takes place through parallel capillaries.
- The perfusion flow can have significant velocity possibly varying in time.
- The diameter of the capillary is comparable with the diameter of the red blood cell. ($\approx 5\mu$m)
- The distribution of the capillaries is regular.

In chapter 2 we first introduce the basic notations of the continuum mixture theory and set up the balance laws. Then we focus on a solid-fluid mixture, formulate the balance laws for each constituent and its consequences for the mixture as

a whole. Using standard thermodynamical arguments we define forms of the constitutive relations for the stresses and interactions so that the entropy inequality will be satisfied. In general, we will obtain a set of nonlinear partial differential equations governing the motion of the mixture.

In chapter 3 we investigate some applications of this description to simple problems. We restrict our considerations to isotropic binary mixture consisting of a solid and a fluid. A problem, in one spatial dimension, of a rectangular slab subjected to stretching with prescribed fluid flux is investigated in the first part. Allowing non-homogeneous deformation in one direction and prescribing homogeneous stretch in the other two directions we obtain a problem formulated as a set of nonlinear ordinary differential equations and solved numerically. The second part is devoted to a problem in two spatial dimensions. A weak formulation of the governing equations is introduced, with finite element discretization based on quadrilateral finite elements with continuous, piecewise biquadratic and bilinear shape functions. An iterative procedure based on the Newton method for solving the resulting set of nonlinear algebraic equations is used. This algorithm is applied to a problem of stretched isotropic slab with perfusion.

CHAPTER 2

Mixture theory

The continuum assumption has proven to be a very useful tool in fluid and solid mechanics. It idealizes the body of interest, consisting of individual particles (atoms or molecules), as a continuous region in space. There has to be certain conditions in order to use this approach, namely the characteristic size of the body and the disturbances imposed on the body have to be "much" greater than the size of the individual particles.

Similar ideas support the mixture theory. In the mixture, each of the constituents is treated as a continuum occupying the whole volume of the mixture. The connection with the real mixture can be established through a homogenization procedure. Clearly, the validity of such treatment can be expected only when the characteristic size of the structural constitution of the mixture (i.e. atomic or molecular level mixture, size of the particles of granular mixture, or size of the pores in porous media) is "much" smaller than the characteristic size of the mixture body and the size of the disturbances imposed on the body.

1. Preliminary definitions

Let us consider a mixture of N constituents. For each constituent there is a reference configuration denoted by $\Omega^\alpha \subset \mathbb{R}^3, \alpha = 1, \dots, N$ and a sufficiently smooth, one to one mapping of the reference configuration Ω^α to the current configuration $\Omega_t^\alpha \subset \mathbb{R}^3$.

$$\vec{\chi}^\alpha : \Omega^\alpha \times [0,T] \mapsto \Omega_t^\alpha, \quad \alpha = 1, \dots, N. \tag{1}$$

The mixture occupies a region $\Omega \subseteq \bigcap_{\alpha=1}^N \Omega_t^\alpha$ in the three dimensional Euclidean space $\mathbb{R}^3$. Co-occupancy, the basic assumption of the mixture theory, requires that for any point in the mixture $\vec{x} \in \Omega$ and time $t \in [0,T]$ there exist material points $\vec{X}^\alpha \in \Omega^\alpha, \alpha = 1, \dots, N$ such that

$$\vec{x} = \vec{\chi}^\alpha(\vec{X}^\alpha, t), \quad \alpha = 1, \dots, N, \tag{2}$$

i.e. the particles of all constituents coexist at a given spatial point.

Let $\mathcal{P} \subset \Omega$ be an arbitrary part of the mixture and let $V(\mathcal{P}) = \int_{\mathcal{P}} dv$ denote the volume occupied by the part $\mathcal{P}$ in the Euclidean space $\mathbb{R}^3$ (i.e. Lebesgue measure). We assume that the mixture configuration is equipped with a mass measure $M(\mathcal{P})$ representing the mass of the part $\mathcal{P}$ and partial mass measures $M^\alpha(\mathcal{P})$ as mass of the α-th constituent in the part $\mathcal{P}$.

Further, we assume that there are measures $V^\alpha(\mathcal{P})$ associated with each constituent which represents the information about the distribution of the material α in the part $\mathcal{P}$. This information has to come from the homogenization process where it can be interpreted as the volume occupied by the α-th constituent in the volume occupied by the part $\mathcal{P}$.

These measures are assumed to be absolutely continuous with respect to the volume measure V, i.e.

$$M(\mathcal{P}) \ll V(\mathcal{P}), \tag{3}$$

$$M^\alpha(\mathcal{P}) \ll V(\mathcal{P}), \tag{4}$$

$$V^\alpha(\mathcal{P}) \ll V(\mathcal{P}), \tag{5}$$

and the partial mass measure M^α to be absolutely continuous with respect to the mixture mass and the partial volume measure V^α of the same constituent

$$M^\alpha(\mathcal{P}) \ll M(\mathcal{P}), \tag{6}$$

$$M^\alpha(\mathcal{P}) \ll V^\alpha(\mathcal{P}). \tag{7}$$

Then there exist densities corresponding to these measures by the Radon-Nikodým theorem. The mixture mass density ϱ and the constituent partial mass density ϱ^α satisfy

$$M(\mathcal{P}) = \int_{\mathcal{P}} \varrho \, dv, \tag{8}$$

$$M^\alpha(\mathcal{P}) = \int_{\mathcal{P}} \varrho^\alpha \, dv. \tag{9}$$

and the distribution density ϕ^α satisfies

$$V^\alpha(\mathcal{P}) = \int_{\mathcal{P}} \phi^\alpha \, dv. \tag{10}$$

In the mixture theory the quantity ϕ^α is usually called volume fraction.

For a non-reacting mixture the assumption that the partial mass and distribution measures are additive is adopted. We require that the mass of the mixture in the part $\mathcal{P}$ is equal to the sum of the partial masses

$$\sum_{\alpha=1}^{N} M^\alpha(\mathcal{P}) = M(\mathcal{P}) \Rightarrow \sum_{\alpha=1}^{N} \varrho^\alpha = \varrho. \tag{11}$$

For the distribution measures we require the sum of the partial distribution measures to be equal to the mixture volume

$$\sum_{\alpha=1}^{N} V^\alpha(\mathcal{P}) = V(\mathcal{P}) \Rightarrow \sum_{\alpha=1}^{N} \phi^\alpha = 1. \tag{12}$$

Finally, the assumption that the partial mass measures M^α are absolutely continuous with respect to the corresponding partial distribution measure V^α, gives rise to the densities denoted by ϱ_t^α and defined by

$$M^\alpha(\mathcal{P}) = \int_{\mathcal{P}} \varrho_t^\alpha \, dV^\alpha = \int_{\mathcal{P}} \varrho_t^\alpha \phi^\alpha \, dv, \tag{13}$$

using (9) we obtain

$$\varrho^\alpha = \varrho_t^\alpha \phi^\alpha. \tag{14}$$

The quantity ϱ_t^α is usually called the true density of the constituent.

In the mixture theory the material distribution ϕ^α is additional independent field which have to be determined. One way to eliminate these additional variables is to make some additional assumption or impose an additional constraint. The simplest possibility available is to assume that the true densities are constant.

Now, we can define the mechanical fields as in classical continuum mechanics. For each constituent the velocity field is defined as

$$\vec{v}^{\alpha} = \frac{\partial \vec{\chi}^{\alpha}}{\partial t}, \tag{15}$$

and the deformation gradient for each constituent with respect to its reference configuration is defined as

$$\boldsymbol{F}^{\alpha} = \frac{\partial \vec{\chi}^{\alpha}}{\partial \vec{X}}. \tag{16}$$

Let us adopt following notations for the derivatives. Any field quantity φ with values in some space Y (i.e. scalar, vector or tensor valued) can be expressed in the Eulerian description as a function of the spatial position $\vec{x} \in \Omega$

$$\varphi = \tilde{\varphi}(\vec{x}, t) : \Omega \times [0, T] \mapsto Y.$$

Then we define following notations for the derivatives of the field φ

$$\frac{\partial \varphi}{\partial t} := \frac{\partial \tilde{\varphi}}{\partial t}, \tag{17}$$

$$\operatorname{grad} \varphi = \frac{\partial \varphi}{\partial \vec{x}} := \frac{\partial \tilde{\varphi}}{\partial \vec{x}}. \tag{18}$$

In the case of Lagrangian description we have to specify which material is taken as reference. Let α be the constituent we take as a reference then for any $\vec{X} \in \Omega^{\alpha}$ we can express the quantity φ as

$$\varphi = \overset{\alpha}{\varphi}(\vec{X}, t) : \Omega^{\alpha} \times [0, T] \mapsto Y,$$

and we define the derivatives of the field φ as

$$\frac{d^{\alpha} \varphi}{dt} := \frac{\partial \overset{\alpha}{\varphi}}{\partial t}, \tag{19}$$

$$\operatorname{Grad}^{\alpha} \varphi = \frac{\partial \varphi}{\partial \vec{X}^{\alpha}} := \frac{\partial \overset{\alpha}{\varphi}}{\partial \vec{X}}. \tag{20}$$

These two descriptions can be related to each other through following relations

$$\overset{\alpha}{\varphi}(\vec{X}, t) = \tilde{\varphi}(\vec{\chi}^{\alpha}(\vec{X}, t), t), \tag{21}$$

$$\frac{d^{\alpha} \varphi}{dt} = \frac{\partial \varphi}{\partial t} + (\operatorname{grad} \varphi) \vec{v}^{\alpha}, \tag{22}$$

$$\operatorname{Grad}^{\alpha} \varphi = (\operatorname{grad} \varphi) \boldsymbol{F}^{\alpha}. \tag{23}$$

2. Balance laws of general mixture

The balance laws for each constituent are postulated in a similar way as in the case of single continuum. Let $\mathcal{P} \subset \Omega$ be a fixed region occupied by the mixture and $\vec{n}$ denotes the unit, outward normal vector to the boundary $\partial \mathcal{P}$ of the part $\mathcal{P}$.

2.1. Balance of mass. In the case of multicomponent mixture we can include a mass inter-conversion between the constituents. Let m^{α} denote a mass source of the α-th constituent. Then the mass balance can be stated as

$$\frac{\partial}{\partial t} \int_{\mathcal{P}} \varrho^{\alpha} \, dv + \int_{\partial \mathcal{P}} \varrho^{\alpha} \vec{v}^{\alpha} \cdot \vec{n} \, da = \int_{\mathcal{P}} m^{\alpha} \, dv. \tag{24}$$

If all the fields are sufficiently smooth this equation can be written in local form as

$$\frac{\partial \varrho^\alpha}{\partial t} + \operatorname{div} \varrho^\alpha \vec{v}^\alpha = m^\alpha. \tag{25}$$

Further requirement that the mass conversion is only between the constituents but the overall mixture mass is conserved leads to

$$\sum_{\alpha=1}^{N} m^\alpha = 0. \tag{26}$$

It is useful to derive the mass balance equation from the Lagrangian point of view. Let $\mathcal{P}^\alpha \subset \Omega^\alpha$ be a fixed set of particles belonging to the material α. Then $\chi^\alpha(\mathcal{P}^\alpha) \subset \Omega$ is a region occupied by these particles at current time, and the balance of mass can be expressed as

$$\frac{d^\alpha}{dt} \int_{\chi^\alpha(\mathcal{P}^\alpha)} \varrho^\alpha \, dv = \int_{\chi^\alpha(\mathcal{P}^\alpha)} m^\alpha \, dv, \tag{27}$$

which in local form can be written as

$$\frac{d^\alpha}{dt}(\varrho^\alpha \det \boldsymbol{F}^\alpha) = m^\alpha \det \boldsymbol{F}^\alpha. \tag{28}$$

2.2. Balance of linear momentum. Let $\boldsymbol{\sigma}^\alpha$ denote the Cauchy stress tensor associated with the α-th constituent, representing the surface forces per mixture unit area, $\vec{f}^\alpha$ be the part of the body force acting on the α-th constituent per its unit mass, and $\vec{b}^\alpha$ denote the momentum source of the α-th component from the rest of the mixture. The balance of linear momentum is stated as

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\mathcal{P}} \varrho^\alpha \vec{v}^\alpha \, dv + \int_{\partial\mathcal{P}} \varrho^\alpha \vec{v}^\alpha \otimes \vec{v}^\alpha \vec{n} \, da = \int_{\partial\mathcal{P}} (\boldsymbol{\sigma}^\alpha)^T \, \vec{n} \, da \\ + \int_{\mathcal{P}} \left(\varrho^\alpha \vec{f}^\alpha + \vec{b}^\alpha \right) dv. \end{aligned} \tag{29}$$

The local form of the linear momentum balance is

$$\frac{\partial \varrho^\alpha \vec{v}^\alpha}{\partial t} + \operatorname{div} \varrho^\alpha \vec{v}^\alpha \otimes \vec{v}^\alpha = \operatorname{div} (\boldsymbol{\sigma}^\alpha)^T + \varrho^\alpha \vec{f}^\alpha + \vec{b}^\alpha. \tag{30}$$

For the whole mixture the action-reaction principle implies

$$\sum_{\alpha=1}^{N} \vec{b}^\alpha = 0. \tag{31}$$

2.3. Balance of angular momentum. We assume that the material is non-polar, i.e. there are no external sources of angular momentum. Let $\vec{p}^\alpha$ denote the angular momentum interaction between the α-th component and the rest of the mixture. The balance of angular momentum is

