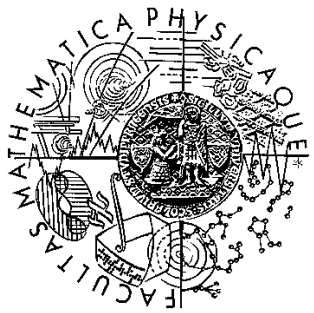

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Fluid structure interaction with applications in biomechanics

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CHAPTER 1

Introduction

The ability to model and predict the mechanical behavior of biological systems is very important in several areas of bio-engineering and medicine. For example, a good mathematical model for a biological tissue could be used in such areas as early recognition or prediction of heart muscle failure, advanced design of new treatments and operative procedures, and the understanding of atherosclerosis and associated problems. Other possible applications include development of virtual reality programs for training new surgeons or designing new operative procedures (see Miga et al. [1998], Paulsen et al. [1999]), and last but not least the design of medical instruments or artificial replacements with mechanical and other properties as close as possible to the original parts (see Zoppou et al. [1997]). These are some of the areas where a good mathematical model of soft tissue and its interactions is essential for success.

The task of modeling a biological system in general includes many different kinds of interactions. Apart of the purely mechanical behavior like viscous fluid flow and elastic deformations it includes also a very complicated chemical and electric interactions.

In this work we focus on the purely mechanical interactions involving viscous fluid and elastic solids and we derive two kinds of models in the framework of continuum mechanics suitable for two different problems encountered in biomechanics. Yet we try to keep the formulation general enough to allow for later inclusion of additional interactions like chemical reactions or electric excitation.

The only assumptions we use, beside the ones of continuum mechanics, are the incompressibility of the materials and isothermality of the processes. These assumptions are considered reasonable and are frequently used in the area of biomechanics. Only for the numerical computations we restrict our problems to one and two dimensions due to the large computational time to solve the nonlinear problems. Specifically we do not restrict our model to small deformations or thin shell approximation.

1. Notations

t	time
$\vec{x}$	spatial position
$\vec{X}$	referential position
Ω_t	spatial region ocupied by the body at the time t
Ω	referential configuration of the body
$\partial\mathcal{P}$	boundary of a region $\mathcal{P}$
$\vec{n}_{\mathcal{P}}$	the unit, outward normal vector to the boundary of $\mathcal{P}$
$\vec{\chi}(\vec{X},t)$	deformation (motion) of the body
$\vec{u}(\vec{X},t)$	displacement
$\vec{v}(\vec{x},t)$	velocity
$\boldsymbol{F}$	deformation gradient
$J=\det\boldsymbol{F}$	determinant of the deformation gradient
$\boldsymbol{C}=\boldsymbol{F}^T\boldsymbol{F}$	right Cauchy-Green deformation tensor
ϱ	density of the material
$\boldsymbol{\sigma}$	Cauchy stress tensor
$\boldsymbol{P}=J\boldsymbol{\sigma}\boldsymbol{F}^{-T}$	the first Piola-Kirchhoff stress tensor
$\frac{\partial}{\partial t}$	time derivative with respect to the spatial coordinates
$\int f\,dv$	volume integration in the spatial coordinates
$\int f\,da$	area integration in the spatial coordinates
$\vec{f}\cdot\vec{g}=\sum_i f_i g_i$	scalar product of two vectors
$\boldsymbol{G}\cdot\boldsymbol{H}=\sum_{ij} G_{ij}H_{ij}$	scalar product of two tensors
$[\operatorname{grad}\phi]_i=\frac{\partial\phi}{\partial x_i}$,	gradient operator with respect to the spatial coordinates ap-
$[\operatorname{grad}\vec{\xi}]_{ij}=\frac{\partial\xi_i}{\partial x_j}$,	plied to a scalar, vector and tensor
$[\operatorname{grad}\boldsymbol{T}]_{ijk}=\frac{\partial T_{ij}}{\partial x_k}$	
$\operatorname{div}\vec{\xi}=\sum_i\frac{\partial\xi_i}{\partial x_i}$,	divergence operator with respect to the spatial coordinates
$[\operatorname{div}\boldsymbol{T}]_i=\sum_j\frac{\partial T_{ij}}{\partial x_j}$	applied to a vector and tensor
$\frac{d}{dt}$	time derivative with respect to the referential coordinates
$\int f\,dV$	volume integration in the referential coordinates
$\int f\,dA$	area integration in the referential coordinates
$[\nabla\phi]_i=\frac{\partial\phi}{\partial X_i}$,	gradient operator with respect to the referential coordinates
$[\nabla\vec{\xi}]_{ij}=\frac{\partial\xi_i}{\partial X_j}$,	applied to a scalar, vector and tensor
$[\nabla\boldsymbol{T}]_{ijk}=\frac{\partial T_{ij}}{\partial X_k}$	
$\operatorname{Div}\vec{\xi}=\sum_i\frac{\partial\xi_i}{\partial X_i}$,	divergence operator with respect to the referential coordi-
$[\operatorname{Div}\boldsymbol{T}]_i=\sum_j\frac{\partial T_{ij}}{\partial X_j}$	nates applied to a vector and tensor
$\|\cdot\|$	the Euclidean norm
$\|\cdot\|_X$	norm in the space X

CHAPTER 2

Classical fluid structure interaction

1. Overview

Both problems of viscous fluid flow and problems of elastic body deformation have been studied separately for over hundred years in great detail. But there are many problems encountered in real life where an interaction between those two medias is of great importance. Typical example of such a problem is the area of aeroelasticity. In the case of biomechanics such interaction is encountered especially when dealing with the blood circulatory system. Problem of a pulsative flow in an elastic tube, flow through the heart flaps, flow in the heart chambers are some of the examples. In all these problems we have to deal with large deformations of an elastic solid interacting with an unsteady, very often periodic, fluid flow.

1.1. Fluid structure models. There have been several different approaches to the problem of fluid-structure interaction. Most notably the work of Peskin and McQueen [1989], Peskin [1982], Peskin and McQueen [1980], Peskin [1977] where an immersed boundary method was developed and applied to a three-dimensional model of the heart. In this model they consider a set of one-dimensional elastic fibers immersed in three-dimensional fluid region and using parallel supercomputer they were able to model the pulse of the heart ventricle. Their method can capture the anisotropy caused by the muscle fibers.

A fluid-structure model with the wall modeled as a thin shell was used to model the left heart ventricle in Costa et al. [1996a,b] and Quarteroni et al. [2000], Quarteroni [2001]. In Heil [1997, 1998] similar approach was used to model a flow in a collapsible tubes. In these models the wall is modeled by two-dimensional thin shell which can be modified to capture the anisotropy of the muscle. In reality the thickness of the wall can be significant and very important. For example in arteries the wall thickness can be up to 30% of the diameter and its local thickening can be the cause of an aneurysm creation. In the case of heart ventricle the thickness of the wall is also significant and also the direction of the muscle fibers changes through the wall.

1.2. Numerical solutions. The solution strategy mostly used is to decouple the problem into the fluid part and solid part, for each of those parts to use some well established method of solution and the interaction is introduced as external boundary conditions or volume forces in each of the subproblems. This has a great advantage that there are many well tested finite element based numerical methods for separate problems of fluid flow and elastic deformation, on the other hand the treatment of the interface and the interaction is problematic.

1.3. Theoretical results. The theoretical investigation of the fluid structure interaction problems is complicated by the need of mixed description. While for the solid part the natural view is the material (Lagrangian) description for the fluid it is the spatial (Eulerian) description. In the case of their combination some kind of mixed description (usually refered to as the arbitrary Lagrangian-Eulerian description) has to be used which brings additional nonlinearity into the resulting equations.

In Le Tallec and Mani [2000] a time dependent, linearized model of interaction between a viscous fluid and an elastic shell in small displacement approximation and its discretization is analyzed. The problem is further simplified by neglecting all changes in the geometry configuration. Under these simplifications by using the energy estimates they are able to show that the proposed formulation is well

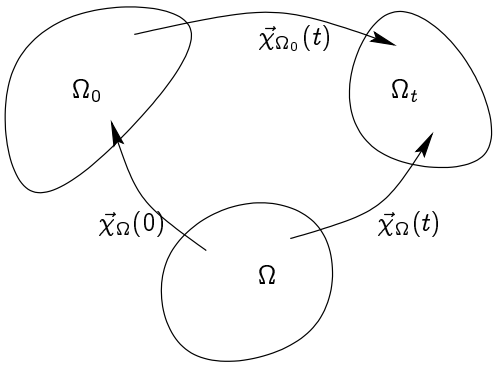

FIGURE 1. The referential domain Ω, initial Ω_0 and current state Ω_t and relations between them. The identification $\Omega \equiv \Omega_0$ is adopted in this text.

posed and a global weak solution exists. Further they show that an independent discretization by standard mixed finite elements for the fluid and by nonconforming DKT finite elements for the shell together with backward or central difference approximation of the time derivatives converges to the solution of the continuous problem.

In Rumpf [1998] a steady problem of equilibrium of an elastic fixed obstacle surrounded by a viscous fluid is studied. Existence of an equilibrium state is show with the displacement and velocity in $C^{2,\alpha}$ and pressure in $C^{1,\alpha}$ under assumption of small data in $C^{2,\alpha}$ and the domain boundaries of class C^3.

For basic introduction and complete reference of continuum theory see Gurtin [1981], Truesdell [1991], Maršík [1999], Haupt [2000]. Its application in biomechanics are presented in Fung [1993] and Maršík and Dvořák [1998] for example. We will mention in the following sections the basic notation and setup used in this work.

2. Continuum theory

Let $\Omega \subset \mathbb{R}^3$ be a reference configuration of a given body, possibly an abstract one. Let $\Omega_t \subset \mathbb{R}^3$ be a configuration of this body at time t. Then one-to-one, sufficiently smooth mapping $\vec{\chi}_\Omega$ of the reference configuration Ω to the current configuration

$$\vec{\chi}_\Omega : \Omega \times [0,T] \mapsto \Omega_t, \tag{1}$$

describes the motion of the body, see figure 1. The mapping $\vec{\chi}_\Omega$ depends on the choice of the reference configuration Ω which can be fixed in a various ways. Here we think of Ω to be the initial (stress-free) configuration Ω_0. Thus, if not emphasized, we mean by $\vec{\chi}$ exactly $\vec{\chi}_\Omega = \vec{\chi}_{\Omega_0}$.