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\mathcal{P}} \vec{x} \times \varrho^\alpha \vec{v}^\alpha \, dv + \int_{\partial\mathcal{P}} \vec{x} \times \varrho^\alpha \vec{v}^\alpha \otimes \vec{v}^\alpha \vec{n} \, da = \int_{\partial\mathcal{P}} \vec{x} \times (\boldsymbol{\sigma}^\alpha)^T \, \vec{n} \, da \\ + \int_{\mathcal{P}} \vec{x} \times \varrho^\alpha \vec{f}^\alpha \, dv \\ + \int_{\mathcal{P}} \vec{p}^\alpha \, dv. \end{aligned} \tag{32}$$

Again expressed in local form it can be written as

$$\frac{\partial \vec{x} \times \varrho^\alpha \vec{v}^\alpha}{\partial t} + \operatorname{div}\left(\vec{x} \times \varrho^\alpha \vec{v}^\alpha \otimes \vec{v}^\alpha\right) = \operatorname{div}\left(\vec{x} \times \left(\boldsymbol{\sigma}^\alpha\right)^T\right) + \vec{x} \times \varrho^\alpha \vec{f}^\alpha + \vec{p}^\alpha. \tag{33}$$

For the mixture as a whole the interactions are assumed to be balanced, which implies

$$\sum_{\alpha=1}^{N} \vec{p}^\alpha = 0. \tag{34}$$

This also implies that the total stress of the mixture, defined as $\boldsymbol{\sigma} = \sum_{\alpha=1}^{N} \boldsymbol{\sigma}^\alpha$, is symmetric

$$\boldsymbol{\sigma} = \left(\boldsymbol{\sigma}\right)^T. \tag{35}$$

If the internal angular momentum interactions $\vec{p}^\alpha$ are assumed to be caused by the momentum interactions, i.e.

$$\vec{p}^\alpha = \vec{x} \times \vec{b}^\alpha, \tag{36}$$

then the partial stresses are also symmetric

$$\boldsymbol{\sigma}^\alpha = \left(\boldsymbol{\sigma}^\alpha\right)^T. \tag{37}$$

2.4. Balance of energy. Let e^α be the internal energy associated with the α-th constituent per its unit mass, $\vec{q}^\alpha$ be the heat flux per unit area of the constituent, r^α be the heat source per unit mass of the constituent and ε^α be the energy interaction among the constituent and the rest of the mixture. Denoting the total constituent energy per its mass as $E^\alpha = e^\alpha + \frac{1}{2}\left|\vec{v}^\alpha\right|^2$, the balance of energy is introduced as

$$\begin{aligned} \frac{\partial}{\partial t}\int_{\mathcal{P}} \varrho^\alpha E^\alpha \, dv + \int_{\partial\mathcal{P}} \varrho^\alpha E^\alpha \vec{v}^\alpha \cdot \vec{n} \, da = & \int_{\partial\mathcal{P}} \left(\left(\boldsymbol{\sigma}^\alpha\right)^T \vec{v}^\alpha - \vec{q}^\alpha\right) \cdot \vec{n} \, da \\ & + \int_{\mathcal{P}} \left(\varrho^\alpha \vec{f}^\alpha \cdot \vec{v}^\alpha + \varrho^\alpha r^\alpha\right) dv \\ & + \int_{\mathcal{P}} \varepsilon^\alpha \, dv. \end{aligned} \tag{38}$$

Writing this in local form and using the mass and momentum balance equations we obtain

$$\begin{aligned} \frac{\partial \varrho^\alpha e^\alpha}{\partial t} + \operatorname{div} \varrho^\alpha e^\alpha \vec{v}^\alpha = & \operatorname{tr}\left(\left(\boldsymbol{\sigma}^\alpha\right)^T \operatorname{grad} \vec{v}^\alpha\right) - \operatorname{div} \vec{q}^\alpha + \varrho^\alpha r^\alpha \\ & - \vec{b}^\alpha \cdot \vec{v}^\alpha + \frac{1}{2} m^\alpha \left|\vec{v}^\alpha\right|^2 + \varepsilon^\alpha. \end{aligned} \tag{39}$$

Again, we require the energy interaction between the constituents to be in balance, i.e.

$$\sum_{\alpha=1}^{N} \varepsilon^\alpha = 0. \tag{40}$$

2.5. Balance of entropy. Finally, the balance of entropy is postulated. Let η^α be the constituent entropy per its unit mass, T^α be a temperature assigned to each constituent and s^α be the entropy exchange among the constituent and the rest of the mixture, then the entropy balance is

(41)
$$\frac{\partial}{\partial t}\int_{\mathcal{P}} \varrho^\alpha \eta^\alpha \, dv + \int_{\partial\mathcal{P}} \varrho^\alpha \eta^\alpha \vec{v}^\alpha \cdot \vec{n} \, da = -\int_{\partial\mathcal{P}} \frac{\vec{q}^\alpha}{T^\alpha} \cdot \vec{n} \, da + \int_{\mathcal{P}} \left(\frac{\varrho^\alpha r^\alpha}{T^\alpha} + s^\alpha \right) dv,$$

or in local form

$$\frac{\partial \varrho^\alpha \eta^\alpha}{\partial t} + \operatorname{div} \varrho^\alpha \eta^\alpha \vec{v}^\alpha = -\operatorname{div} \frac{\vec{q}^\alpha}{T^\alpha} + \frac{\varrho^\alpha r^\alpha}{T^\alpha} + s^\alpha. \tag{42}$$

Summing these equations together over all constituents we obtain

$$\frac{\partial \sum_\alpha \varrho^\alpha \eta^\alpha}{\partial t} + \operatorname{div} \sum_\alpha \varrho^\alpha \eta^\alpha \vec{v}^\alpha = \operatorname{div} \sum_\alpha \frac{\vec{q}^\alpha}{T^\alpha} + \sum_\alpha \frac{\varrho^\alpha r^\alpha}{T^\alpha} + \sum_\alpha s^\alpha. \tag{43}$$

The second law of thermodynamics says that the total entropy production is non-negative, implying

$$\sum_{\alpha=1}^{N} s^\alpha \geq 0. \tag{44}$$

In the rest of this work we assume that all the processes are isothermal, all the constituents are at the same temperature T and there is no mass conversion between the constituents, i.e. m^α is identically zero for all constituents. Introducing the Helmholtz potential $\Psi^\alpha = e^\alpha - T\eta^\alpha$ and using (39), (43) and (44) we obtain the reduced entropy inequality in the following form

$$\frac{\partial \sum_\alpha \varrho^\alpha \Psi^\alpha}{\partial t} + \operatorname{div} \sum_\alpha \varrho^\alpha \Psi^\alpha \vec{v}^\alpha - \operatorname{tr}\left((\boldsymbol{\sigma}^\alpha)^T \operatorname{grad} \vec{v}^\alpha \right) + \vec{b}^\alpha \cdot \vec{v}^\alpha \leq 0. \tag{45}$$

This inequality is later used to narrow the class of constitutive relations for the stresses $\boldsymbol{\sigma}^\alpha$ and momentum interactions $\vec{b}^\alpha$.

3. Binary mixture of a solid and a fluid

Let us now focus on the steady state diffusion of one fluid through an incompressible homogeneous elastic solid subjected to a finite deformation, without external body forces $\vec{f}^\alpha = \vec{0}$. The motion of each constituent of the mixture is described by mappings

$$\vec{\chi}^s : \Omega^s \times [0,T] \mapsto \Omega \tag{46}$$

$$\vec{\chi}^f : \Omega^f \times [0,T] \mapsto \Omega \tag{47}$$

where the superscripts s and f refer to quantities associated with the solid and fluid, respectively. At each time $t \in [0,T]$, the mappings are one-to-one and sufficiently smooth to render the various mathematical operations meaningful.

3.1. Balance equations. Following the definitions from the previous section we denote by ϱ^s and ϱ^f the partial densities of the solid and fluid, respectively. The total density of the mixture is given by $\varrho = \varrho^s + \varrho^f$. The deformation gradient for the solid is given by $\boldsymbol{F} = \frac{\partial \vec{x}^s}{\partial \vec{X}^s}$, and the velocity vector of the solid $\vec{v}^s = \frac{\partial \vec{x}^s}{\partial t}$. In the absence of inter conversion between the constituents, the equation for the conservation of mass for the solid is

$$\frac{\partial \varrho^s}{\partial t} + \operatorname{div}(\varrho^s \vec{v}^s) = 0. \tag{48}$$

The conservation of mass (48) can also be expressed in the Lagrangian form

$$\varrho^s \det \boldsymbol{F} = \varrho_0^s, \tag{49}$$

where ϱ_0^s is the density of the solid in the undeformed state. Let $\boldsymbol{\sigma}^s$ and $\boldsymbol{\sigma}^f$ denote the partial stresses associated with the solid and fluid constituents, respectively, and $\vec{b}^s$ and $\vec{b}^f$ denote the interactive forces acting on the solid and fluid. The balance of linear momentum for the solid is given by

$$\frac{\partial(\varrho^s \vec{v}^s)}{\partial t} + \operatorname{div}(\varrho^s \vec{v}^s \otimes \vec{v}^s) = \operatorname{div} \boldsymbol{\sigma}^s + \vec{b}^s. \tag{50}$$

For the fluid constituent we define the velocity $\vec{v}^f = \frac{\partial \vec{x}^f}{\partial t}$ and the conservation of mass and the balance of linear momentum for this constituent are

$$\frac{\partial \varrho^f}{\partial t} + \operatorname{div}(\varrho^f \vec{v}^f) = 0, \tag{51}$$

$$\frac{\partial(\varrho^f \vec{v}^f)}{\partial t} + \operatorname{div}(\varrho^f \vec{v}^f \otimes \vec{v}^f) = \operatorname{div} \boldsymbol{\sigma}^f + \vec{b}^f. \tag{52}$$

By the action-reaction principle we have $\vec{b} = \vec{b}^f = -\vec{b}^s$. For the mixture, we define the total stress $\boldsymbol{\sigma} = \boldsymbol{\sigma}^s + \boldsymbol{\sigma}^f$ which is symmetric as required by the balance of the angular momentum.

The entropy inequality for the mixture is used to restrict the class of constitutive forms for $\boldsymbol{\sigma}^s$, $\boldsymbol{\sigma}^f$ and $\vec{b}$. The reduced entropy inequality (45) for the binary mixture can be written as

$$\begin{aligned} &\frac{\partial(\varrho\Psi)}{\partial t} + \operatorname{div}(\varrho^s \Psi^s \vec{v}^s + \varrho^f \Psi^f \vec{v}^f) \\ &\qquad - \operatorname{tr}\left(\boldsymbol{\sigma}^s \operatorname{grad} \vec{v}^s + \boldsymbol{\sigma}^f \operatorname{grad} \vec{v}^f\right) + \vec{b} \cdot (\vec{v}^s - \vec{v}^f) \leq 0, \end{aligned} \tag{53}$$

where $\varrho\Psi = \varrho^s \Psi^s + \varrho^f \Psi^f$. For a detailed discussion of the above derivations and related issues we refer to [**23**], [**33**] and [**22**].

3.2. Kinematical constraints. The distribution of each constituent is described by the quantities ϕ^s and ϕ^f and can be interpreted as the volumetric amount of the constituent per unit volume of the mixture. The volume additivity constraint for binary mixture

$$\phi^s + \phi^f = 1, \tag{54}$$

can be interpreted as requirement that the mixture consists only of these two components, not allowing voids. The true density of each constituent, denoted by ϱ_t^s and ϱ_t^f satisfy

$$\varrho^s = \phi^s \varrho_t^s, \tag{55}$$

$$\varrho^f = \phi^f \varrho_t^f. \tag{56}$$

So far the quantities $\varrho^s, \varrho^f, \phi^s, \phi^f$ are independent unknown fields and we have to add some more additional assumptions to be able to determine them.

The approach adopted in this work assumes that the true densities of both constituents are constant, and effectively eliminates for example the densities ϱ^s, ϱ^f by (55) and (56), leaving us with the volume fractions ϕ^s, ϕ^f as the unknowns. Then the volume additivity requirement (54) has to be considered as an additional kinematical constraint. It implies together with equations (48), (51) and (54) that

$$\operatorname{div}(\phi^s \vec{v}^s + \phi^f \vec{v}^f) = 0. \tag{57}$$

This constraint, written in integral form as

$$\int_{\partial \mathcal{P}} (\phi^s \vec{v}^s \cdot \vec{n} + \phi^f \vec{v}^f \cdot \vec{n})\, da = 0, \tag{58}$$

is equivalent to the requirement that the mixture as a whole preserves its volume or that the mixture can undergo only isochoric deformations in the sense that at any time instant the net volumetric flux through the boundary $\partial\mathcal{P}$ of the mixture part $\mathcal{P}$ is zero. Such mixture can be called an incompressible mixture. This constraint will be included into the formulation with corresponding Lagrange multiplier.

Other possible approach is to keep the quantities $\varrho^s, \varrho^f, \phi^s, \phi^f$ as independent and assume the two additional equations (54) and (57). In the case of binary mixture this seems to give sufficient number of equations. The resulting model would be volume preserving mixture of compressible components.

3.3. Constitutive equations. We will restrict our considerations to isothermal, non-reacting mixtures where the solid constituent is homogeneous and hyperelastic. For such mixtures it can be shown, using the second law of thermodynamics and the postulate of material frame indifference, that the Helmholtz potential can be expressed as a function of the deformation tensor $\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}$ and the fluid volume fraction ϕ^f, i.e.,

$$\Psi = \hat{\Psi}(\boldsymbol{C}, \phi^f). \tag{59}$$

Using techniques that are standard in continuum mechanics (see [**22**]) we obtain an expression for the partial stresses $\boldsymbol{\sigma}^s$ and $\boldsymbol{\sigma}^f$ and the interaction force $\vec{b}$ such that the entropy inequality is satisfied for all admissible processes. Using equation (53), the assumption that the temperature T is constant and incorporating the volume additivity constraint (57) by the method of Lagrange multiplier we obtain

$$\begin{aligned} \frac{\partial(\varrho\Psi)}{\partial t} + \operatorname{div}(\varrho^s \Psi^s \vec{v}^s + \varrho^f \Psi^f \vec{v}^f) - \operatorname{tr}\left(\boldsymbol{\sigma}^s \operatorname{grad} \vec{v}^s + \boldsymbol{\sigma}^f \operatorname{grad} \vec{v}^f\right) \\ + \vec{b} \cdot (\vec{v}^s - \vec{v}^f) + p \operatorname{div}(\phi^s \vec{v}^s + \phi^f \vec{v}^f) \leq 0, \end{aligned} \tag{60}$$

where the Lagrange multiplier is denoted by p. Inserting the function $\hat{\Psi}(\boldsymbol{C}, \phi^f)$ for Ψ and using the chain rule and the balance laws (49), (50), (51), (52) and (54) we

obtain

$$\begin{aligned}
&\operatorname{tr}\Big(\big(\boldsymbol{\sigma}^s - \frac{\varrho^f\varrho^s}{\varrho}(\hat\Psi^s-\hat\Psi^f)\boldsymbol{I} + p\phi^s\boldsymbol{I} - 2\varrho\boldsymbol{F}\frac{\partial\hat\Psi}{\partial\boldsymbol{C}}\boldsymbol{F}^T\big)\nabla\vec v^s\\
&\qquad + \big(\boldsymbol{\sigma}^f + \frac{\varrho^f\varrho^s}{\varrho}(\hat\Psi^s-\hat\Psi^f)\boldsymbol{I} + p\phi^f\boldsymbol{I} + \varrho\phi^f\frac{\partial\hat\Psi}{\partial\phi^f}\boldsymbol{I}\big)\nabla\vec v^f\Big)\\
&\quad -\Big(\vec b + \nabla\big(\frac{\varrho^f\varrho^s}{\varrho}(\hat\Psi^s-\hat\Psi^f)\big) + p\nabla\phi^s - \varrho\frac{\partial\hat\Psi}{\partial\phi^f}\nabla\phi^f + \varrho^f\nabla\hat\Psi\Big)\cdot(\vec v^s-\vec v^f) \geq 0.
\end{aligned} \tag{61}$$

Now if we choose the partial stresses to be given by

$$\boldsymbol{\sigma}^s = \frac{\varrho^s\varrho^f}{\varrho}(\hat\Psi^s-\hat\Psi^f)\boldsymbol{I} - \phi^s p\boldsymbol{I} + 2\varrho\boldsymbol{F}\frac{\partial\hat\Psi}{\partial\boldsymbol{C}}\boldsymbol{F}^T + \hat{\boldsymbol{\sigma}}^s \tag{62}$$

$$\boldsymbol{\sigma}^f = -\frac{\varrho^s\varrho^f}{\varrho}(\hat\Psi^s-\hat\Psi^f)\boldsymbol{I} - \phi^f p\boldsymbol{I} - \varrho\phi^f\frac{\partial\hat\Psi}{\partial\phi^f}\boldsymbol{I} + \hat{\boldsymbol{\sigma}}^f \tag{63}$$

and the interaction force to have the form

$$\vec b = -\nabla\left(\frac{\varrho^s\varrho^f}{\varrho}(\hat\Psi^s-\hat\Psi^f)\right) - p\nabla\phi^s + \varrho\frac{\partial\hat\Psi}{\partial\phi^f}\nabla\phi^f - \varrho^f\nabla\hat\Psi + \hat{\vec b}, \tag{64}$$

where $\hat{\boldsymbol{\sigma}}^f$, $\hat{\boldsymbol{\sigma}}^s$, $\hat{\vec b}$ are the parts of the partial stresses that depend only on the dynamical variables, i.e. $\vec v^s - \vec v^f$, $\nabla\vec v^s$, $\nabla\vec v^f$, then the entropy inequality reduces to

$$\operatorname{tr}\big(\hat{\boldsymbol{\sigma}}^s\boldsymbol{D}^s + \hat{\boldsymbol{\sigma}}^f\boldsymbol{D}^f + \hat{\boldsymbol{\sigma}}^s(\boldsymbol{W}^s-\boldsymbol{W}^f)\big) - \hat{\vec b}\cdot(\vec v^s-\vec v^f) \geq 0, \tag{65}$$

where $\boldsymbol{D}^s$, $\boldsymbol{W}^s$, $\boldsymbol{D}^f$, $\boldsymbol{W}^f$ denote the symmetric and skew-symmetric part of the velocity gradients.

Note that the terms containing the partial Helmholtz potentials $\hat\Psi^s$ and $\hat\Psi^f$ in the equations (62), (63) and (64) will cancel in the equations of motion and even in the expression for total stress $\boldsymbol{\sigma}$.

For example, assuming $\hat{\boldsymbol{\sigma}}^f$, $\hat{\boldsymbol{\sigma}}^s$, $\hat{\vec b}$ to be linear functions of their arguments we have (see. [**36**])

$$\hat{\boldsymbol{\sigma}}^s = \gamma_{11}\operatorname{tr}\boldsymbol{D}^s\boldsymbol{I} + 2\mu_{11}\boldsymbol{D}^s + \gamma_{12}\operatorname{tr}\boldsymbol{D}^f\boldsymbol{I} + 2\mu_{12}\boldsymbol{D}^f - c(\boldsymbol{W}^s-\boldsymbol{W}^f) \tag{66}$$

$$\hat{\boldsymbol{\sigma}}^f = \gamma_{21}\operatorname{tr}\boldsymbol{D}^s\boldsymbol{I} + 2\mu_{21}\boldsymbol{D}^s + \gamma_{22}\operatorname{tr}\boldsymbol{D}^f\boldsymbol{I} + 2\mu_{22}\boldsymbol{D}^f + c(\boldsymbol{W}^s-\boldsymbol{W}^f) \tag{67}$$

$$\hat{\vec b} = \phi^s\phi^f\boldsymbol{\alpha}(\vec v^s-\vec v^f) \tag{68}$$

where γ and μ are restricted by equation (65) to be

$$\mu_{11} \geq 0, \qquad \mu_{22} \geq 0, \tag{69}$$

$$\gamma_{11} + \frac{2}{3}\mu_{11} \geq 0, \qquad \gamma_{22} + \frac{2}{3}\mu_{22} \geq 0, \tag{70}$$

$$4\mu_{11}\mu_{22} \geq (\mu_{12}+\mu_{21})^2, \tag{71}$$

$$4(\gamma_{11}+\frac{2}{3}\mu_{11})(\gamma_{22}+\frac{2}{3}\mu_{22}) \geq (\gamma_{12}+\frac{2}{3}\mu_{12}+\gamma_{21}+\frac{2}{3}\mu_{21})^2, \tag{72}$$

$$c \geq 0, \tag{73}$$

and $\boldsymbol{\alpha}$ is positive-definite tensor, in general (see [**37**] for the derivation of these results).

For isotropic materials the dependence of $\hat{\Psi}$ on $\boldsymbol{C}$ is only through the following three invariants

$$I_C = \operatorname{tr} \boldsymbol{C}, \qquad II_C = \frac{1}{2}(\operatorname{tr}^2 \boldsymbol{C} - \operatorname{tr} \boldsymbol{C}^2), \qquad III_C = \det \boldsymbol{C}. \tag{74}$$

Combining (54), (49), (55), (56) and using the definitions of III_C and $\boldsymbol{C}$, we can see that the fluid volume fraction ϕ^f can be expressed as a function of the third invariant III_C, i.e.,

$$\phi^f = 1 - \phi^s = 1 - \frac{\phi_0^s}{\sqrt{III_C}}. \tag{75}$$

This allows us to eliminate either ϕ^f or III_C in the constitutive relation for Ψ in favor of the other or ϕ^s. On the other hand, since we incorporated the constraint (54) in our formulation via a Lagrange multiplier, we can treat the quantities ϕ^f and ϕ^s as independent and the equation (54) is considered as an additional equation to be added to the balance equations.

For simplicity, let us set

$$\hat{\boldsymbol{\sigma}}^s = 0, \qquad \hat{\boldsymbol{\sigma}}^f = 0, \qquad \hat{\vec{b}} = \phi^s \phi^f \boldsymbol{\alpha}(\vec{v}^s - \vec{v}^f)$$

where $\boldsymbol{\alpha}$ is positive definite tensor, and denote

$$\bar{\boldsymbol{\sigma}}^s = \boldsymbol{\sigma}^s - \frac{\varrho^s \varrho^f}{\varrho}(\Psi^s - \Psi^f)\boldsymbol{I}, \tag{76}$$

$$\bar{\boldsymbol{\sigma}}^f = \boldsymbol{\sigma}^f + \frac{\varrho^s \varrho^f}{\varrho}(\Psi^s - \Psi^f)\boldsymbol{I}, \tag{77}$$

$$\bar{\vec{b}} = \vec{b} + \nabla\left(\frac{\varrho^s \varrho^f}{\varrho}(\Psi^s - \Psi^f)\right). \tag{78}$$

We shall assume that $\Psi = \bar{\Psi}(I_C, II_C, \phi^s, \phi^f)$. Then equations (62) and (63) yield

$$\bar{\boldsymbol{\sigma}}^s = -\phi^s(p + \varrho\frac{\partial \bar{\Psi}}{\partial \phi^s})\boldsymbol{I} + 2\varrho\left[\frac{\partial \bar{\Psi}}{\partial I_C}\boldsymbol{B} + \frac{\partial \bar{\Psi}}{\partial II_C}(I_C \boldsymbol{B} - \boldsymbol{B}^2)\right], \tag{79}$$

$$\bar{\boldsymbol{\sigma}}^f = -\phi^f(p + \varrho\frac{\partial \bar{\Psi}}{\partial \phi^f})\boldsymbol{I}, \tag{80}$$

$$\bar{\vec{b}} = -(p + \varrho\frac{\partial \bar{\Psi}}{\partial \phi^f})\nabla\phi^s - \varrho^f \nabla\bar{\Psi} + \phi^s\phi^f\boldsymbol{\alpha}(\vec{v}^s - \vec{v}^f). \tag{81}$$

Now, introducing $\bar{p} = p + \varrho\frac{\partial \bar{\Psi}}{\partial \phi^f}$ and $\tilde{\Psi}(I_C, II_C, \phi^s) = \bar{\Psi}(I_C, II_C, \phi^s, 1 - \phi^s)$ we finally obtain

$$\bar{\boldsymbol{\sigma}}^s = -\phi^s(\bar{p} + \varrho\frac{\partial \tilde{\Psi}}{\partial \phi^s})\boldsymbol{I} + 2\varrho\left[\frac{\partial \tilde{\Psi}}{\partial I_C}\boldsymbol{B} + \frac{\partial \tilde{\Psi}}{\partial II_C}(I_C \boldsymbol{B} - \boldsymbol{B}^2)\right], \tag{82}$$

$$\bar{\boldsymbol{\sigma}}^f = -\phi^f \bar{p}\boldsymbol{I}, \tag{83}$$

$$\bar{\vec{b}} = -\bar{p}\nabla\phi^s - \varrho^f\nabla\tilde{\Psi} + \phi^s\phi^f\boldsymbol{\alpha}(\vec{v}^s - \vec{v}^f). \tag{84}$$

Substituting these constitutive equations into the balance of linear momentum for the constituents, we obtain

$$\frac{\partial(\varrho^s \vec{v}^s)}{\partial t} + \operatorname{div}(\varrho^s \vec{v}^s \otimes \vec{v}^s) = -\phi^s \nabla \bar{p} + \operatorname{div} \boldsymbol{\sigma}^E + \varrho^f \nabla \tilde{\Psi} - \phi^s \phi^f \boldsymbol{\alpha}(\vec{v}^s - \vec{v}^f), \tag{85}$$

$$\frac{\partial(\varrho^f \vec{v}^f)}{\partial t} + \operatorname{div}(\varrho^f \vec{v}^f \otimes \vec{v}^f) = -\phi^f \nabla \bar{p} - \varrho^f \nabla \tilde{\Psi} + \boldsymbol{\alpha} \phi^s \phi^f (\vec{v}^s - \vec{v}^f), \tag{86}$$

with $\boldsymbol{\sigma}^E = 2\varrho\big[\frac{\partial \tilde{\Psi}}{\partial I_C}\boldsymbol{B} + \frac{\partial \tilde{\Psi}}{\partial II_C}(I_C \boldsymbol{B} - \boldsymbol{B}^2)\big] - \varrho \phi^s \frac{\partial \tilde{\Psi}}{\partial \phi^s}\boldsymbol{I}$. These equations together with the constraint (54), mass balance equations (49) and (51), suitable boundary conditions and constitutive relation for the function $\tilde{\Psi}$ form complete problem to solve.

CHAPTER 3

Applications

1. One dimensional problem

1.1. Determination of the response functions. In this section, we discuss a procedure for determining the response functions for a solid infused with a fluid. The analysis proceeds very much along the lines of the work of Rivlin and his co-workers [**45**] for purely elastic solids. Our aim is to provide hints for experiments with biological tissues. In such cases the fluid part of the mixture is diffusing with speed given by biological conditions, e.g. blood flow in capillaries, and for which the inertial effects might be significant.