If we denote by $\vec{X}$ a material point in the reference configuration Ω then the position of this point at time t is given by

$$\vec{x} = \vec{\chi}(\vec{X}, t). \tag{2}$$

Next, the mechanical fields describing the deformation are defined. The displacement field

$$\vec{u}(\vec{X},t) = \vec{\chi}(\vec{X},t) - \vec{X}. \tag{3}$$

The velocity field is defined as the time derivative of the position

$$\vec{v} = \frac{\partial \vec{\chi}}{\partial t}, \tag{4}$$

and the deformation gradient with respect to the reference configuration is defined as the spatial derivative

$$\boldsymbol{F} = \frac{\partial \vec{\chi}}{\partial \vec{X}}. \tag{5}$$

By J we denote the determinant of $\boldsymbol{F}$, i.e.,

$$J = \det \boldsymbol{F}. \tag{6}$$

Let us adopt following useful notations for some derivatives. Any field quantity φ with values in some vector space Y (i.e. scalar, vector or tensor valued) can be expressed in the Eulerian description as a function of the spatial position $\vec{x} \in \mathbb{R}^3$

$$\varphi = \tilde{\varphi}(\vec{x},t) : \Omega_t \times [0,T] \mapsto Y.$$

Then we define following notations for the derivatives of the field φ

$$\frac{\partial \varphi}{\partial t} := \frac{\partial \tilde{\varphi}}{\partial t}, \tag{7}$$

$$\operatorname{grad} \varphi = \frac{\partial \varphi}{\partial \vec{x}} := \frac{\partial \tilde{\varphi}}{\partial \vec{x}}. \tag{8}$$

In the case of Lagrangian description we consider the quantity φ to be defined on the reference configuration Ω, then for any $\vec{X} \in \Omega$ we can express the quantity φ as

$$\varphi = \bar{\varphi}(\vec{X},t) : \Omega \times [0,T] \mapsto Y,$$

and we define the derivatives of the field φ as

$$\frac{d\varphi}{dt} := \frac{\partial \bar{\varphi}}{\partial t}, \tag{9}$$

$$\nabla \varphi = \frac{\partial \varphi}{\partial \vec{X}} := \frac{\partial \bar{\varphi}}{\partial \vec{X}}. \tag{10}$$

These two descriptions can be related to each other through following relations

$$\bar{\varphi}(\vec{X},t) = \tilde{\varphi}(\vec{\chi}(\vec{X},t),t), \tag{11}$$

$$\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial t} + (\operatorname{grad} \varphi)\vec{v}, \tag{12}$$

$$\nabla \varphi = (\operatorname{grad} \varphi)\boldsymbol{F}, \tag{13}$$

$$\int_{\Omega_t} \varphi \, dv = \int_{\Omega} \varphi J \, dV \tag{14}$$

$$\frac{d\boldsymbol{F}}{dt} = \nabla \vec{v}, \tag{15}$$

$$\frac{\partial J}{\partial \boldsymbol{F}} = J \boldsymbol{F}^{-T}, \tag{16}$$

$$\frac{dJ}{dt} = J \operatorname{div} \vec{v}. \tag{17}$$

For the formulation of the balance laws we will need to express a time derivatives of some integrals. The following series of equalities obtained by using the previously stated relations will be useful

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t}\varphi\,dv=\frac{d}{dt}\int_{\Omega}\varphi J\,dV&=\int_{\Omega}\frac{d}{dt}\left(\varphi J\right)\,dV\\&=\int_{\Omega_t}\left(\frac{d\varphi}{dt}+\varphi\,\mathrm{div}\,\vec{v}\right)\,dv\\&=\int_{\Omega_t}\left(\frac{\partial\varphi}{\partial t}+\mathrm{div}\left(\varphi\vec{v}\right)\right)\,dv\\&=\int_{\Omega_t}\frac{\partial\varphi}{\partial t}\,dv+\int_{\partial\Omega_t}\varphi\vec{v}\cdot\vec{n}\,da\\&=\frac{\partial}{\partial t}\int_{\Omega_t}\varphi\,dv+\int_{\partial\Omega_t}\varphi\vec{v}\cdot\vec{n}\,da.\end{aligned}\tag{18}$$

And also the Piola identity will be used

$$\mathrm{Div}(J\boldsymbol{F}^{-T})=\vec{0},\tag{19}$$

which can be checked by differentiating the left hand side and using (17) together with an identity obtained by differentiating the relation $\boldsymbol{F}\boldsymbol{F}^{-1}=\boldsymbol{I}$.

3. Balance laws

In this section we will formulate the balance relations for mass and momentum in three forms: the Eulerian, the Lagrangian and the arbitrary Eulerian-Lagrangian (ALE) description.

The Eulerian (or spatial) description is well suited for a problem of fluid flowing through some spatially fixed region. In such a case the material particles can enter and leave the region of interest. The fundamental quantity describing the motion is the velocity vector.

On the other hand the Lagrangian (or referential) description is well suited for a problem of deforming a given body consisting of a fixed set of material particles. In this case the actual boundary of the body can change its its shape. The fundamental quantity describing the motion in this case is the vector of displacement from the referential state.

In the case of fluid-structure interaction problem we can still use the Lagrangian description for the deformation of the solid part. The fluid flow now takes place in a domain with boundary given by the deformation of the structure which can change in time and is influenced back by the fluid flow. The mixed ALE description of the fluid has to be used in this case. The fundamental quantity describing the motion of the fluid is still the velocity vector but the description is accompanied by a certain displacement field which describes the change of the fluid domain. This displacement field has no connection to the fluid velocity field and the purpose of its introduction is to provide a transformation of the current fluid domain and corresponding governing equations to some fixed reference domain. This method is sometimes called a pseudo-solid mapping method [see Sackinger et al., 1996].

The balance relations for energy and entropy are formulated in the Eulerian description which is sufficient for our purpose since they are used only in the specification of the constitutive relations in section 4.1

3.1. Balance of mass. Let $\mathcal{P} \subset \mathbb{R}^3$ be a fixed region in space (control volume) with the boundary $\partial\mathcal{P}$ and unit outward normal vector $\vec{n}_{\mathcal{P}}$, such that

$$\mathcal{P} \subset \Omega_t \quad \text{for all } t \in [0,T].$$

Let ϱ denotes the mass density of the material. Then the balance of mass in the region $\mathcal{P}$ can be written as

$$\frac{\partial}{\partial t}\int_{\mathcal{P}} \varrho \, dv + \int_{\partial\mathcal{P}} \varrho\vec{v}\cdot\vec{n}_{\mathcal{P}}\, da = 0. \tag{20}$$

If all the fields are sufficiently smooth this equation can be written in local form with respect to the current configuration as

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho\vec{v}) = 0. \tag{21}$$

It will be useful to derive the mass balance equation from the Lagrangian point of view. Let $\mathcal{Q} \subset \Omega$ be a fixed set of particles. Then $\vec{\chi}(\mathcal{Q},t) \subset \Omega_t$ is a region occupied by these particles at the time t, and the balance of mass can be expressed as

$$\frac{d}{dt}\int_{\vec{\chi}(\mathcal{Q},t)} \varrho \, dv = 0, \tag{22}$$

which in local form with respect to the reference configuration can be written as

$$\frac{d}{dt}(\varrho J) = 0. \tag{23}$$

In the case of arbitrary Lagrangian-Eulerian description we take a region $\mathcal{R} \in \mathbb{R}^3$ which is itself moving independently of the motion of the body. Let the motion of the control region $\mathcal{R}$ be described by a given mapping

$$\vec{\zeta}_{\mathcal{R}} : \mathcal{R}\times[0,T] \mapsto \mathcal{R}_t, \qquad \mathcal{R}_t \subset \Omega_t \quad \forall t \in [0,T],$$

with the corresponding velocity $\vec{v}_{\mathcal{R}} = \frac{\partial \vec{\zeta}_{\mathcal{R}}}{\partial t}$, deformation gradient $\boldsymbol{F}_{\mathcal{R}} = \frac{\partial \vec{\zeta}_{\mathcal{R}}}{\partial \vec{X}}$ and its determinant $J_{\mathcal{R}} = \det \boldsymbol{F}_{\mathcal{R}}$. The balance equation can be written as

$$\frac{\partial}{\partial t}\int_{\mathcal{R}_t} \varrho \, dv + \int_{\partial\mathcal{R}_t} \varrho(\vec{v}-\vec{v}_{\mathcal{R}})\cdot\vec{n}_{\mathcal{R}_t}\, da = 0, \tag{24}$$

this can be viewed as Eulerian description with moving spatial coordinate system or as a grid deformation in the context of the finite element method. In order to obtain a local form of the balance relation we need to transform the integration to the fixed spatial region $\mathcal{R}$

$$\frac{\partial}{\partial t}\int_{\mathcal{R}} \varrho J_{\mathcal{R}} \, dv + \int_{\partial\mathcal{R}} \varrho(\vec{v}-\vec{v}_{\mathcal{R}})\cdot\boldsymbol{F}_{\mathcal{R}}^{-T}\vec{n}_{\mathcal{R}} J_{\mathcal{R}}\, da = 0, \tag{25}$$

then the local form is

$$\frac{\partial}{\partial t}(\varrho J_{\mathcal{R}}) + \operatorname{div}\left(\varrho J_{\mathcal{R}}(\vec{v}-\vec{v}_{\mathcal{R}})\boldsymbol{F}_{\mathcal{R}}^{-T}\right) = 0. \tag{26}$$

The two previous special formulations can be now recovered. If the region $\mathcal{R}$ is not moving in space, i.e. $\mathcal{R} = \mathcal{R}_t, \forall t \in [0,T]$, then $\vec{\zeta}_{\mathcal{R}}$ is the identity mapping, $\boldsymbol{F}_{\mathcal{R}} = \boldsymbol{I}, J_{\mathcal{R}} = 1, \vec{v}_{\mathcal{R}} = \vec{0}$ and (26) reduces to (21). While if the region $\mathcal{R}$ moves exactly with the material, i.e. $\vec{\zeta}_{\mathcal{R}} = \vec{\chi}|_{\mathcal{R}}$ then $\boldsymbol{F}_{\mathcal{R}} = \boldsymbol{F}, J_{\mathcal{R}} = J, \vec{v}_{\mathcal{R}} = \vec{v}$ and (26) reduces to (23).