Let us consider one-dimensional steady diffusion through a biaxially stretched rectangular slab $\Omega_0 = [-L, L] \times [-H, H] \times [-L, L]$, see figure 1, with a motion of the form

$$x_1 = f(X_1), \qquad x_2 = \lambda_2 X_2, \qquad x_3 = \lambda_3 X_3, \tag{87}$$

where $f(X_1)$ is an unknown function that allows for a non-homogeneous deformation in the X_1 direction, and λ_2, λ_3 are prescribed positive constants. The fluid velocity is assumed to be $\vec{v}^f(x_1) = (v_1^f(x_1), 0, 0)$. The deformation gradient $\boldsymbol{F}$ and the deformation tensors $\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}$ and $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$ have the forms

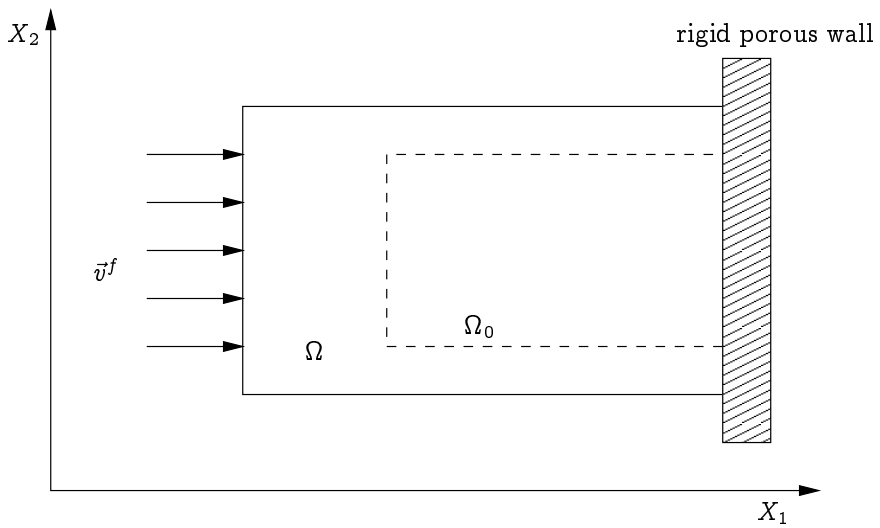

FIGURE 1. Undeformed and deformed configuration of the slab.

$$\boldsymbol{F} = \begin{pmatrix} f' & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \boldsymbol{C} = \boldsymbol{B} = \begin{pmatrix} f'^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}.$$

The invariants of $\boldsymbol{C}$ are

$$I_C = f'^2 + \lambda_2^2 + \lambda_3^2, \qquad II_C = f'^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + f'^2\lambda_3^2, \qquad III_C = f'^2\lambda_2^2\lambda_3^2.$$

Next, consider the following nondimensional quantities

$$x_1{}^\star = \frac{x_1}{H}, \qquad X_1{}^\star = \frac{X_1}{H}, \qquad f^\star = \frac{f}{H}, \qquad \varrho^\star = \frac{\varrho}{\varrho_t^f},$$

$$v_1^{f\star} = \frac{v_1^f}{V}, \qquad \Psi^\star = \frac{\Psi}{V^2}, \qquad p^\star = \frac{p}{\varrho_t^f V^2}, \qquad \alpha^\star = \frac{\alpha H}{V},$$

where H is the undeformed thickness of the slab and V is a characteristic fluid velocity. Nondimensional quantities are used in the rest of this study, thus we omit the superscript * below.

Suppose that the Helmholtz potential is a function of the first two invariants of the deformation tensor $\boldsymbol{C}$ and the volume fraction ϕ^s, i.e. $\Psi = \tilde{\Psi}(I_C, II_C, \phi^s)$. Thus, from (82) and (83) the total stress $\boldsymbol{\sigma}$ can be written as

$$\boldsymbol{\sigma} = -\big(p + \varrho\phi^s \frac{\partial\tilde{\Psi}}{\partial\phi^s}\big)\boldsymbol{I} + 2\varrho\left[\frac{\partial\tilde{\Psi}}{\partial I_C}\boldsymbol{B} + \frac{\partial\tilde{\Psi}}{\partial II_C}(I_C\boldsymbol{B} - \boldsymbol{B}^2)\right].$$

where the unknown response functions $\frac{\partial\tilde{\Psi}}{\partial I_C}$, $\frac{\partial\tilde{\Psi}}{\partial II_C}$ and $\frac{\partial\tilde{\Psi}}{\partial\phi^s}$ can depend on I_C, II_C, ϕ^s. In particular, for the deformation (87) we have

$$\sigma_{11} = -\tilde{p} + 2\varrho\left[\frac{\partial\tilde{\Psi}}{\partial I_C}f'^2 + \frac{\partial\tilde{\Psi}}{\partial II_C}(f'^2\lambda_2^2 + f'^2\lambda_3^2)\right], \tag{88}$$

$$\sigma_{22} = -\tilde{p} + 2\varrho\left[\frac{\partial\tilde{\Psi}}{\partial I_C}\lambda_2^2 + \frac{\partial\tilde{\Psi}}{\partial II_C}(f'^2\lambda_2^2 + \lambda_2^2\lambda_3^2)\right], \tag{89}$$

$$\sigma_{33} = -\tilde{p} + 2\varrho\left[\frac{\partial\tilde{\Psi}}{\partial I_C}\lambda_3^2 + \frac{\partial\tilde{\Psi}}{\partial II_C}(f'^2\lambda_3^2 + \lambda_2^2\lambda_3^2)\right], \tag{90}$$

where $\tilde{p} = p + \varrho\phi^s\frac{\partial\tilde{\Psi}}{\partial\phi^s}$. Now forming the equations for the stress differences we can eliminate the isotropic part of the stress

$$\sigma_{11} - \sigma_{22} = 2\varrho\left[\frac{\partial\tilde{\Psi}}{\partial I_C}(f'^2 - \lambda_2^2) + \frac{\partial\tilde{\Psi}}{\partial II_C}(f'^2\lambda_3^2 - \lambda_2^2\lambda_3^2)\right], \tag{91}$$

$$\sigma_{22} - \sigma_{33} = 2\varrho\left[\frac{\partial\tilde{\Psi}}{\partial I_C}(\lambda_2^2 - \lambda_3^2) + \frac{\partial\tilde{\Psi}}{\partial II_C}(f'^2\lambda_2^2 - f'^2\lambda_3^2)\right], \tag{92}$$

and thus solve for two of the response functions

$$\frac{\partial\tilde{\Psi}}{\partial I_C} = \frac{1}{2\varrho(f'^2 - \lambda_3^2)}\left[\frac{f'^2(\sigma_{11} - \sigma_{22})}{f'^2 - \lambda_2^2} - \frac{\lambda_3^2(\sigma_{22} - \sigma_{33})}{\lambda_2^2 - \lambda_3^2}\right], \tag{93}$$

$$\frac{\partial\tilde{\Psi}}{\partial II_C} = \frac{1}{2\varrho(f'^2 - \lambda_3^2)}\left[\frac{\sigma_{11} - \sigma_{22}}{f'^2 - \lambda_2^2} - \frac{\sigma_{22} - \sigma_{33}}{\lambda_2^2 - \lambda_3^2}\right]. \tag{94}$$

This is of the same form as for an elastic solid, but in the case of a mixture the quantities on the right hand side are functions of spatial position X_1. Of course, we seek the response functions as a function of I_C, II_C, ϕ^s.

Conservation of mass for the fluid yields (51)

$$\phi^f v_1^f = F_0, \tag{95}$$

where the constant F_0 can be measured as it is the mass flow rate of the fluid. Using this fact we can integrate the momentum equation for the mixture, obtained by summing the constituents momentum balances (50) and (52)

$$\phi^f \frac{dv_1^f}{dx_1} v_1^f - \frac{d\sigma_{11}}{dx_1} = 0, \tag{96}$$

to get

$$F_0 v_1^f - \sigma_{11} = M_0, \tag{97}$$

where the constant M_0 can be determined by measuring the quantities on the left hand side at one point. Then if it is possible to measure the fluid velocity at any other point X_1 in the slab this equation allows us to determine the stress σ_{11} at the same point. We can see that neglecting the inertial term will result in constant stress σ_{11} throughout the specimen while in the case of significant inertial effects the stress σ_{11} will vary with X_1.

Since the values of response functions $\frac{\partial \tilde{\Psi}}{\partial I_C}$ and $\frac{\partial \tilde{\Psi}}{\partial II_C}$ are determined by equations (93) and (94) we can write (97) as

$$\tilde{p} = M_0 - F_0 v_1^f - \sigma_{11}^E, \tag{98}$$

where $\sigma_{11}^E = 2\varrho \left[\frac{\partial \tilde{\Psi}}{\partial I_C} f'^2 + \frac{\partial \tilde{\Psi}}{\partial II_C} (f'^2 \lambda_2^2 + f'^2 \lambda_3^2) \right]$ and the quantity $\tilde{p}$ can be determined as follows.

The balance of linear momentum for the fluid (52) is

$$\phi^f \frac{dv_1^f}{dx_1} v_1^f = -\phi^f \frac{dp}{dx_1} + \phi^f \frac{d\tilde{\Psi}}{dx_1} - \alpha \phi^s \phi^f v_1^f. \tag{99}$$

By using our definition of $\tilde{p} = p + \varrho \phi^s \frac{\partial \tilde{\Psi}}{\partial \phi^s}$ and expanding the derivative of the Helmholtz potential, this equation can be rewritten as

$$\frac{dv_1^f}{dx_1} v_1^f = -\frac{dp}{dx_1} - \frac{p - \tilde{p}}{\varrho \phi^s} \frac{d\phi^s}{dx_1} + \frac{\partial \tilde{\Psi}}{\partial I_C} \frac{dI_C}{dx_1} + \frac{\partial \tilde{\Psi}}{\partial II_C} \frac{dII_C}{dx_1} - \alpha \phi^s v_1^f. \tag{100}$$

Assuming that the drag coefficient α is known, this equation can be used to obtain a differential equation for $\frac{dp}{dx_1}$

$$\frac{dp}{dx_1} + \frac{p}{\varrho \phi^s} \frac{d\phi^s}{dx_1} = -\frac{dv_1^f}{dx_1} v_1^f + \frac{\tilde{p}}{\varrho \phi^s} \frac{d\phi^s}{dx_1} + \frac{\partial \tilde{\Psi}}{\partial I_C} \frac{dI_C}{dx_1} + \frac{\partial \tilde{\Psi}}{\partial II_C} \frac{dII_C}{dx_1} - \alpha \phi^s v_1^f. \tag{101}$$

Denoting the right hand side of (101) by $Q(x_1)$, which consists of already determined quantities, we have

$$\frac{dp}{dx_1} + p \frac{d}{dx_1} (\ln \phi^s - \ln \varrho) = Q(x_1), \tag{102}$$

which we can solve for p to obtain

$$p(x_1) = \frac{\varrho}{\phi^s} \left(\int \frac{\phi^s}{\varrho} Q(x_1)\, dx_1 + c \right), \tag{103}$$

the c being the integration constant and could be determined by specifying one of the partial stresses at any point. Consequently we can determine the third response function as

$$\frac{\partial \tilde{\Psi}}{\partial \phi^s} = \frac{\tilde{p} - p}{\varrho \phi^s}.$$

1.2. Illustrative example of swelling. The steady diffusion of a fluid through a non-linearly elastic slab has been studied by Shi et al. [**46**] and Reynolds and Humphrey [**38**] within the context of mixture theory. The study by Shi et al. ignores the inertial effects in the flow of the fluid and also assumes that the solid is in equilibrium. The interactive term includes the effects due to drag and also incorporates effects due to density gradients and gradients in the Helmholtz potential. The more recent study by Reynolds and Humphrey ignores inertial terms in the balance of linear momentum for the fluid and neglects the effects due to the gradients of the Helmholtz potential in the interaction term. The main difference between the two studies, however, is the manner in which the boundary conditions are enforced. Shi et. al. generate boundary conditions through a thermodynamic argument based on the notion of a saturated mixture, while Reynolds and Humphrey prescribe boundary conditions directly for the velocity of the fluid. Here, we re-visit the problem, including inertial effects and retaining the effects of the gradient of the Helmholtz potential in the interaction term.

We are looking for a steady state solution, i.e., $\vec{v}^s = 0$ and $\frac{\partial \vec{v}^f}{\partial t} = 0$, of the one-dimensional problem introduced in the previous section. We assume the following form for the Helmholtz potential

$$\tilde{\Psi}(I_C, \phi^s) = c_1(I_C - 3) + c_2 \ln(\phi^s). \tag{104}$$

Due to the choice of the deformation, the deformation gradient $\boldsymbol{F}$ and the right and left Cauchy-Green stretch tensors $\boldsymbol{B}$, $\boldsymbol{C}$ are diagonal and depend only on X_1 and consequently, the stress tensor $\boldsymbol{\sigma}^E$ is diagonal and depends on X_1. By (75) the mass density of the fluid ϱ^f also depends only on X_1. Assuming that the fluid velocity is only in the x_1 direction (which is the same as X_1), i.e. $\vec{v}^f(x_1) = (v_1^f(x_1), 0, 0)$, equation (51) yields

$$\frac{d(\phi^f v_1^f)}{dx_1} = 0, \tag{105}$$

which allows us to express the fluid velocity and its spatial derivative in terms of the volume fraction of the fluid.