3.2. Balance of linear momentum. The balance of linear momentum is postulated in a similar way. Let $\boldsymbol{\sigma}$ denote the Cauchy stress tensor field, representing the surface forces per unit area, $\vec{f}$ be the body forces acting on the material per its unit mass. Then the balance of linear momentum in the Eulerian description is stated as

$$\frac{\partial}{\partial t}\int_{\mathcal{P}} \varrho\vec{v}\,dv + \int_{\partial\mathcal{P}} \varrho\vec{v}\otimes\vec{v}\vec{n}_{\mathcal{P}}\,da = \int_{\partial\mathcal{P}} \boldsymbol{\sigma}^T\vec{n}_{\mathcal{P}}\,da + \int_{\mathcal{P}} \varrho\vec{f}\,dv. \tag{27}$$

The local form of the linear momentum balance is

$$\frac{\partial\varrho\vec{v}}{\partial t} + \operatorname{div}(\varrho\vec{v}\otimes\vec{v}) = \operatorname{div}\boldsymbol{\sigma}^T + \varrho\vec{f}, \tag{28}$$

or with the use of (21) we can write

$$\varrho\frac{\partial\vec{v}}{\partial t} + \varrho(\operatorname{grad}\vec{v})\vec{v} = \operatorname{div}\boldsymbol{\sigma}^T + \varrho\vec{f}. \tag{29}$$

From the Lagrangian point of view the momentum balance relation is

$$\frac{d}{dt}\int_{\vec{\chi}(Q,t)} \varrho\vec{v}\,dv = \int_{\partial\vec{\chi}(Q,t)} \boldsymbol{\sigma}^T\vec{n}_{\vec{\chi}(Q,t)}\,da + \int_{\vec{\chi}(Q,t)} \varrho\vec{f}\,dv. \tag{30}$$

Let us denote by $\boldsymbol{P} = J\boldsymbol{\sigma}^T\boldsymbol{F}^{-T}$ the first Piola-Kirchhoff stress tensor [see Gurtin, 1981], then the local form of the momentum balance is

$$\frac{d}{dt}(\varrho J\vec{v}) = \operatorname{Div}\boldsymbol{P} + \varrho J\vec{f}, \tag{31}$$

or using (23) we can write

$$\varrho J\frac{d\vec{v}}{dt} = \operatorname{Div}\boldsymbol{P} + \varrho J\vec{f}. \tag{32}$$

In the arbitrary Lagrangian-Eulerian formulation we obtain

$$\frac{\partial}{\partial t}\int_{\mathcal{R}_t} \varrho\vec{v}\,dv + \int_{\partial\mathcal{R}_t} \varrho\vec{v}\otimes(\vec{v}-\vec{v}_{\mathcal{R}})\vec{n}_{\mathcal{R}_t}\,da = \int_{\partial\mathcal{R}_t} \boldsymbol{\sigma}^T\vec{n}_{\mathcal{R}_t}\,da + \int_{\mathcal{R}_t} \varrho\vec{f}\,dv, \tag{33}$$

which in the local form gives

$$\frac{\partial\varrho J_{\mathcal{R}}\vec{v}}{\partial t} + \operatorname{div}\left(\varrho J_{\mathcal{R}}\vec{v}\otimes(\vec{v}-\vec{v}_{\mathcal{R}})\boldsymbol{F}_{\mathcal{R}}^{-T}\right) = \operatorname{div}\left(J_{\mathcal{R}}\boldsymbol{\sigma}^T\boldsymbol{F}_{\mathcal{R}}^{-T}\right) + \varrho J_{\mathcal{R}}\vec{f}, \tag{34}$$

or with the use of (26) we can write

$$\varrho J_{\mathcal{R}}\frac{\partial\vec{v}}{\partial t} + \varrho J_{\mathcal{R}}(\operatorname{grad}\vec{v})\boldsymbol{F}_{\mathcal{R}}^{-T}(\vec{v}-\vec{v}_{\mathcal{R}}) = \operatorname{div}\left(J_{\mathcal{R}}\boldsymbol{\sigma}^T\boldsymbol{F}_{\mathcal{R}}^{-T}\right) + \varrho J_{\mathcal{R}}\vec{f}. \tag{35}$$

3.3. Balance of angular momentum. For the angular momentum balance we assume that there are no external or internal sources of angular momentum, then it follows that the Cauchy stress tensor has to be symmetric [see Truesdell, 1991], i.e.

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T. \tag{36}$$

3.4. Balance of energy. Let e be the internal energy per its unit mass, $\vec{q}$ be the heat flux per unit area, r be the heat source per unit mass. Denoting the total energy per mass as $E = e + \frac{1}{2}|\vec{v}|^2$, the balance of energy is stated as

$$\frac{\partial}{\partial t}\int_{\mathcal{P}} \varrho E \, dv + \int_{\partial\mathcal{P}} \varrho E \vec{v}\cdot\vec{n}\, da = \int_{\partial\mathcal{P}} (\boldsymbol{\sigma}\vec{v} - \vec{q})\cdot\vec{n}\, da + \int_{\mathcal{P}} \left(\varrho \vec{f}\cdot\vec{v} + \varrho r\right) dv. \tag{37}$$

Writing this in local form and using the mass and momentum balance equations (21) and (28) we obtain

$$\frac{\partial \varrho e}{\partial t} + \operatorname{div}(\varrho e \vec{v}) = \operatorname{tr}(\boldsymbol{\sigma}\operatorname{grad}\vec{v}) - \operatorname{div}\vec{q} + \varrho r. \tag{38}$$

3.5. Balance of entropy. Finally, the balance of entropy is introduced. Let η be the entropy per its unit mass, T be the temperature and s be the entropy production, then the entropy balance is

$$\frac{\partial}{\partial t}\int_{\mathcal{P}} \varrho\eta \, dv + \int_{\partial\mathcal{P}} \varrho\eta\vec{v}\cdot\vec{n}\, da = -\int_{\partial\mathcal{P}} \frac{\vec{q}}{T}\cdot\vec{n}\, da + \int_{\mathcal{P}} \left(\frac{\varrho r}{T} + s\right) dv, \tag{39}$$

or in local form

$$\frac{\partial \varrho\eta}{\partial t} + \operatorname{div}(\varrho\eta\vec{v}) = -\operatorname{div}\frac{\vec{q}}{T} + \frac{\varrho r}{T} + s. \tag{40}$$

The second law of thermodynamics says that the total entropy production is non-negative, implying

$$s \geq 0. \tag{41}$$

We introduce the Helmholtz potential $\Psi = e - T\eta$, then using (38) and (41) we obtain the entropy inequality in the following form

$$\eta\left(\frac{\partial \varrho T}{\partial t} + \operatorname{div}(\varrho T\vec{v})\right) + \frac{\partial \varrho\Psi}{\partial t} + \operatorname{div}(\varrho\Psi\vec{v}) + \frac{\vec{q}}{T}\cdot\operatorname{grad} T - \operatorname{tr}(\boldsymbol{\sigma}\operatorname{grad}\vec{v}) \leq 0. \tag{42}$$

This inequality is used to narrow the class of constitutive relations for the stresses $\boldsymbol{\sigma}$ in the section 4.1.

4. Problem formulation

Our goal is to model an object (large blood vessel, heart ventricle) where an elastic, deformable body interacts with viscous fluid. The formulation should be general enough to include possibly additional effects and characteristics found in biological models like a chemical reactions, transport of electric excitation and active behavior of the tissue.

At this point we make a few assumptions that will allow us to deal with the task of setting up a tractable problem. We will use the superscripts ^{s} and ^{f} to denote the quantities connected with the solid and fluid. Let us assume that the both materials are incompressible, which is well accepted approximation in biomechanics and denote the constant densities of each material by ϱ^f, ϱ^s. Further we assume all the processes to be isothermal, meaning that the temperature T is constant, which is also well accepted approximation for certain biomechanical processes.

We denote by Ω_t^f the domain occupied by the fluid and Ω_t^s by the solid at time $t \in [0, T]$. Let $\Gamma_t^0 = \bar{\Omega}_t^f \cap \bar{\Omega}_t^s$ be the part of the boundary where the solid interacts

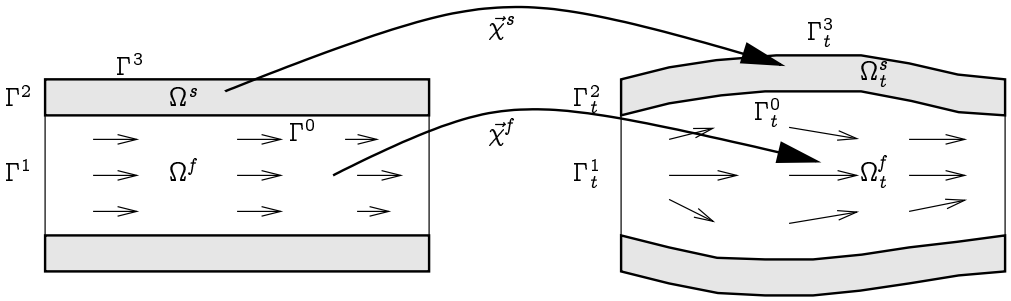

FIGURE 2. Undeformed (original) and deformed (current) configurations.

with the fluid and $\Gamma^i_t, i = 1, 2, 3$ be the remaining external boundaries of the solid and the fluid as depicted in figure 2.