The balance of linear momentum for the solid (50) implies that the Lagrange multiplier p is a function of x_1 alone and reduces to

$$0 = -\phi^s \frac{dp}{dx_1} + \frac{d\sigma_{11}^E}{dx_1} - \phi^f \frac{d\tilde{\Psi}}{dx_1} + \alpha \phi^s \phi^f v_1^f. \tag{106}$$

The balance of linear momentum for the mixture, obtained by summing the momentum balance equations of both constituents (50) and (52) together reduces to

$$\phi^f \frac{dv_1^f}{dx_1} v_1^f = -\frac{dp}{dx_1} + \frac{d\sigma_{11}^E}{dx_1}. \tag{107}$$

a	the full problem with inertia and $\nabla\Psi$ term
b	inertia neglected
c	$\nabla\Psi$ term neglected
d	inertia and $\nabla\Psi$ term neglected

TABLE 1. Figure captions explanation.

Now, multiplying (107) by ϕ^s and subtracting (106) we can eliminate the Lagrange multiplier p and obtain

$$\phi^s \frac{dv_1^f}{dx_1} v_1^f = -\frac{d\sigma_{11}^E}{dx_1} + \frac{d\tilde{\Psi}}{dx_1} - \alpha\phi^s v_1^f. \tag{108}$$

Next, subtracting (106) from (107) we obtain

$$\frac{dv_1^f}{dx_1} v_1^f = \frac{dp}{dx_1} + \frac{d\tilde{\Psi}}{dx_1} - \alpha\phi^s v_1^f. \tag{109}$$

The next step is to express all the terms in terms of the unknown functions f' and p in the reference coordinates. Using (105) the inertial term can be rewritten as

$$\frac{dv_1^f}{dx_1} v_1^f = -\frac{1}{f'}\frac{F_0^2}{\phi^{f3}}\frac{d\phi^f}{dX_1} = -\frac{f''}{f'^2}\frac{F_0^2}{\phi^{f2}}\frac{\phi^s}{\phi^f}, \tag{110}$$

and using (104) to express the derivative of the σ_{11}^E and $\tilde{\Psi}$ we have

$$\frac{d\sigma_{11}^E}{dx_1} = \frac{1}{f'}\frac{d}{dX_1}(c_1\varrho f'^2 - c_2\varrho) = \frac{f''}{f'^2}\left[c_1 f'^2(1+\varrho) - c_2(1-\varrho)\right], \tag{111}$$

$$\frac{d\tilde{\Psi}}{dx_1} = \frac{1}{f'}\frac{d}{dX_1}\left[c_1(I_C - 3) + c_2\ln(\phi^s)\right] = \frac{f''}{f'^2}(c_1 f'^2 - c_2), \tag{112}$$

where $\varrho = \phi^f + \frac{\varrho_t^s}{\varrho_t^f}\phi^s$, $\phi^f = 1 - \phi^s$ and $\phi^s = \frac{\phi_0^s}{f'\lambda_2\lambda_3}$. Substituting this into the (108) we have a first order ordinary differential equation for the unknown function $f'(X_1)$ and then (109) gives us a first order ordinary differential equation for p. The resulting system can be written as

$$f'' = G_1(f')$$
$$p' = G_2(f').$$

To obtain the solution for f we need two boundary conditions:

- the boundary displacement $f(X_1 = -1) = -1$,
- the inlet fluid velocity $v_1^f(X_1 = -1)$, which is measurable.

If prescribed it gives us the value of $f'(X_1 = -1)$ through (95) and (75) via

$$f' = \frac{\phi_0^s}{\lambda_2\lambda_3}\left(\frac{1}{1 - \frac{F_0}{v_1^f}}\right)$$

To find the corresponding p we need one additional boundary condition on stress

- the total stress value $\sigma_{11}(X_1 = -1)$ can be prescribed.

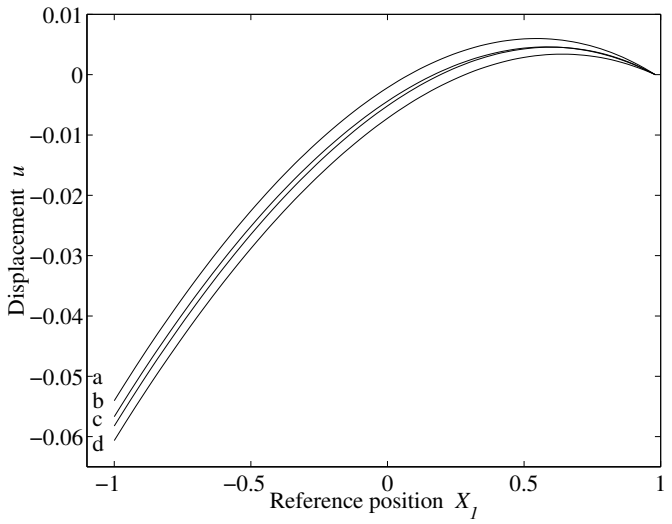

FIGURE 2. Displacement as a function of reference position X_1.

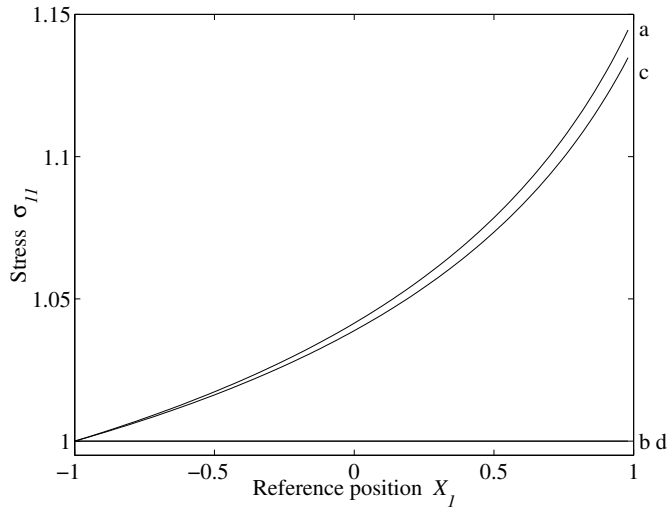

FIGURE 3. Total stress σ_{11} as a function of reference position X_1.

This choice of the boundary conditions allows us to solve the resulting ordinary differential equations as an initial value problem by a standard method such as Runge-Kutta.

For illustration, the material parameters used in our computations are

$$\phi_0^s = 0.85, \quad \frac{\varrho_t^s}{\varrho_t^f} = 1.1, \quad \alpha = 1, \quad c_1 = 1, \quad c_2 = 1$$

and the prescribed stretch ratios and boundary conditions are

$$\lambda_2 = 1.1, \quad \lambda_3 = 0.9, \quad \phi^f(-1) = 0.2, \quad v_1^f(-1) = 1, \quad \sigma_{11}(-1) = 1.$$

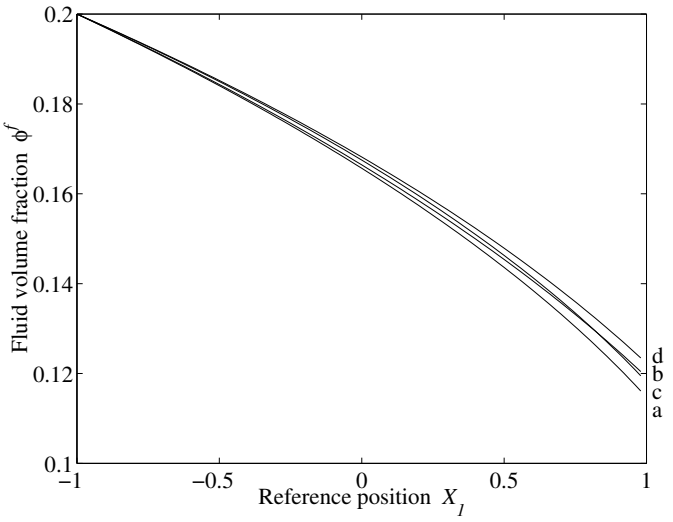

FIGURE 4. Volume fraction as a function of reference position X_1.

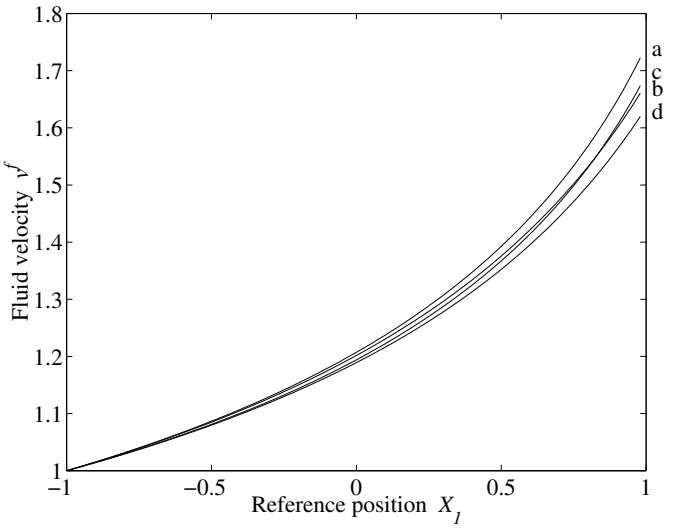

FIGURE 5. Fluid velocity as a function of reference position X_1.

We have compared the solutions of the problem with neglected inertia or $\nabla\Psi$ terms to the solution of the complete problem. The captions used in all figures for each solution are explained in table 1.

As expected, the influence of the fluid inertia is insignificant for small prescribed inflow velocities, but for higher fluid velocities the effects can be significant. In figures 2, 3, 4 and 5 we show displacement of the solid u_1^s, total stress σ_{11}, fluid volume fraction ϕ^f and fluid velocity v_1^f throughout the slab. First, we can see in the figure 3 that the stress σ_{11} varies with the position X_1 when the inertial term is included (cases **a**, **c**) but the relative difference compared to its value is quite

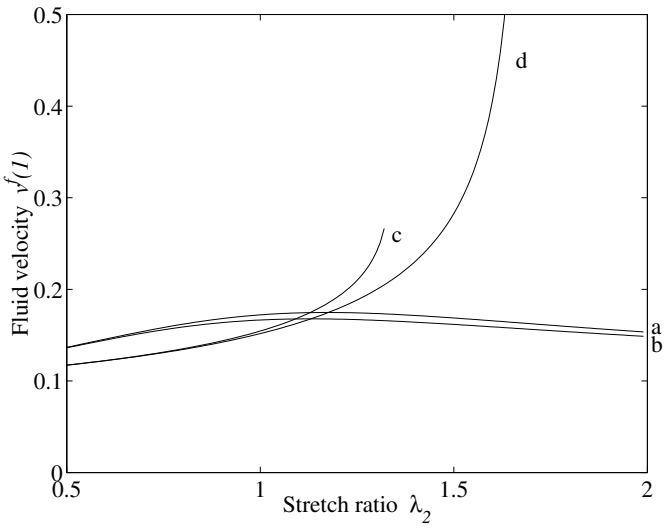

FIGURE 6. Outflow velocity $v_1^f(1)$ as a function of stretch ratio λ_2.

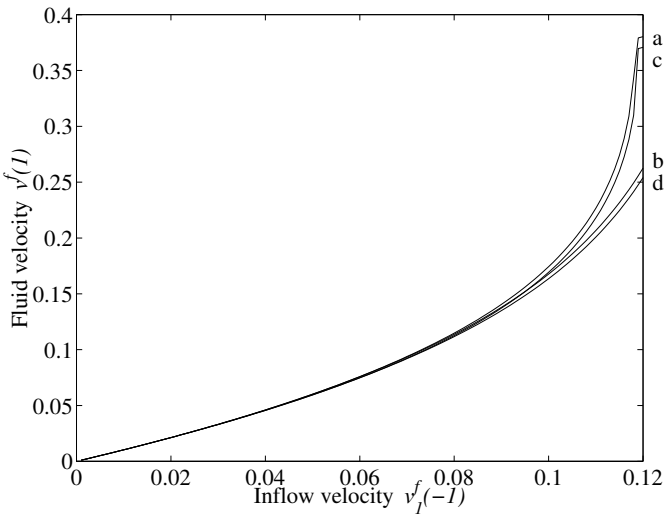

FIGURE 7. Outflow velocity $v_1^f(1)$ as a function of inflow velocity $v_1^f(-1)$.

small (0.1%). In the cases **b** and **d** the stress σ_{11} is constant trough the domain. The relative difference for the displacement of the end point, fluid outflow velocity, and volume fraction are of order of 10%.

In figure 6 we depict the computed values of outflow fluid velocity at the end-point $X_1 = 1$ while varying the stretch ratio λ_2 in the range from 0.5 to 2. In this case we can see that neglecting the term $\nabla\Psi$ in the interaction force (cases **c** and **d**) can significantly change the trend of the dependence, especially for the fluid velocity and volume fraction, while the influence on displacement and stress values is negligible.

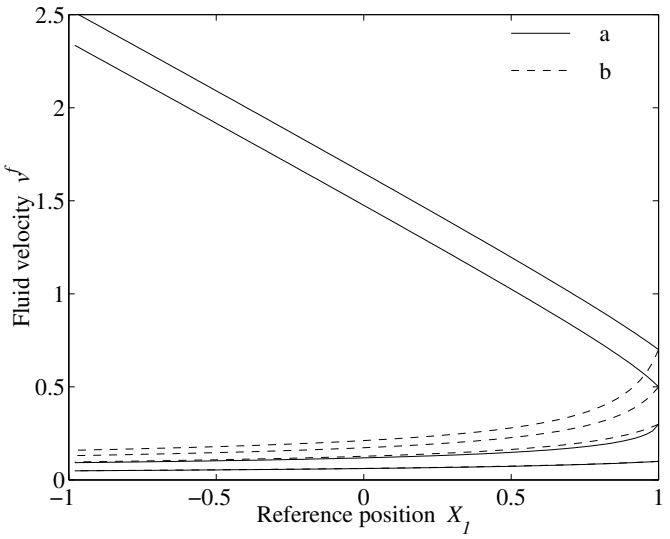

FIGURE 8. Fluid velocity as a function of reference position X_1.

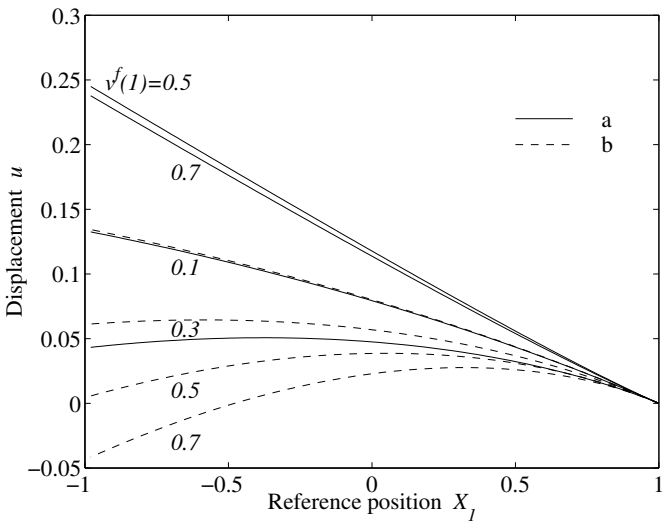

FIGURE 9. Displacement as a function of reference position X_1.

In figure 7 we compare the computed values of outflow fluid velocity at the endpoint $X_1 = 1$ as a function of the prescribed inflow velocity $v_1^f(-1)$ which is varied from 0 up to 0.12. When including the inertia term, then for certain interval of values of the inflow velocity the solution does not exists on the whole spatial interval $(-1, 1)$. There is a solution for which the fluid volume fraction decreases to zero inside the slab, which is a singular point for the fluid mass conservation equation (95). In figures 8 and 9 we apply the boundary conditions at the point $X_1 = 1$, i.e. the outflow velocity is prescribed. If the inertia term is included

we can observe in figure 8 a jump in the inflow velocity as the outflow velocity is increased. This jump occurs for higher value of the velocity boundary condition when the drag coefficient α is decreased, i.e. weaker solid fluid interaction. In figure 9 the displacement u_1^s of the solid is show for the same values of the outflow velocity boundary condition as in figure 8.

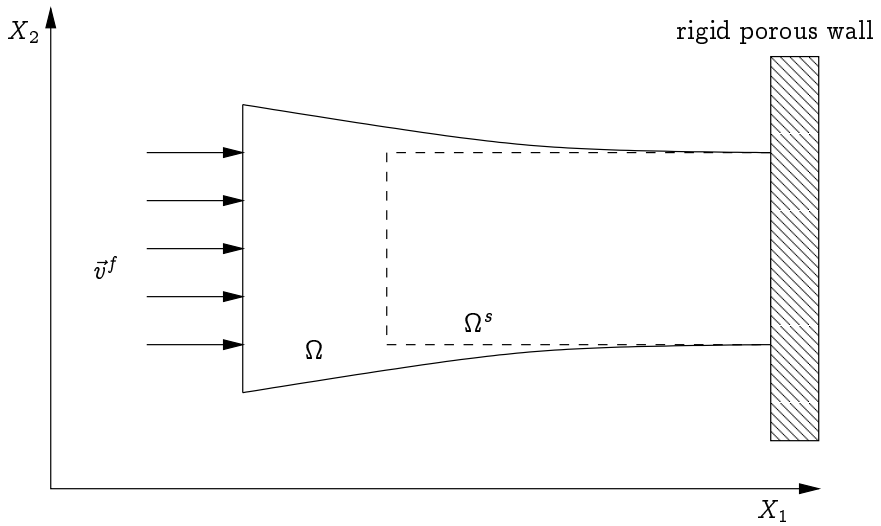

FIGURE 10. Undeformed and deformed configuration in two dimensional problem.

2. Two dimensional problem

In this section we formulate and propose a solution algorithm for a steady state, two-dimensional problem of stretched rectangular slab of solid-fluid mixture. Two or three dimensional problems for solid fluid mixtures in connection with hydrated biological tissue were recently presented in [**8**, **39**, **40**, **41**, **42**, **43**] for biphasic mixture to model behavior of articular cartilage under compression. In all these works the nonlinear inertial of the fluid is neglected.

Let $\vec{u}$ denotes the solid displacement, $\vec{v}$ denotes the fluid velocity, ϕ is the fluid volume fraction and p is the Lagrange multiplier associated with the kinematical constraint (57). Let $\Omega^s = [-L, L] \times [-H, H] \times [-L, L]$ be the reference domain occupied by the solid. The deformed domain, shown in figure 10, is

$$\Omega = \{x = X + \vec{u}(X), \forall X \in \Omega^s\}. \tag{113}$$

The deformation is assumed to be of a form

$$x_1 = X_1 + u_1(X_1, X_2), \qquad x_2 = X_2 + u_2(X_1, X_2), \qquad x_3 = \lambda_3 X_3, \tag{114}$$

where λ_3 is prescribed positive constant. The fluid velocity is assumed to be

$$\vec{v}(x_1) = (v_1(x_1, x_2), v_2(x_1, x_2), 0). \tag{115}$$

Let the symbol ∇ stands for Grad^s, then $\boldsymbol{F} = \boldsymbol{I} + \nabla\vec{u}$ and our task is to find $(\vec{u}, \vec{v}, \phi, p)$ such that

$$\phi(\operatorname{grad}\vec{v})\vec{v} + \operatorname{grad} p - \operatorname{div}\boldsymbol{\sigma}^E = 0 \tag{116}$$

$$(\operatorname{grad}\vec{v})\vec{v} + \operatorname{grad}(p + \Psi) + (1 - \phi)\boldsymbol{\alpha}\vec{v} = 0 \tag{117}$$

$$(1 - \phi)\det\boldsymbol{F} = \phi_0^s \tag{118}$$

$$\operatorname{div}(\phi\vec{v}) = 0 \tag{119}$$

holds in domain Ω, where ϕ_0^s is the volume fraction of the solid in the reference state and the Helmholtz potential Ψ is given by constitutive equation as a function of ϕ and $\boldsymbol{F}$. The Cauchy stress tensor $\boldsymbol{\sigma}$ is then given by

$$\boldsymbol{\sigma}^E = (\phi + \beta(1-\phi))\frac{\partial \Psi}{\partial \boldsymbol{F}}\boldsymbol{F}^T, \tag{120}$$

where $\beta = \frac{\varrho_t^s}{\varrho_t^f}$ is the true mass ratio. By $\boldsymbol{S}$ we will denote the Piola-Kirchoff stress tensor

$$\boldsymbol{S} = -p(\det \boldsymbol{F})\boldsymbol{F}^{-T} + \boldsymbol{S}^E, \tag{121}$$

$$\boldsymbol{S}^E = (\phi + \beta(1-\phi))(\det \boldsymbol{F})\frac{\partial \Psi}{\partial \boldsymbol{F}}. \tag{122}$$

Let the boundary $\partial\Omega$ be divided into three disjoint parts $\partial\Omega = \bigcup_{i=1}^3 \Gamma_i$. Let $\vec{n}$ be the unit normal vector to the boundary $\partial\Omega$. Then if $\vec{t}$ is given traction on the boundary, $\vec{u}_B$ is given solid boundary displacement and $\vec{v}_B$ is given fluid velocity at the boundary the possible boundary conditions are

$$\frac{1}{3}\operatorname{tr}\boldsymbol{\sigma} = p_B, \quad \vec{u} = \vec{u}_B \quad \text{on } \Gamma_1, \tag{123}$$

$$-p\vec{n} + \boldsymbol{\sigma}^E\vec{n} = \vec{t}, \quad \vec{v} = \vec{v}_B \quad \text{on } \Gamma_2, \tag{124}$$

$$\vec{u} = \vec{u}_B, \quad \vec{v} = \vec{v}_B \quad \text{on } \Gamma_3. \tag{125}$$

We avoid prescribing partial stresses on the boundary since it is not clear weather such values can be obtained by measurements or how it should be partitioned into the partial stresses. Instead, we focus on the prescription of the fluid velocity at the boundary. There are other possible boundary conditions, apart of prescribing the fluid velocity. We may prescribe

$$\phi\vec{v} = \vec{f}_B, \tag{126}$$

or the total fluid flux through certain part of the boundary

$$\int_\Gamma \phi\vec{v}\cdot\vec{n}\,da = f_B, \tag{127}$$

which would be convenient as an outflow boundary condition. On the other hand the implementation of these conditions in the solution process is more complicated.

2.1. Weak formulation. In order to apply the finite element method we formulate our problem in a weak sense. Let us define spaces U, V, M and P as follows

$$U = \{\vec{u} \in [W^{1,2}(\Omega^s)]^2, \vec{u} = \vec{0} \text{ on } \Gamma_1 \cup \Gamma_3\} \tag{128}$$

$$V = \{\vec{v} \in [W^{1,2}(\Omega^s)]^2, \vec{v} = 0 \text{ on } \Gamma_2 \cup \Gamma_3\} \tag{129}$$

$$M = \{\phi \in W^{1,2}(\Omega^s), 0 \le \phi \le 1 \text{ in } \Omega\} \tag{130}$$

$$P = \{p \in L^2(\Omega^s)\} \tag{131}$$

Multiplying equations (116)–(119) by a test functions $(\vec{\zeta}, \vec{\xi}, \eta, \gamma)$, integrating over the domain Ω and using per-partes integration we obtain

$$\int_\Omega \phi \frac{\partial \vec{v}}{\partial \vec{x}} \vec{v} \cdot \vec{\zeta}\, dx - \int_\Omega p \operatorname{div}(\phi\vec{\zeta})\, dx + \int_\Omega \operatorname{tr}\left(\boldsymbol{\sigma}^E \frac{\partial \vec{\zeta}}{\partial \vec{x}}\right) dx - \int_{\Gamma_2} \vec{t} \cdot \vec{\zeta}\, da = 0, \tag{132}$$

$$\int_\Omega \frac{\partial \vec{v}}{\partial \vec{x}} \vec{v} \cdot \vec{\xi}\, dx - \int_\Omega (p + \Psi) \operatorname{div} \vec{\xi}\, dx + \int_\Omega (1-\phi)\boldsymbol{\alpha}\vec{v} \cdot \vec{\xi}\, dx + \int_{\Gamma_1} (p_B - \frac{1}{3}\operatorname{tr}\boldsymbol{\sigma} + \Psi)\vec{n} \cdot \vec{\xi}\, da = 0, \tag{133}$$

$$\int_\Omega \left(1 - \phi - \frac{\phi_0^s}{\det \boldsymbol{F}}\right) \eta\, dx = 0, \tag{134}$$

$$\int_\Omega \operatorname{div}(\phi\vec{v})\gamma\, dx = 0. \tag{135}$$

In these equation, the integration is done over the yet unknown domain Ω so it is convenient to transform the integration to the reference domain Ω^s which yields

$$\int_{\Omega^s} \phi \nabla \vec{v} \operatorname{cof} \boldsymbol{F}^T \vec{v} \cdot \vec{\zeta}\, dX - \int_{\Omega^s} p \operatorname{cof} \boldsymbol{F} \cdot \nabla\vec{\zeta}\, dX + \int_{\Omega^s} \boldsymbol{S}^E \cdot \nabla\vec{\zeta}\, dX - \int_{\Gamma_2} \vec{t} \operatorname{cof} \boldsymbol{F}\vec{N} \cdot \vec{\zeta}\, dA = 0, \tag{136}$$

$$\int_{\Omega^s} \nabla\vec{v} \operatorname{cof} \boldsymbol{F}\vec{v} \cdot \vec{\xi}\, dX - \int_{\Omega^s} (p + \Psi) \operatorname{cof} \boldsymbol{F} \cdot \nabla\vec{\xi}\, dX + \int_{\Omega^s} (1-\phi)\boldsymbol{\alpha}\vec{v} \cdot \vec{\xi} \det \boldsymbol{F}\, dX + \int_{\Gamma_1} (p_B - \frac{1}{3}\operatorname{tr}\boldsymbol{\sigma} + \Psi) \operatorname{cof} \boldsymbol{F}\vec{N} \cdot \vec{\xi}\, dA = 0, \tag{137}$$

$$\int_{\Omega^s} ((1-\phi)\det \boldsymbol{F} - \phi_0^s)\, \eta\, dX = 0, \tag{138}$$

$$\int_{\Omega^s} \nabla(\phi\vec{v}) \cdot \operatorname{cof} \boldsymbol{F}\gamma\, dX = 0, \tag{139}$$

where $\operatorname{cof} \boldsymbol{F} = \det \boldsymbol{F}\boldsymbol{F}^{-T}$ and the inner product for two tensors is defined as $\boldsymbol{F} \cdot \boldsymbol{G} = \operatorname{tr}(\boldsymbol{F}^T\boldsymbol{G})$. Then our task is to find $(\vec{u}, \vec{v}, \phi, p)$ such that $(\vec{u} - \vec{u}_B, \vec{v} - \vec{v}_B, \phi, p) \in U \times V \times M \times P$ and equations (136)–(139) are satisfied for all $(\vec{\zeta}, \vec{\xi}, \eta, \gamma) \in U \times V \times M \times P$.