Let the deformation of the solid part be described by the mapping $\vec{\chi}^s$

$$\vec{\chi}^s : \Omega^s \times [0, T] \mapsto \Omega^s_t, \tag{43}$$

with the corresponding displacement $\vec{u}^s$ and the velocity $\vec{v}^s$ given by

$$\vec{u}^s(\vec{X}, t) = \vec{\chi}^s(\vec{X}, t) - \vec{X}, \tag{44}$$

$$\vec{v}^s(\vec{X}, t) = \frac{\partial \vec{\chi}^s}{\partial t}(\vec{X}, t). \tag{45}$$

The fluid flow is described by the velocity field $\vec{v}^f$ defined on the fluid domain Ω^f_t

$$\vec{v}^f(\vec{x}, t) : \Omega^f_t \times [0, T] \mapsto \mathbb{R}^3. \tag{46}$$

Further we define the auxiliary mapping, denoted by $\vec{\zeta}^f$, to describe the change of the fluid domain and corresponding displacement $\vec{u}^f$ by

$$\vec{\zeta}^f : \Omega^f \times [0, T] \mapsto \Omega^f_t, \tag{47}$$

$$\vec{u}^f(\vec{X}, t) = \vec{\zeta}^f(\vec{X}, t) - \vec{X}. \tag{48}$$

We require that the mapping $\vec{\zeta}^f$ is sufficiently smooth, one to one and has to satisfy

$$\vec{\zeta}^f(\vec{X}, t) = \vec{\chi}^s(\vec{X}, t), \quad \forall (\vec{X}, t) \in \Gamma^0 \times [0, T]. \tag{49}$$

In the context of the finite element method this will describe the artificial mesh deformation inside the fluid region and it will be constructed as a solution to a suitable boundary value problem with (49) as the boundary condition.

The momentum and mass balance of the fluid in the time dependent fluid domain according to (26) and (34) are

$$\varrho^f \frac{\partial \vec{v}^f}{\partial t} + \varrho^f (\operatorname{grad} \vec{v}^f)(\vec{v}^f - \frac{\partial \vec{u}^f}{\partial t}) = \operatorname{div} \boldsymbol{\sigma}^f \qquad \text{in } \Omega^f_t, \tag{50}$$

$$\operatorname{div} \vec{v}^f = 0 \qquad \text{in } \Omega^f_t, \tag{51}$$

together with the momentum (28) and mass (21) balance of the solid in the solid domain

$$\varrho^s \frac{\partial \vec{v}^s}{\partial t} + \varrho^s (\operatorname{grad} \vec{v}^s)\vec{v}^s = \operatorname{div} \boldsymbol{\sigma}^s \qquad \text{in } \Omega_t^s, \tag{52}$$

$$\operatorname{div} \vec{v}^s = 0 \qquad \text{in } \Omega_t^s. \tag{53}$$

The interaction is due to the exchange of momentum through the common part of the boundary Γ_t^0. On this part we require that the forces are in balance, i.e.

$$\boldsymbol{\sigma}^f \vec{n} = \boldsymbol{\sigma}^s \vec{n} \qquad \text{on } \Gamma_t^0. \tag{54}$$

Further, we prescribe the no slip boundary condition for the fluid on the interface. This is expressed by

$$\vec{v}^f = \vec{v}^s \qquad \text{on } \Gamma_t^0. \tag{55}$$

The remaining external boundary conditions can be of the following kind. A pressure boundary condition on the fluid inflow and outflow part Γ_t^1

$$\boldsymbol{\sigma}^f \vec{n} = p_B \vec{n} \text{ on } \Gamma_t^1, \tag{56}$$

with p_B given value. Alternatively we can prescribe a Dirichlet type boundary condition on the inflow or outflow part Γ_t^1

$$\vec{v}^f = \vec{v}_B \text{ on } \Gamma_t^1, \tag{57}$$

where $\vec{v}_B$ is given. The Dirichlet boundary condition is prescribed for the solid displacement at the part Γ_t^2

$$\vec{u}^s = \vec{0} \text{ on } \Gamma_t^2, \tag{58}$$

and the stress free boundary condition for the solid is applied at the part Γ_t^3

$$\boldsymbol{\sigma}^s \vec{n} = \vec{0} \text{ on } \Gamma_t^3. \tag{59}$$

We introduce the domain $\Omega = \Omega^f \cup \Omega^s$, where Ω^f, Ω^s are the domains occupied by the fluid and solid in the initial undeformed state, and two fields defined on this domain as

$$\vec{u} : \Omega \times [0,T] \to \mathbb{R}^3,$$
$$\vec{v} : \Omega \times [0,T] \to \mathbb{R}^3,$$

such that the field $\vec{v}$ represents the velocity at the given point and $\vec{u}$ the displacement on the solid part and the artificial displacement in the fluid part, taking care of the fact that the fluid domain is changing with time,

$$\vec{v} = \begin{cases} \vec{v}^s & \text{on } \Omega^s, \\ \vec{v}^f & \text{on } \Omega^f, \end{cases} \tag{60}$$

$$\vec{u} = \begin{cases} \vec{u}^s & \text{on } \Omega^s, \\ \vec{u}^f & \text{on } \Omega^f. \end{cases} \tag{61}$$

Due to the conditions (49) and (55) both fields are continuous across the interface Γ_t^0 and we can define global quantities on Ω as the deformation gradient and its determinant

$$\boldsymbol{F} = \boldsymbol{I} + \nabla \vec{u}, \tag{62}$$

$$J = \det \boldsymbol{F}. \tag{63}$$

Using this notation the solid balance laws (52) and (53) can be expressed in the Lagrangian formulation with the initial configuration Ω^s as reference, cf. (31),

$$J\varrho^s \frac{d\vec{v}}{dt} = \operatorname{Div} \boldsymbol{P}^s \qquad \text{in } \Omega^s, \tag{64}$$

$$J = 1 \qquad \text{in } \Omega^s. \tag{65}$$

The fluid equations (50) and (51) are already expressed in the arbitrary Lagrangian-Eulerian formulation with respect to the time dependent region Ω_t^f, now we transform the equations to the fixed initial region Ω^f by the mapping ζ^f defined by (47)

$$\varrho^f \frac{\partial \vec{v}}{\partial t} + \varrho^f (\nabla \vec{v}) \boldsymbol{F}^{-1} (\vec{v} - \frac{\partial \vec{u}}{\partial t}) = J^{-1} \operatorname{Div}(J \boldsymbol{\sigma}^f F^{-T}) \qquad \text{in } \Omega^f, \tag{66}$$

$$\operatorname{Div}(J \vec{v} \boldsymbol{F}^{-T}) = 0 \qquad \text{in } \Omega^f. \tag{67}$$

It remains to prescribe some relation for the mapping ζ^f. In terms of the corresponding displacement $\vec{u}^f$ we formulate some simple relation together with the Dirichlet boundary conditions required by (49), for example

$$\frac{\partial \vec{u}}{\partial t} = \Delta \vec{u} \qquad \text{in } \Omega^f, \tag{68}$$

$$\vec{u} = \vec{u}^s \qquad \text{on } \Gamma^0, \tag{69}$$

$$\vec{u} = \vec{0} \qquad \text{on } \Gamma^1. \tag{70}$$

The complete set of the equations can be written as

$$\frac{\partial \vec{u}}{\partial t} = \begin{cases} \vec{v} & \text{in } \Omega^s, \\ \Delta \vec{u} & \text{in } \Omega^f, \end{cases} \tag{71}$$

$$\frac{\partial \vec{v}}{\partial t} = \begin{cases} \frac{1}{J\varrho^s} \operatorname{Div} \boldsymbol{P}^s & \text{in } \Omega^s, \\ -(\nabla \vec{v}) \boldsymbol{F}^{-1} (\vec{v} - \frac{\partial \vec{u}}{\partial t}) + \frac{1}{J\varrho^f} \operatorname{Div}(J \boldsymbol{\sigma}^f \boldsymbol{F}^{-T}) & \text{in } \Omega^f, \end{cases} \tag{72}$$

$$0 = \begin{cases} J - 1 & \text{in } \Omega^s, \\ \operatorname{Div}(J \vec{v} \boldsymbol{F}^{-T}) & \text{in } \Omega^f, \end{cases} \tag{73}$$

with the initial conditions

$$\vec{u}(0) = \vec{0} \qquad \text{in } \Omega, \tag{74}$$

$$\vec{v}(0) = \vec{v}_0 \qquad \text{in } \Omega, \tag{75}$$

and boundary conditions

$$\vec{u} = \vec{0}, \quad \vec{v} = \vec{v}_B \qquad \text{on } \Gamma^1, \tag{76}$$

$$\vec{u} = \vec{0} \qquad \text{on } \Gamma^2, \tag{77}$$

$$\boldsymbol{\sigma}^s \vec{n} = \vec{0} \qquad \text{on } \Gamma^3. \tag{78}$$

4.1. Constitutive equations. Further we need to specify the constitutive relations for the stress tensors. For the fluid we use the incompressible Newtonian relation

$$\boldsymbol{\sigma}^f = -p^f \boldsymbol{I} + \mu(\operatorname{grad} \vec{v}^f + \operatorname{grad}^T \vec{v}^f), \tag{79}$$

where μ represents the viscosity of the fluid and p^f is the Lagrange multiplier corresponding to the incompressibility constraint (51).

For the solid part we assume that it can be described by an incompressible hyper-elastic material. We specify the Helmholtz potential Ψ and the solid stress is given by

$$\boldsymbol{\sigma}^s = -p^s \boldsymbol{I} + \varrho^s \frac{\partial \Psi}{\partial \boldsymbol{F}} \boldsymbol{F}^T, \tag{80}$$

the first Piola-Kirchhoff stress tensor is then given by

$$\boldsymbol{P}^s = -J p^s \boldsymbol{F}^{-T} + J \varrho^s \frac{\partial \Psi}{\partial \boldsymbol{F}}, \tag{81}$$

where p^s is the Lagrange multiplier corresponding to the incompressibility constraint (65).