2.2. Finite element discretization. The reference domain Ω^s is approximated by a domain Ω_h with piecewise linear boundary. The interior is divided by regular quadrilateral mesh into convex quadrilateral elements. Any two elements have in common either whole side, vertex or are disjoint. Let the set of all quadrilaterals in Ω_h be denoted by $\mathcal{T}_h$ and let $\tilde{T} = (-1, 1)^2$ be the reference quadrilateral.

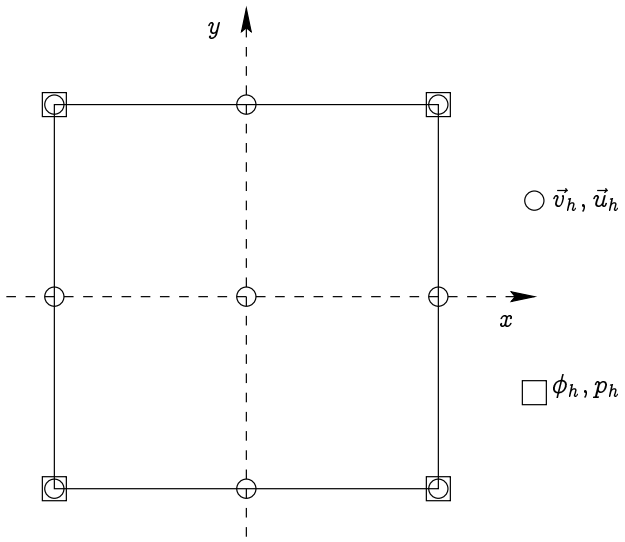

FIGURE 11. Degrees of freedom on the reference element and local coordinate system.

For each element $T \in \mathcal{T}_h$ there is a bilinear one to one mapping on to the reference element $\tilde{T}$. The spaces U, V, M resp. P are approximated by finite element spaces

$$U_h = \{\vec{u}_h \in [C(\Omega_h)]^2, \vec{u}_h/T \in [Q_2(T)]^2 \quad \forall T \in \mathcal{T}_h, \vec{u}_h = \vec{0} \text{ on } \Gamma_1 \cup \Gamma_3\}, \tag{140}$$

$$V_h = \{\vec{v}_h \in [C(\Omega_h)]^2, \vec{v}_h/T \in [Q_2(T)]^2 \quad \forall T \in \mathcal{T}_h, \vec{v}_h = 0 \text{ on } \Gamma_1 \cup \Gamma_3\}, \tag{141}$$

$$M_h = \{\phi_h \in C(\Omega_h), \phi_h/T \in Q_1(T) \quad \forall T \in \mathcal{T}_h, 0 \leq \phi_h \leq 1\}, \tag{142}$$

$$P_h = \{p_h \in C(\Omega_h), p_h/T \in Q_1(T) \quad \forall T \in \mathcal{T}_h\}. \tag{143}$$

Where $Q_n(T)$ denotes the space of polynomial functions, defined on the quadrilateral T, which transformed to the reference element $\tilde{T}$ are polynomials of order at most n in each reference coordinate. The location of local degrees of freedom on the reference element is shown in figure 11. The resulting discrete problem is obtained by taking usual nodal basis of the space $U_h \times V_h \times M_h \times P_h$ and using the elements of this basis as test functions $(\vec{\zeta}, \vec{\xi}, \eta, \gamma)$ in (136–139). This set of non-linear algebraic equations can be written as

$$\vec{\mathcal{R}}(\vec{X}) = \vec{0}, \tag{144}$$

where $\vec{X} = (\vec{u}_h, \vec{v}_h, \phi_h, p_h)$ is the vector of unknown components.

Next, we investigate the linearization of (144) around some point $\vec{X} = (\vec{u}, \vec{v}, \phi, p)$. Let us use the following notation

$$(f, g) = \int_{\Omega^s} f \cdot g \, dX,$$

for f and g being vector or tensor quantities. We define the following bilinear forms as variations of the momentum equation (136) with respect to each of the

unknowns

$$
\begin{aligned}
a_{u,u}(\vec{w},\vec{\zeta}) = &\left(\phi\nabla\vec{v}(\frac{\partial \operatorname{cof} \boldsymbol{F}^T}{\partial \boldsymbol{F}}\nabla\vec{w})\vec{v},\vec{\zeta}\right) - \left(p\frac{\partial \operatorname{cof} \boldsymbol{F}}{\partial \boldsymbol{F}}\nabla\vec{w},\nabla\vec{\zeta}\right) \\
&+ \left(\varrho\frac{\partial^2\Psi}{\partial \boldsymbol{F}^2}\nabla\vec{w},\nabla\vec{\zeta}\right) - \int_{\Gamma_2} \vec{t}\frac{\partial \operatorname{cof} \boldsymbol{F}}{\partial \boldsymbol{F}}\nabla\vec{w}\vec{N}\cdot\vec{\zeta}\,dA,
\end{aligned} \tag{145}
$$

$$
a_{u,v}(\vec{w},\vec{\zeta}) = \left(\phi\nabla\vec{w}\operatorname{cof}\boldsymbol{F}^T\vec{v},\vec{\zeta}\right) + \left(\phi\nabla\vec{v}\operatorname{cof}\boldsymbol{F}^T\vec{w},\vec{\zeta}\right), \tag{146}
$$

$$
b_{u,\phi}(w,\vec{\zeta}) = \left(w\nabla\vec{v}\operatorname{cof}\boldsymbol{F}^T\vec{v},\vec{\zeta}\right) + \left((1-\beta)w\frac{\partial\Psi}{\partial\boldsymbol{F}},\vec{\zeta}\right), \tag{147}
$$

$$
c_{u,p}(w,\vec{\zeta}) = \left(w\operatorname{cof}\boldsymbol{F},\nabla\vec{\zeta}\right), \tag{148}
$$

variations of the momentum equation (137) with respect to each of the unknowns

$$
\begin{aligned}
a_{v,u}(\vec{w},\vec{\xi}) = &\left(\nabla\vec{v}(\frac{\partial \operatorname{cof} \boldsymbol{F}^T}{\partial \boldsymbol{F}}\nabla\vec{w})\vec{v},\vec{\xi}\right) - \left((p+\Psi)\frac{\partial \operatorname{cof} \boldsymbol{F}}{\partial \boldsymbol{F}}\nabla\vec{w},\nabla\vec{\xi}\right) \\
&- \left((\frac{\partial\Psi}{\partial\boldsymbol{F}}\cdot\nabla\vec{w})\operatorname{cof}\boldsymbol{F},\nabla\vec{\xi}\right) + \left((\operatorname{cof}F\cdot\nabla\vec{w})(1-\phi)\boldsymbol{\alpha}\vec{v},\vec{\xi}\right) \\
&+ \int_{\Gamma_1}(p_B - \frac{1}{3}\operatorname{tr}\boldsymbol{\sigma} + \Psi)\frac{\partial \operatorname{cof} \boldsymbol{F}}{\partial \boldsymbol{F}}\nabla\vec{w}\vec{N}\cdot\vec{\xi}\,dA \\
&+ \int_{\Gamma_1}(p_B - \frac{1}{3}\operatorname{tr}(\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{F}}\nabla\vec{w}) + \frac{\partial\Psi}{\partial\boldsymbol{F}}\cdot\nabla\vec{w})\operatorname{cof}\boldsymbol{F}\vec{N}\cdot\vec{\xi}\,dA,
\end{aligned} \tag{149}
$$

$$
\begin{aligned}
a_{v,v}(\vec{w},\vec{\xi}) = &\left(\nabla\vec{w}\operatorname{cof}\boldsymbol{F}^T\vec{v},\vec{\xi}\right) + \left(\nabla\vec{v}\operatorname{cof}\boldsymbol{F}^T\vec{w},\vec{\xi}\right) \\
&+ \left(\det\boldsymbol{F}(1-\phi)\boldsymbol{\alpha}\vec{w},\vec{\xi}\right),
\end{aligned} \tag{150}
$$

$$
b_{v,\phi}(w,\vec{\xi}) = -\left(w\det\boldsymbol{F}\boldsymbol{\alpha}\vec{v},\vec{\xi}\right), \tag{151}
$$

$$
c_{v,p}(w,\vec{\xi}) = \left(w\operatorname{cof}\boldsymbol{F},\nabla\vec{\xi}\right), \tag{152}
$$

variations of (138) with respect to each of the unknowns

$$
a_{\phi,u}(\vec{w},\eta) = ((1-\phi)\operatorname{cof}\boldsymbol{F},\nabla\vec{w}\eta), \tag{153}
$$

$$
b_{\phi,\phi}(w,\eta) = -(\det\boldsymbol{F}w,\eta), \tag{154}
$$

and variations of (139) with respect to each of the unknowns

$$
a_{p,u}(\vec{w},\gamma) = \left(\nabla(\phi\vec{v}),\frac{\partial \operatorname{cof} \boldsymbol{F}}{\partial \boldsymbol{F}}\nabla\vec{w}\gamma\right), \tag{155}
$$

$$
a_{p,v}(\vec{w},\gamma) = (\nabla(\phi\vec{w}),\operatorname{cof}\boldsymbol{F}\gamma), \tag{156}
$$

$$
b_{p,\phi}(w,\eta) = (\nabla(w\vec{v}),\operatorname{cof}\boldsymbol{F}\gamma). \tag{157}
$$

Then the linearization of (144) around solution $\vec{X} = (\vec{u}_h,\vec{v}_h,\phi_h,p_h)$ is

$$
a_{u,u}(\delta\vec{u},\vec{\zeta}) + a_{u,v}(\delta\vec{v},\vec{\zeta}) + b_{u,\phi}(\delta\phi,\vec{\zeta}) + c_{u,p}(\delta p,\vec{\zeta}) = 0, \tag{158}
$$

$$
a_{v,u}(\delta\vec{u},\vec{\xi}) + a_{v,v}(\delta\vec{v},\vec{\xi}) + b_{v,\phi}(\delta\phi,\vec{\xi}) + c_{v,p}(\delta p,\vec{\xi}) = 0, \tag{159}
$$

$$
a_{\phi,u}(\delta\vec{u},\eta) + b_{\phi,\phi}(\delta\phi,\eta) = 0, \tag{160}
$$

$$
a_{p,u}(\delta\vec{u},\gamma) + a_{p,v}(\delta\vec{v},\gamma) + b_{p,\phi}(\delta\phi,\gamma) = 0, \tag{161}
$$

for all $(\vec{\zeta}, \vec{\xi}, \eta, \gamma) \in U_h \times V_h \times M_h \times P_h$. This can be rewritten in the corresponding matrix notation. Find $\delta\vec{X}$ such that

$$\left[\frac{\partial \vec{\mathcal{R}}}{\partial \vec{X}}(\vec{X})\right] \delta\vec{X} = 0, \tag{162}$$

where the Jacobian matrix $\left[\frac{\partial \vec{\mathcal{R}}}{\partial \vec{X}}(\vec{X})\right]$ has following structure

$$\left[\frac{\partial \vec{\mathcal{R}}}{\partial \vec{X}}(\vec{X})\right] = \begin{pmatrix} \boldsymbol{A}_{u,u} & \boldsymbol{A}_{u,v} & \boldsymbol{B}_{u,\phi} & \boldsymbol{C} \\ \boldsymbol{A}_{v,u} & \boldsymbol{A}_{v,v} & \boldsymbol{B}_{v,\phi} & \boldsymbol{C} \\ (1-\phi)\boldsymbol{C}^T & \boldsymbol{0} & \boldsymbol{B}_{\phi,\phi} & \boldsymbol{0} \\ \boldsymbol{A}_{p,u} & \phi\boldsymbol{C}^T & \boldsymbol{B}_{p,\phi} & \boldsymbol{0} \end{pmatrix} \tag{163}$$

where each block of the matrix corresponds to one of the bilinear forms defined above, for example $\boldsymbol{A}_{u,u}$ is such that $\boldsymbol{A}_{u,u}\delta\vec{u} \cdot \vec{\zeta} = a_{u,u}(\delta\vec{u}, \vec{\zeta})$, etc. This matrix has typical structure of constraint system with zero diagonal block.

2.3. Solution algorithm. The system (144) of nonlinear algebraic equations is solved using Newton method as the basic iteration. One step of the Newton iteration is described in figure 12. The convergence of this basic iteration can be

1. Let $\vec{X}^n$ be some starting guess.
2. Set the residuum vector $\vec{R}^n = \vec{\mathcal{R}}(\vec{X}^n)$ and the tangent matrix $\boldsymbol{A} = \frac{\partial \vec{\mathcal{R}}}{\partial \vec{X}}(\vec{X}^n)$.
3. Solve for the correction $\delta\vec{X}$
$$\boldsymbol{A}\delta\vec{X} = \vec{R}^n.$$
4. Update the solution $\vec{X}^{n+1} = \vec{X}^n - \delta\vec{X}$.

FIGURE 12. One step of the Newton method.

characterized by the following standard result.

THEOREM 1. *Let $\vec{X}$ be a solution of $\vec{F}(\vec{X}) = 0$ and $\frac{\partial \vec{F}}{\partial \vec{X}}(\vec{X}^n)$ is invertible and locally Lipschitz continuous. Then, if $\vec{X}^0$ is sufficiently close to $\vec{X}$, the Newton algorithm has the following property*

$$\|\vec{X}^{n+1} - \vec{X}\| \leq c\|\vec{X}^n - \vec{X}\|^2. \tag{164}$$

We can see that this gives us quadratic convergence provided that the initial guess is sufficiently close to the solution. To ensure the convergence globally two improvements of this basic iteration are used.