The Helmholtz potential can be expressed as a function of different quantities

$$\Psi = \hat{\Psi}(\boldsymbol{F}) = \hat{\Psi}(\boldsymbol{I} + \nabla \vec{u}),$$

but due to the principle of material frame indifference the Helmholtz potential Ψ depends on the deformation only through the right Cauchy-Green deformation tensor $\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}$ [see Gurtin, 1981]

$$\Psi = \tilde{\Psi}(\boldsymbol{C}). \tag{82}$$

A certain coerciveness condition is usually imposed on the form of the Helmholtz potential

$$\bar{\Psi}(\nabla \vec{u}(\vec{X}, t)) \geq a \left\| \nabla \vec{u}(\vec{X}, t) \right\|^2 - b(\vec{X}), \tag{83}$$

where a is a positive constant and $b \in L^1(\Omega^s)$.

Typical examples for the Helmholtz potential used for isotropic materials like rubber are the neo-Hookean model given by

$$\tilde{\Psi} = c_1 (I_C - 3), \tag{84}$$

or the Mooney-Rivlin material

$$\tilde{\Psi} = c_1 (I_C - 3) + c_2 (II_C - 3), \tag{85}$$

where $I_C = \operatorname{tr} \boldsymbol{C}, II_C = \operatorname{tr} \boldsymbol{C}^2 - \operatorname{tr}^2 \boldsymbol{C}, III_C = \det \boldsymbol{C}$ are the invariants of the right Cauchy-Green deformation tensor $\boldsymbol{C}$ and c_i are some material constants.

In the case of material anisotropy we can use

$$\tilde{\Psi} = c_1 (I_C - 3) + c_2 (II_C - 3) + c_3 (|\boldsymbol{F}\vec{a}| - 1)^2, \tag{86}$$

with $\vec{a}$ being the preferred material direction. The term $|\boldsymbol{F}\vec{a}|$ represents the extension in the direction $\vec{a}$ and when rewritten as $(a \cdot \boldsymbol{C}a)^{\frac{1}{2}}$ we can see that it is a function of $\boldsymbol{C}$. In Humphrey et al. [1990a,b] a similar material relation of the form

$$\tilde{\Psi} = c_1 \left(\exp \left(b_1 (I_C - 3) \right) - 1 \right) + c_2 \left(\exp \left(b_2 (|\boldsymbol{F}\vec{a}| - 1) \right) - 1 \right) \tag{87}$$

has been proposed to describe a passive behavior of the muscle tissue. Adding to any form of Ψ a term like $f(t, \vec{x})(|\boldsymbol{F}\vec{a}| - 1)$ one can model the active behavior of a material and then the system can be coupled with additional models of chemical and electric activation of the active response of the tissue, see Maurel et al. [1998].

With a suitable choice of the material parameters the entropy inequality and the balance of energy is automatically satisfied. The reduced entropy inequality is obtained from (42) by using the assumptions of isothermality and incompressibility

$$\varrho \frac{d\Psi}{dt} - \operatorname{tr} (\boldsymbol{\sigma} \operatorname{grad} \vec{v}) - p \operatorname{div} \vec{v} \leq 0. \tag{88}$$

For the fluid we have the constitutive equation (79) and $\Psi = 0$. Then to satisfy the reduced entropy inequality (88) it is sufficient to require

$$\mu > 0. \tag{89}$$

In the solid region with the hyper-elastic material assumptions (80) and (82) the left hand side of the entropy inequality (88) is identically equal to 0.

4.2. Weak formulation. We non-dimensionalize all the quantities by a given characteristic length L and speed V as follows

$$\begin{aligned} \hat{t} &= t\frac{V}{L}, & \hat{\vec{x}} &= \frac{\vec{x}}{L}, \\ \hat{\vec{u}} &= \frac{\vec{u}}{L}, & \hat{\vec{v}} &= \frac{\vec{v}}{V}, \\ \hat{\boldsymbol{\sigma}}^s &= \boldsymbol{\sigma}^s \frac{L}{\varrho^f V^2}, & \hat{\boldsymbol{\sigma}}^f &= \boldsymbol{\sigma}^f \frac{L}{\varrho^f V^2}, \\ \hat{\mu} &= \frac{\mu}{\varrho^f V L}, & \hat{\Psi} &= \Psi \frac{L}{\varrho^f V^2}, \end{aligned}$$

further using the same symbols, without the hat, for the non-dimensional quantities and denoting by $\beta = \frac{\varrho^s}{\varrho^f}$ the densities ratio. The non-dimensionalized system with the choice of material relations, (79) for viscous fluid and (81) for the hyper-elastic solid is

$$\frac{\partial \vec{u}}{\partial t} = \begin{cases} \vec{v} & \text{in } \Omega^s, \\ \Delta \vec{u} & \text{in } \Omega^f, \end{cases} \tag{90}$$

$$\frac{\partial \vec{v}}{\partial t} = \begin{cases} \frac{1}{\beta} \operatorname{Div}\left(-Jp^s \boldsymbol{F}^{-T} + \frac{\partial \Psi}{\partial \boldsymbol{F}}\right) & \text{in } \Omega^s, \\ -(\nabla \vec{v})\boldsymbol{F}^{-1}(\vec{v} - \frac{\partial \vec{u}}{\partial t}) & \\ \quad + \operatorname{Div}\left(-Jp^f \boldsymbol{F}^{-T} + J\mu \nabla \vec{v} \boldsymbol{F}^{-1} \boldsymbol{F}^{-T}\right) & \text{in } \Omega^f, \end{cases} \tag{91}$$

$$0 = \begin{cases} J - 1 & \text{in } \Omega^s, \\ \operatorname{Div}(J\vec{v}\boldsymbol{F}^{-T}) & \text{in } \Omega^f, \end{cases} \tag{92}$$

and the boundary conditions

$$\boldsymbol{\sigma}^f \vec{n} = \boldsymbol{\sigma}^s \vec{n} \quad \text{on } \Gamma_t^0, \tag{93}$$

$$\vec{v} = \vec{v}_B \quad \text{on } \Gamma_t^1, \tag{94}$$

$$\vec{u} = \vec{0} \quad \text{on } \Gamma_t^2, \tag{95}$$

$$\boldsymbol{\sigma}^f \vec{n} = \vec{0} \quad \text{on } \Gamma_t^3. \tag{96}$$

Let $I = [0, T]$ denotes the time interval of interest. We multiply the equations (90)-(92) by the test functions $\vec{\zeta}, \vec{\xi}, \gamma$ such that $\vec{\zeta} = \vec{0}$ on Γ^2, $\vec{\xi} = \vec{0}$ on Γ^1 and integrate over the space domain Ω and the time interval I. Using integration by

parts on some of the terms and the boundary conditions we obtain

$$\begin{aligned}
\int_0^T \int_\Omega \frac{\partial \vec{u}}{\partial t} \cdot \vec{\zeta}\, dV\, dt &= \int_0^T \int_{\Omega^s} \vec{v} \cdot \vec{\zeta}\, dV\, dt \\
&\quad - \int_0^T \int_{\Omega^f} \nabla \vec{u} \cdot \nabla \vec{\zeta}\, dV\, dt, \qquad (97)\\
\int_0^T \int_{\Omega^f} J \frac{\partial \vec{v}}{\partial t} \cdot \vec{\xi}\, dV\, dt &+ \int_0^T \int_{\Omega^s} \beta J \frac{\partial \vec{v}}{\partial t} \cdot \vec{\xi}\, dV\, dt \\
&= - \int_0^T \int_{\Omega^f} J \nabla \vec{v} \boldsymbol{F}^{-1} (\vec{v} - \frac{\partial \vec{u}}{\partial t}) \cdot \vec{\xi}\, dV\, dt \\
&\quad + \int_0^T \int_{\Omega} J p \boldsymbol{F}^{-T} \cdot \nabla \vec{\xi}\, dV\, dt \qquad (98)\\
&\quad - \int_0^T \int_{\Omega^s} \frac{\partial \Psi}{\partial \boldsymbol{F}} \cdot \nabla \vec{\xi}\, dV\, dt \\
&\quad - \int_0^T \int_{\Omega^f} J \mu \nabla \vec{v} \boldsymbol{F}^{-1} \boldsymbol{F}^{-T} \cdot \nabla \vec{\xi}\, dV\, dt, \\
0 &= \int_0^T \int_{\Omega^s} (J-1) \gamma\, dV\, dt \qquad (99)\\
&\quad + \int_0^T \int_{\Omega^f} \mathrm{Div}(J \vec{v} \boldsymbol{F}^{-T}) \gamma\, dV\, dt.
\end{aligned}$$

Transforming some of the integrals to the current domain the momentum balance equation (98) becomes

$$\begin{aligned}
\int_0^T \int_{\Omega_t^f} \frac{\partial \vec{v}}{\partial t} \cdot \vec{\xi}\, dv\, dt &+ \int_0^T \int_{\Omega_t^s} \beta \frac{\partial \vec{v}}{\partial t} \cdot \vec{\xi}\, dv\, dt \\
&= - \int_0^T \int_{\Omega_t^f} \operatorname{grad} \vec{v} (\vec{v} - \frac{\partial \vec{u}}{\partial t}) \cdot \vec{\xi}\, dv\, dt \\
&\quad + \int_0^T \int_{\Omega_t} p \operatorname{div} \vec{\xi}\, dv\, dt \qquad (100)\\
&\quad - \int_0^T \int_{\Omega^s} \frac{\partial \Psi}{\partial \boldsymbol{F}} \cdot \nabla \vec{\xi}\, dV\, dt \\
&\quad - \int_0^T \int_{\Omega_t^f} \mu \operatorname{grad} \vec{v} \cdot \operatorname{grad} \vec{\xi}\, dv\, dt.
\end{aligned}$$

Now, using the velocity $\vec{v}$ as a test function $\vec{\xi}$ in this equation we obtain the balance of the mechanical energy

$$\begin{aligned}\frac{1}{2}\int_0^T\int_{\Omega_t^f}\frac{\partial}{\partial t}\left|\vec{v}\right|^2\,dv\,dt+&\frac{\beta}{2}\int_0^T\int_{\Omega_t^s}\frac{\partial}{\partial t}\left|\vec{v}\right|^2\,dv\,dt\\ &=-\int_0^T\int_{\Omega_t^f}\operatorname{grad}\vec{v}(\vec{v}-\frac{\partial\vec{u}}{\partial t})\cdot\vec{v}\,dv\,dt\\ &\quad-\int_0^T\int_{\Omega^s}\frac{\partial\Psi}{\partial\boldsymbol{F}}\cdot\nabla\vec{v}\,dV\,dt\\ &\quad-\int_0^T\int_{\Omega_t^f}\mu\left|\operatorname{grad}\vec{v}\right|^2\,dv\,dt.\end{aligned}\tag{101}$$

The first term on the left hand side of (101) together with the convective term on the right hand side can be rewritten with the help of (17) as

$$\begin{aligned}&\frac{1}{2}\int_0^T\int_{\Omega_t^f}\frac{\partial}{\partial t}\left|\vec{v}\right|^2\,dv\,dt+\int_0^T\int_{\Omega_t^f}\operatorname{grad}\vec{v}(\vec{v}-\frac{\partial\vec{u}}{\partial t})\cdot\vec{v}\,dv\,dt\\ &=\frac{1}{2}\int_0^T\int_{\Omega^f}J\frac{\partial}{\partial t}\left|\vec{v}\right|^2\,dV\,dt+\int_0^T\int_{\Omega_t^f}\operatorname{grad}\vec{v}(\vec{v}-\frac{\partial\vec{u}}{\partial t})\cdot\vec{v}\,dv\,dt\\ &=\frac{1}{2}\int_0^T\int_{\Omega^f}\frac{\partial}{\partial t}\left(J\left|\vec{v}\right|^2\right)\,dV\,dt-\frac{1}{2}\int_0^T\int_{\Omega^f}J\left|\vec{v}\right|^2\operatorname{div}\frac{\partial\vec{u}}{\partial t}\,dV\,dt\\ &\quad+\frac{1}{2}\int_0^T\int_{\Omega_t^f}\operatorname{grad}\left|\vec{v}\right|^2(\vec{v}-\frac{\partial\vec{u}}{\partial t})\,dv\,dt\\ &=\frac{1}{2}\int_0^T\frac{\partial}{\partial t}\int_{\Omega_t^f}\left|\vec{v}\right|^2\,dv\,dt+\frac{1}{2}\int_0^T\int_{\Omega_t^f}\operatorname{div}\left(\left|\vec{v}\right|^2(\vec{v}-\frac{\partial\vec{u}}{\partial t})\right)\,dv\,dt\\ &=\frac{1}{2}\left[\left\|\vec{v}(t)\right\|^2_{L^2(\Omega_t^f)}\right]_0^T+\frac{1}{2}\int_0^T\int_{\partial\Omega_t^f}\left|\vec{v}\right|^2(\vec{v}-\frac{\partial\vec{u}}{\partial t})\cdot\vec{n}_{\Omega_t^f}\,da\,dt,\\ &=\frac{1}{2}\left[\left\|\vec{v}(t)\right\|^2_{L^2(\Omega_t^f)}\right]_0^T+\frac{1}{2}\int_0^T\int_{\Gamma_t^1}\left|\vec{v}_B\right|^2\vec{v}_B\cdot\vec{n}_{\Omega_t^f}\,da\,dt.\end{aligned}\tag{102}$$

where we have used the boundary condition (94) and the fact that $\frac{\partial\vec{u}}{\partial t}=\vec{v}$ on Γ^0 and $\frac{\partial\vec{u}}{\partial t}=\vec{0}$ on Γ^1.

The second term on the right hand side of (101) can be rewritten by use of (15) as

$$\begin{aligned}\int_0^T\int_{\Omega^s}\frac{\partial\Psi}{\partial\boldsymbol{F}}\cdot\nabla\vec{v}\,dV\,dt&=\int_0^T\int_{\Omega^s}\frac{\partial\Psi}{\partial\boldsymbol{F}}\cdot\frac{d\boldsymbol{F}}{dt}\,dV\,dt\\ &=\int_0^T\int_{\Omega^s}\frac{d\Psi}{dt}\,dV\,dt\\ &=\left[\int_{\Omega^s}\Psi(\boldsymbol{F}(t))\,dV\right]_0^T.\end{aligned}\tag{103}$$

Using (102) and (103) in (101) with the initial condition (75) and the fact that $\Psi = 0$ in the initial, undeformed state, we obtain

(104)
$$\frac{1}{2}\left\|\vec{v}(T)\right\|^2_{L^2(\Omega^f_T)} + \frac{\beta}{2}\left\|\vec{v}(T)\right\|^2_{L^2(\Omega^s_T)} + \int_0^T \mu \left\|\operatorname{grad}\vec{v}\right\|^2_{L^2(\Omega^f_t)}\,dt + \int_{\Omega^s} \Psi(\boldsymbol{F}(T))\,dV$$
$$= \frac{1}{2}\left\|\vec{v}_0\right\|^2_{L^2(\Omega^f)} + \frac{\beta}{2}\left\|\vec{v}_0\right\|^2_{L^2(\Omega^s)} - \frac{1}{2}\int_0^T \int_{\Gamma^1_t} \left|\vec{v}_B\right|^2 \vec{v}_B \cdot \vec{n}_{\Omega^f_t}\,da\,dt.$$

Using the assumption (83), setting $c = \min(1, \beta)$ and considering the homogeneous Dirichlet boundary condition $\vec{v}_B = \vec{0}$ we finally obtain an energy estimate of the following form

(105)
$$\frac{c}{2}\left\|\vec{v}(T)\right\|^2_{L^2(\Omega_T)} + \int_0^T \mu \left\|\operatorname{grad}\vec{v}\right\|^2_{L^2(\Omega^f_t)}\,dt + a\left\|\nabla\vec{u}(T)\right\|^2_{L^2(\Omega^s)}$$
$$\leq \left\|b\right\|_{L^1(\Omega^s)} + \frac{1}{2}\left\|\vec{v}_0\right\|^2_{L^2(\Omega^f)} + \frac{\beta}{2}\left\|\vec{v}_0\right\|^2_{L^2(\Omega^s)}.$$

Let us define the following spaces

$$U = \{\vec{u} \in L^\infty(I, [W^{1,2}(\Omega)]^3), \vec{u} = \vec{0} \text{ on } \Gamma^2\},$$
$$V = \{\vec{v} \in L^2(I, [W^{1,2}(\Omega_t)]^3) \cap L^\infty(I, [L^2(\Omega_t)]^3), \vec{v} = \vec{0} \text{ on } \Gamma^1\},$$
$$P = \{p \in L^2(I, L^2(\Omega))\},$$

then the variational formulation of the fluid-structure interaction problem is stated as follows

DEFINITION 106. *Find* $(\vec{u}, \vec{v} - \vec{v}_B, p) \in U \times V \times P$ *such that equations* (97), (98) *and* (99) *are satisfied for all* $(\vec{\zeta}, \vec{\xi}, \gamma) \in U \times V \times P$.

4.3. Discretization. From now on, we restrict ourselves to two dimensions. This restriction is due to an easier presentation and the computational time needed to solve the problem. Apart of these two reasons, the same discretization procedure can be applied to the three dimensional problem.

The time discretization is done by the Crank-Nicholson scheme which is only conditionally stable but it has better conservation property than for example the implicit Euler scheme [see Farhat et al., 1995, Koobus and Farhat, 1999]. The Crank-Nicholson scheme can be obtained by dividing the time interval I into the series of time steps $[t^n, t^{n+1}]$ with step length $k_n = t^{n+1} - t^n$. Assuming that the test functions are piecewise constant on each time step $[t^n, t^{n+1}], \forall n$, writing the weak formulation (97)-(98) for the time interval $[t^n, t^{n+1}]$, approximating the time derivatives by the central differences

(107)
$$\frac{\partial f}{\partial t} \approx \frac{f(t^{n+1}) - f(t^n)}{k_n}$$

and approximating the time integration for the remaining terms by the trapezoidal quadrature rule as

(108)
$$\int_{t^n}^{t^{n+1}} f(t)\,dt \approx \frac{k_n}{2}(f(t^n) + f(t^{n+1})),$$

we obtain the time discretized system. The last equation was taken explicitly for the time t^{n+1} and the corresponding term with the Lagrange multiplier p^n_h in the equation (98) was also taken explicitly.

The discretization in space is done by the finite element method. We approximate the domain Ω by a domain Ω_h with polygonal boundary and by $\mathcal{T}_h$ we denote a set of quadrilaterals covering the domain Ω_h. We assume that $\mathcal{T}_h$ is regular in the sense that any two quadrilateral are disjoint or have a common vertex or a common edge. By $\bar{T} = [-1,1]^2$ we denote the reference quadrilateral.

Our treatment of the problem as a one system suggests to use the same finite elements on both, the solid part and the fluid region. Since both materials are incompressible we have to choose a pair of finite element spaces know to be stable for the problems with incompressibility constraint. One possible choice is the conforming biquadratic, discontinuous bilinear Q_2, P_1 pair, see figure 3 for the location of the degrees of freedom. The local basis on the reference element is given by

$$\begin{aligned} Q_2(\bar{T}) = \mathrm{span}\{&(1-x^2)(1-y^2), \frac{1}{2}(1-x^2)(y-y^2), \frac{1}{2}(1-x^2)(y+y^2), \\ &\frac{1}{2}(x-x^2)(1-y^2), \frac{1}{4}(x-x^2)(y-y^2), \frac{1}{4}(x-x^2)(y+y^2), \\ &\frac{1}{2}(x+x^2)(1-y^2), \frac{1}{4}(x+x^2)(y-y^2), \frac{1}{4}(x+x^2)(y+y^2)\}, \end{aligned} \tag{109}$$

$$P_1(\bar{T}) = \mathrm{span}\{1, x, y\}, \tag{110}$$

this choice results to 39 degrees of freedom on an element in the case of our displacement, velocity, pressure formulation in two dimensions and to 112 degrees of freedom on an element in three dimensions. This seem rather prohibitive, especially for a three dimensional computation.