First, the dumped Newton method with line search improves the chance of convergence by adaptively changing the length of the correction vector. The solution update step in figure 12 is replaced by

$$\vec{X}^{n+1} = \vec{X}^n + \omega\delta\vec{X}, \tag{165}$$

where the parameter ω is found such that certain error measure decreases. One of the possible choices for the quantity to decrease is

$$f(\omega) = \vec{\mathcal{R}}(\vec{X}^n + \omega\delta\vec{X}) \cdot \delta\vec{X}. \tag{166}$$

Since we know

$$f(0) = \vec{\mathcal{R}}(\vec{X}^n) \cdot \delta\vec{X}, \tag{167}$$

and

$$f'(0) = \left[\frac{\partial \vec{\mathcal{R}}}{\partial \vec{X}}(\vec{X}^n)\right] \delta\vec{X} \cdot \delta\vec{X} = \vec{\mathcal{R}}(\vec{X}^n) \cdot \delta\vec{X}, \tag{168}$$

and computing $f(\omega_0)$ for $\omega_0 = -1$ or ω_0 determined adaptively from previous iterations we can approximate $f(\omega)$ by a quadratic function

$$f(\omega) = \frac{f(\omega_0) - f(0)(\omega_0 + 1)}{\omega_0^2}\omega^2 + f(0)(\omega + 1). \tag{169}$$

Then setting

$$\tilde{\omega} = \frac{f(0)\omega_0^2}{f(\omega_0) - f(0)(\omega_0 + 1)}, \tag{170}$$

the new optimal step length $\omega \in [-1, 0]$ is

$$\omega = \begin{cases} -\dfrac{\tilde{\omega}}{2} & \text{if } \dfrac{f(0)}{f(\omega_0)} > 0, \\ -\dfrac{\tilde{\omega}}{2} - \sqrt{\dfrac{\tilde{\omega}^2}{4} - \tilde{\omega}} & \text{if } \dfrac{f(0)}{f(\omega_0)} \leq 0. \end{cases} \tag{171}$$

This line search can be repeated with ω_0 taken as the last ω until, for example, $f(\omega) \leq \frac{1}{2}f(0)$.

Additionally, the continuation method is employed in order to have the starting approximation in the Newton iteration in the range of convergence. In the continuation method the problem $\vec{F}(\vec{X}) = 0$ is replaced by

$$\vec{G}(\vec{X}, \lambda) = 0 \tag{172}$$

where λ is a parameter such that for $\vec{G}(\vec{X}, 0) = 0$ we know the solution, while for $\lambda = 1$ the original problem is recovered

$$\vec{G}(\vec{X}, 1) \equiv \vec{F}(\vec{X}). \tag{173}$$

For example making the boundary conditions to depend on the parameter λ in such a way that for $\lambda = 0$ we have the undeformed, stress free state, and for $\lambda = 1$ we have the original boundary conditions.

In the process of the continuation method, we follow the solution curve given by the initial value problem

$$\frac{d}{ds}\vec{G}(\vec{X}(s), \lambda(s)) = 0, \tag{174}$$

$$(\vec{X}(0), \lambda(0)) = (\vec{X}_0, 0), \tag{175}$$

until the point $\lambda(s) = 1$. The basic method used to solve this problem is the Euler-Newton iteration, outlined in figure 13, where the explicit Euler method is applied to (174) as predictor and then the solution is corrected by the Newton method. This step is repeated until $\lambda^n = 1$. The parameter ω in the update step can be fixed or can be chosen adaptively, for example depending on the number of Newton iterations in the correction step needed to correct the solution.

The jacobian matrix is computed via finite differences. To invert the matrices in the most inner loops, BiCGStab or GMRES methods are used with suitable preconditioning. See [**47, 48**] for further details on these methods. The incomplete LU decomposition is used for preconditioning with suitable ordering of unknowns

1. Let $\vec{X}^n$ be given starting approximation and λ^n the value of the continuation parameter.
2. Predictor step. Solve for $(\dot{\vec{X}}^n, \dot{\lambda^n})$

$$\left[\frac{\partial \vec{G}}{\partial \vec{X}}(\vec{X}^n, \lambda^n)\right] \dot{\vec{X}}^n + \left[\frac{\partial \vec{G}}{\partial \lambda}(\vec{X}^n, \lambda^n)\right] \dot{\lambda^n} = 0, \tag{176}$$

$$\|(\dot{\vec{X}}^n, \dot{\lambda^n})\| = 1. \tag{177}$$

3. Update the solution $(\vec{X}^{n+\frac{1}{2}}, \lambda^{n+1}) = (\vec{X}^n, \lambda^n) + \omega(\dot{\vec{X}}^n, \dot{\lambda^n})$.
4. Correction step. Solve for $\vec{X}^{n+1}$ by Newton iteration with $\vec{X}^{n+\frac{1}{2}}$ as starting guess

$$\vec{G}(\vec{X}^{n+1}, \lambda^{n+1}) = 0. \tag{178}$$

FIGURE 13. One step of Euler-Newton algorithm.

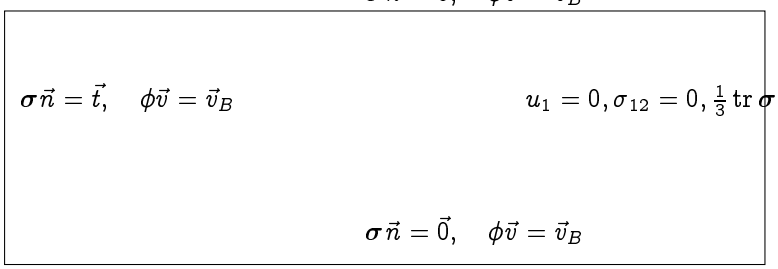

FIGURE 14. Applied boundary condition in the two dimensional problem.

and with allowed fill-in for certain pattern in the zero diagonal block of the jacobian matrix.

2.4. Illustrative example of perfusion. We take two dimensional crossection of the slab along the direction of the perfusion and assume the constitutive relation for the Helmholtz potential

$$\Psi = c_1(I_C - 3) + c_2 \ln(\phi). \tag{179}$$

The boundary conditions applied are shown in figure 14.

In figure 15 is shown the finite element grid on the reference configuration of the solid. The initial solution is taken to be zero displacement and velocity, given constant volume fraction and lagrange multiplier p such that the solution is stress free. The the solution for required values of the boundary conditions is computed by continuation.

In figure 16 we show the finite element grid on the deformed configuration of the solid. We can see that the slab becomes thicker in the X_2 direction at the left end, and thinner at the fluid outflow end. This variation in the thickness is caused by the gradual decrease in the pressure along the fluid flow. Figure 17 shows the velocity field of the perfusion and 18 shows the fluid volume fraction throughout the slab. As in the problem studied in previous section, the fluid velocity increases toward the end of the slab while the volume fraction decreases. In figure 19 the pressure field is shown and in figures 20–22 the components of the stress tensor are

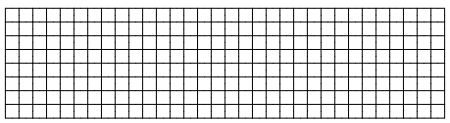

FIGURE 15. Finite element grid on the reference configuration of the solid.

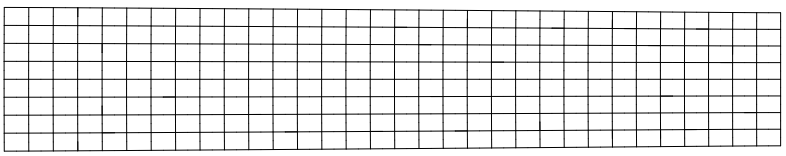

FIGURE 16. Finite element grid on the deformed configuration of the solid.

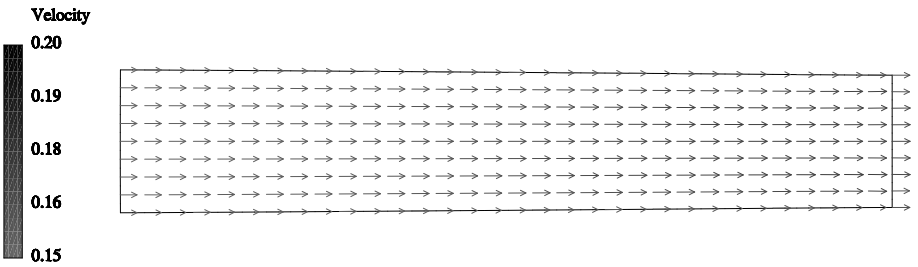

FIGURE 17. Fluid velocity.

depicted. We can notice the presence of stress concentration around the corners of the slab where the type of the boundary condition changes.

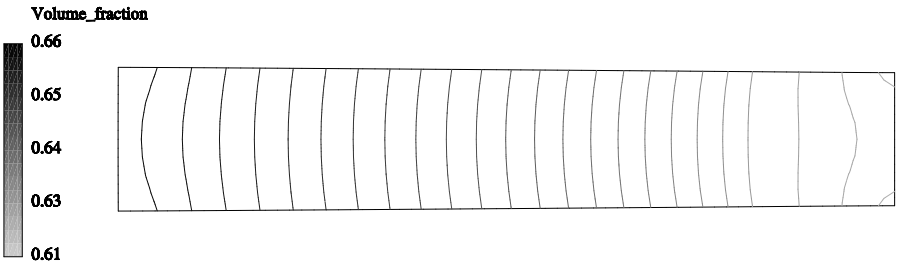

FIGURE 18. Isolines of the fluid volume fraction.

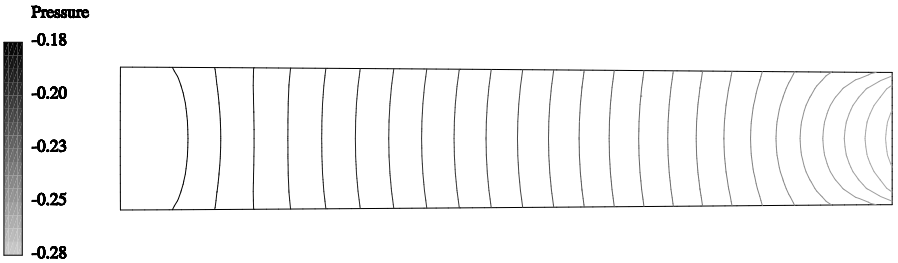

FIGURE 19. Isolines of the pressure field.

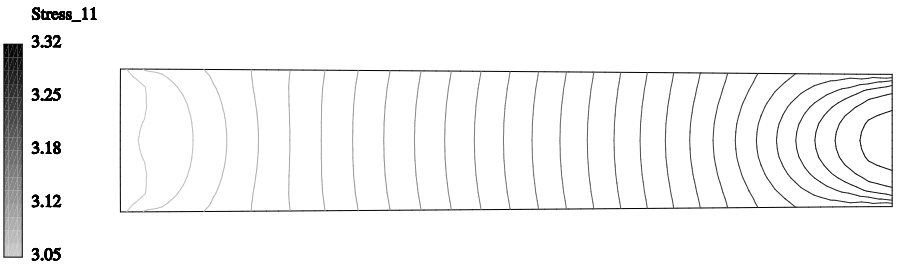

FIGURE 20. Isolines of the stress component σ_{11}.

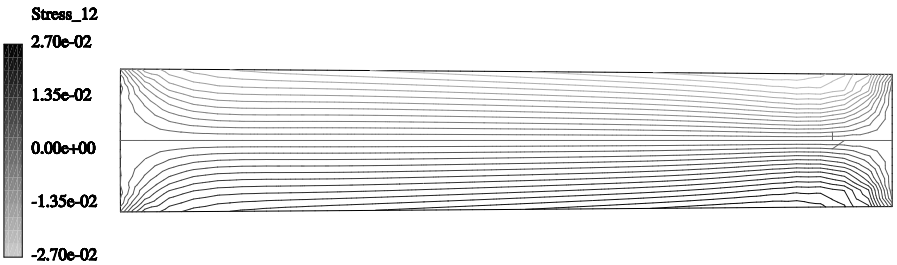

FIGURE 21. Isolines of the stress component σ_{12}.

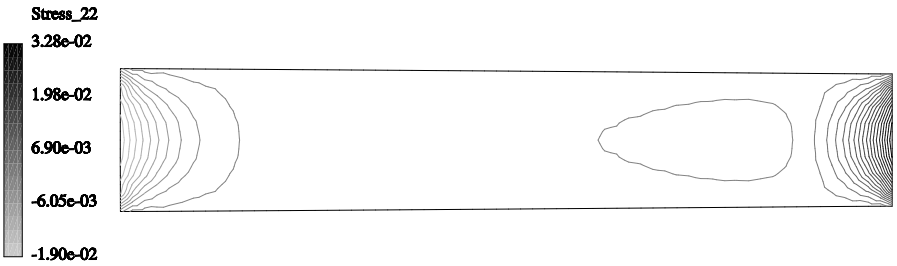

FIGURE 22. Isolines of the stress component σ_{22}.

CHAPTER 4

Conclusion

In this work we have investigate application of mixture theory to modeling of hydrated soft tissues with perfusion. The mixture theory was show to be capable to describe simple finite deformation with simultaneous perfusion of quadrilateral slab. There are still open questions of appropriate constitutive equations and boundary conditions for the constituents and the mixture. One and two dimensional illustrative examples were solve, but further development has to be done to be able to solve more general setting.

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