The other choice is the nonconforming rotated bilinear, constants $\tilde{Q}_1, P_0$ pair, see figure 4 for the location of the degrees of freedom. The local basis on the reference element in this case is given by

$$\begin{aligned} \tilde{Q}_1(\bar{T}) = \mathrm{span}\{&\frac{(1-x^2)+y(y-2)}{4}, \frac{(1-x^2)+y(y+2)}{4}, \\ &\frac{(1-y^2)+x(x-2)}{4}, \frac{(1-y^2)+x(x+2)}{4}\}, \end{aligned} \tag{111}$$

$$P_0(\bar{T}) = \mathrm{span}\{1\}. \tag{112}$$

In this case we have one order lower approximation than in the previous case with 17 degrees of freedom on an element for the displacement, velocity, pressure formulation in two dimensions and only 25 degrees of freedom on an element in three dimensions. This seems more appealing when considering a three dimensional problem. It was shown in Falk [1991] that the rotated bilinear elements $\tilde{Q}_1$ does not preserve the discrete version of the Korn's inequality, which can cause an occurrence of spurious modes in the solution. However, our experiences show that when solving the resulting linear system by a conjugate gradient type method these modes does not appear.

The spaces U, V, P on an interval $[t^n, t^{n+1}]$ would be approximated in the case of Q_2, Q_1 pair as

$$\begin{aligned} U_h &= \{\vec{u}_h \in [C(\Omega_h)]^2, \vec{u}_h|_T \in [Q_2(T)]^2 \quad \forall T \in \mathcal{T}_h, \vec{u}_h = \vec{0} \text{ on } \Gamma_2\}, \\ V_h &= \{\vec{v}_h \in [C(\Omega_h)]^2, \vec{v}_h|_T \in [Q_2(T)]^2 \quad \forall T \in \mathcal{T}_h, \vec{v}_h = 0 \text{ on } \Gamma_1\}, \\ P_h &= \{p_h \in L^2(\Omega_h), p_h|_T \in P_1(T) \quad \forall T \in \mathcal{T}_h\}, \end{aligned}$$

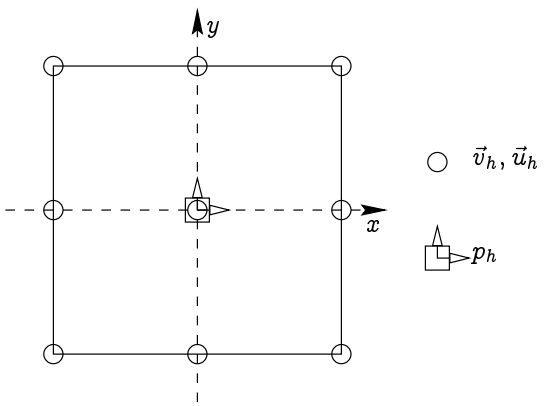

FIGURE 3. Location of the degrees of freedom for the Q_2, P_1 element

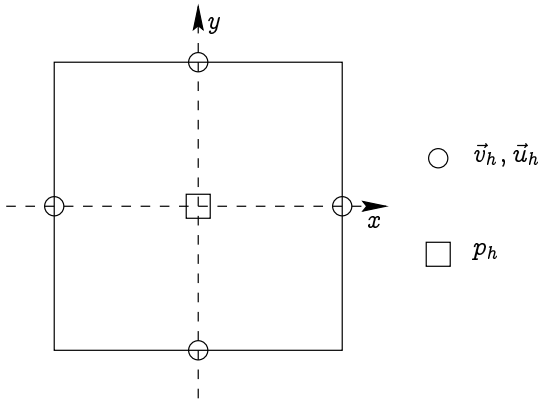

FIGURE 4. Location of the degrees of freedom for the $\tilde{Q}_1, P_0$ element

and in the case of $\tilde{Q}_1, P_0$ elements as

$$U_h = \{\vec{u}_h \in [L^2(\Omega_h)]^2, \quad \vec{u}_h|_T \in [\tilde{Q}_1(T)]^2 \quad \forall T \in \mathcal{T}_h,$$
$$\int_{T\cap S} (\vec{u}_h|_T - \vec{u}_h|_S)\, ds = \vec{0} \quad \forall T, S \in \mathcal{T}_h, \quad \int_{T\cap\Gamma_2} \vec{u}_h\, ds = \vec{0} \quad \forall T \in \mathcal{T}_h\},$$
$$V_h = \{\vec{v}_h \in [L^2(\Omega_h)]^2, \quad \vec{v}_h|_T \in [\tilde{Q}_1(T)]^2 \quad \forall T \in \mathcal{T}_h,$$
$$\int_{T\cap S} (\vec{v}_h|_T - \vec{v}_h|_S)\, ds = \vec{0} \quad \forall T, S \in \mathcal{T}_h, \quad \int_{T\cap\Gamma_1} \vec{v}_h\, ds = \vec{0} \quad \forall T \in \mathcal{T}_h\},$$
$$P_h = \{p_h \in L^2(\Omega_h), p_h|_T \in P_0(T) \quad \forall T \in \mathcal{T}_h\}.$$

An alternative variant of the $\tilde{Q}_1$ finite element space can be obtain by requiring the functions to be continuous with respect to the value at the midpoints of each edge of the grid instead of the continuity with respect to the average value over each edge.

Let us denote by $\vec{u}_h^n$ the approximation of $\vec{u}(t^n)$, $\vec{v}_h^n$ the approximation of $\vec{v}(t^n)$ and p_h^n the approximation of $p(t^n)$. Further we will use following shorthand

notation

$$\boldsymbol{F}^n = \boldsymbol{I} + \nabla \vec{u}_h^n, \quad J^n = \det \boldsymbol{F}^n \quad J^{n+\frac{1}{2}} = \frac{1}{2}(J^n + J^{n+1}),$$
$$(f,g) = \int_\Omega f \cdot g \, dV, \quad (f,g)_s = \int_{\Omega^s} f \cdot g \, dV, \quad (f,g)_f = \int_{\Omega^f} f \cdot g \, dV,$$

f, g being scalars, vectors or tensors.

Writing down the discrete equivalent of the equations (97)-(99) yields

$$(113) \quad (\vec{u}_h^{n+1}, \vec{\eta}) - \frac{k_n}{2}\left\{ (\vec{v}_h^{n+1}, \vec{\eta})_s + (\nabla \vec{u}_h^{n+1}, \nabla \vec{\eta})_f \right\} - (\vec{u}_h^n, \vec{\eta}) - \frac{k_n}{2}\left\{ (\vec{v}_h^n, \vec{\eta})_s + (\nabla \vec{u}_h^n, \nabla \vec{\eta})_f \right\} = 0,$$

$$(114) \quad \left(J^{n+\frac{1}{2}} \vec{v}_h^{n+1}, \vec{\xi}\right)_f + \beta \left(\vec{v}_h^{n+1}, \vec{\xi}\right)_s - k_n \left(J^{n+1} p_h^{n+1} (\boldsymbol{F}^{n+1})^{-T}, \nabla \vec{\xi}\right)_s + \frac{k_n}{2}\left\{ \left(\frac{\partial \Psi}{\partial \boldsymbol{F}}(\nabla \vec{u}_h^{n+1}), \nabla \vec{\xi}\right)_s + \mu \left(J^{n+1} \nabla \vec{v}_h^{n+1} (\boldsymbol{F}^{n+1})^{-1}, \nabla \vec{\xi} (\boldsymbol{F}^{n+1})^{-1}\right)_f + \left(J^{n+1} \nabla \vec{v}_h^{n+1} (\boldsymbol{F}^{n+1})^{-1} \vec{v}_h^{n+1}, \vec{\xi}\right)_f \right\} - \frac{1}{2}\left(J^{n+1} \nabla \vec{v}_h^{n+1} (\boldsymbol{F}^{n+1})^{-1} (\vec{u}_h^{n+1} - \vec{u}_h^n), \vec{\xi}\right)_f - \left(J^{n+\frac{1}{2}} \vec{v}_h^n, \vec{\xi}\right)_f - \beta \left(\vec{v}_h^n, \vec{\xi}\right)_s + \frac{k_n}{2}\left\{ \left(\frac{\partial \Psi}{\partial \boldsymbol{F}}(\nabla \vec{u}_h^n), \nabla \vec{\xi}\right)_s + \mu \left(J^n \nabla \vec{v}_h^n (\boldsymbol{F}^n)^{-1}, \nabla \vec{\xi} (\boldsymbol{F}^n)^{-1}\right)_f + \left(J^n \nabla \vec{v}_h^n (\boldsymbol{F}^n)^{-1} \vec{v}_h^n, \vec{\xi}\right)_f \right\} + \frac{1}{2}\left(J^n \nabla \vec{v}_h^n (\boldsymbol{F}^n)^{-1} (\vec{u}_h^{n+1} - \vec{u}_h^n), \vec{\xi}\right)_f = 0,$$

$$(115) \quad \left(J^{n+1} - 1, \gamma\right)_s + \left(J^{n+1} \nabla \vec{v}_h^{n+1} (\boldsymbol{F}^{n+1})^{-1}, \gamma\right)_f = 0.$$

Using the basis of the spaces U_h, V_h, P_h as the test functions $\vec{\zeta}, \vec{\xi}, \gamma$ we obtain a nonlinear algebraic set of equations. In each time step we have to find $\vec{X} = (\vec{u}_h^{n+1}, \vec{v}_h^{n+1}, p_h^{n+1}) \in U_h \times V_h \times P_h$ such that

$$(116) \quad \vec{\mathcal{R}}(\vec{X}) = \vec{0},$$

where $\vec{\mathcal{R}}$ represents the system (113–115).

4.4. Solution algorithm. The system (116) of nonlinear algebraic equations is solved using Newton method as the basic iteration. One step of the Newton iteration can be written as

$$(117) \quad \vec{X}^{n+1} = \vec{X}^n - \left[\frac{\partial \vec{\mathcal{R}}}{\partial \vec{X}}(\vec{X}^n)\right]^{-1} \vec{\mathcal{R}}(\vec{X}^n)$$

The convergence of this basic iteration can be characterized by the following standard result.

THEOREM 1. *Let $\vec{X}$ be a solution of $\vec{F}(\vec{X}) = 0$ and $\frac{\partial \vec{F}}{\partial \vec{X}}(\vec{X}^n)$ is invertible and locally Lipschitz continuous. Then, if $\vec{X}^0$ is sufficiently close to $\vec{X}$, the Newton algorithm has the following property*

$$\|\vec{X}^{n+1} - \vec{X}\| \leq c\|\vec{X}^n - \vec{X}\|^2. \tag{118}$$

We can see that this gives us quadratic convergence provided that the initial guess is sufficiently close to the solution. To ensure the convergence globally some improvements of this basic iteration are used.

The dumped Newton method with line search improves the chance of convergence by adaptively changing the length of the correction vector. The solution update step in the Newton method (117) is replaced by

$$\vec{X}^{n+1} = \vec{X}^n + \omega\delta\vec{X}, \tag{119}$$

where the parameter ω is found such that certain error measure decreases. One of the possible choices for the quantity to decrease is

$$f(\omega) = \vec{\mathcal{R}}(\vec{X}^n + \omega\delta\vec{X}) \cdot \delta\vec{X}. \tag{120}$$

Since we know

$$f(0) = \vec{\mathcal{R}}(\vec{X}^n) \cdot \delta\vec{X}, \tag{121}$$

and

$$f'(0) = \left[\frac{\partial\vec{\mathcal{R}}}{\partial\vec{X}}(\vec{X}^n)\right]\delta\vec{X} \cdot \delta\vec{X} = \vec{\mathcal{R}}(\vec{X}^n) \cdot \delta\vec{X}, \tag{122}$$

and computing $f(\omega_0)$ for $\omega_0 = -1$ or ω_0 determined adaptively from previous iterations we can approximate $f(\omega)$ by a quadratic function

$$f(\omega) = \frac{f(\omega_0) - f(0)(\omega_0 + 1)}{\omega_0^2}\omega^2 + f(0)(\omega + 1). \tag{123}$$

Then setting

$$\tilde{\omega} = \frac{f(0)\omega_0^2}{f(\omega_0) - f(0)(\omega_0 + 1)}, \tag{124}$$

the new optimal step length $\omega \in [-1, 0]$ is

$$\omega = \begin{cases} -\dfrac{\tilde{\omega}}{2} & \text{if } \dfrac{f(0)}{f(\omega_0)} > 0, \\ -\dfrac{\tilde{\omega}}{2} - \sqrt{\dfrac{\tilde{\omega}^2}{4} - \tilde{\omega}} & \text{if } \dfrac{f(0)}{f(\omega_0)} \leq 0. \end{cases} \tag{125}$$

This line search can be repeated with ω_0 taken as the last ω until, for example, $f(\omega) \leq \frac{1}{2}f(0)$. By this we can enforce a monotonous convergence of the approximation $\vec{X}^n$.

An adaptive time-step selection was found to help in the nonlinear convergence. A heuristic algorithm was used to correct the time-step length according to the convergence of the nonlinear iterations in the previous time-step. If the convergence was close to quadratic, i.e. only up to three Newton steps were needed to obtain required precision, the time step could be slightly increased, otherwise the time-step length was reduced.

1. Let $\vec{X}^n$ be some starting guess.
2. Set the residuum vector $\vec{R}^n = \vec{\mathcal{R}}(\vec{X}^n)$ and the tangent matrix $\boldsymbol{A} = \frac{\partial \vec{\mathcal{R}}}{\partial \vec{X}}(\vec{X}^n)$.
3. Solve for the correction $\delta\vec{X}$
$$\boldsymbol{A}\delta\vec{X} = \vec{R}^n.$$
4. Find optimal step length ω.
5. Update the solution $\vec{X}^{n+1} = \vec{X}^n - \omega\delta\vec{X}$.

FIGURE 5. One step of the Newton method with the line search.

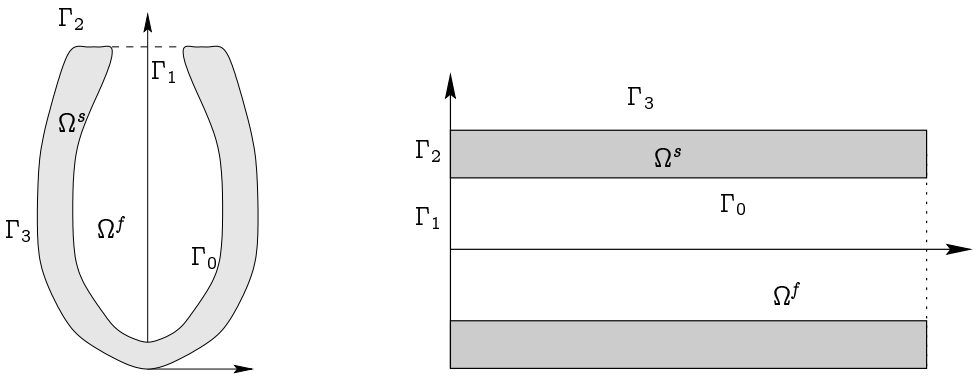

FIGURE 6. Schematic view of the ventricle and elastic tube geometries.

The structure of the Jacobian matrix $\frac{\partial \vec{\mathcal{R}}}{\partial \vec{X}}$ is

$$\frac{\partial \vec{\mathcal{R}}}{\partial \vec{X}}(\vec{X}) = \begin{pmatrix} A_{vv} & A_{vu} & B_v \\ A_{uv} & A_{uu} & B_u \\ B_v^T & B_u^T & 0 \end{pmatrix}, \tag{126}$$

and since we know the sparsity pattern of the matrix in advance it can be computed by finite differences from the residual vector $\vec{\mathcal{R}}(\vec{X})$

$$\left[\frac{\partial \vec{\mathcal{R}}}{\partial \vec{X}}\right]_{ij}(\vec{X}^n) \approx \frac{[\vec{\mathcal{R}}]_i(\vec{X}^n + \alpha_j \vec{e}_j) - [\vec{\mathcal{R}}]_i(\vec{X}^n - \alpha_j \vec{e}_j)}{2\alpha_j}, \tag{127}$$

where $\vec{e}_j$ are the unit basis vectors in $\mathbb{R}^n$ and coefficients α_j are adaptively taken according to the change in the solution in the previous time step.

The linear problems are solved by BiCGStab iterations with ILU preconditioner with allowed certain fill-in for the diagonal blocks. The algorithms used are described in Barrett et al. [1994] and the implementation was taken from Bramley and Wang [1997].

5. Applications

The numerical simulation of the cardiovascular hemodynamics has become a useful tool for deeper understanding of the onset of diseases of the human circulatory system, as for example blood cell and intima damages in stenosis, aneurysm rupture, evaluation of the new surgery techniques of heart, arteries and veins. To

obtain more information about the time course of relevant hemodynamical parameters like blood pressure and flow, residual volumes, elastic modulus etc. in the heart ventricles and atrias, the main arteries (e.g. aorta, carotid etc.) it is necessary to simulate the cardiovascular system as a whole. There has been developed a complete model of the cardiovascular system by Převorovská and Maršík [2000]. It consists of fourteen components and is elaborated with the baroreflex control and the respiratory effect. However, the components of this system are treated as a simple objects governed by the system of empirical ordinary differential equations. In order to obtain more realistic view of the the cardiovascular system behavior, detailed and precise model of key components has to be included. This concerns basically to the left and right atrias, left and right ventricles and main arteries.

In the next section we present two-dimensional computations of the flow in ellipsoidal cavity and through a channel with elastic walls. In both cases the flow is driven by changing pressure at the inflow part of the boundary while the elastic part of the boundary is either fixed or stress free.

5.1. Heart ventricle. The left heart ventricle is a approximately ellipsoidal void surrounded by the heart muscle. By a periodic contraction of the muscle, the volume of the ventricle is periodically increased and decreased. By these contraction and by the help of two valves the blood is pumped to the aorta.

In our two-dimensional computations we use an ellipsoidal cavity, see figure 6, with prescribed time-dependent pressure boundary condition at the fluid boundary part Γ^1.

$$p(t) = \sin t \quad \text{on } \Gamma^1 \tag{128}$$

The material of the solid wall is modeled by the neo-Hookean constitutive relation (84).

In figure 7 the volume change of the cavity as a function of the time and the average pressure inside the cavity versus the volume of the cavity is shown. We can see that after the initial cycle which was started from the undeformed configuration the system comes to a time periodic solution. In figure 8, the left panel shows the trajectory of a material point on the solid fluid interface and the right panel shows the corresponding velocity for the same material point.

The figure 9 shows the computational grid for the maximal and minimal volume configuration of the cavity and figure 10 shows the velocity field of the fluid for the same configurations. One of the important characteristics is the shear stress exerted by the fluid flow on the wall material. This is shown on the figure 11 for three different times. In figures 12 a series of snapshots of the pressure distribution throughout the pulse period is shown. Finally, in figure 13 a computation leading to a collapse of the solid walls near the entry is shown. Such a case leads to a breakdown in the algorithm.

5.2. Elastic tube. The second application is the simulation of a flow in an elastic tube. The flow in the tube is driven by a time-dependent pressure difference between the ends of the channel. Such flow is also interesting to investigate in the presence of some constriction as a stenosis, which is shown in figures 18.

For the flow in the tube without any constriction the time dependence of the fluid volume inside the tube is shown together with the pressure volume diagram in the figure 14. The trajectory of a material point on the solid fluid interface is show in the figure 15 on the left panel. The right panel shows the velocity of the

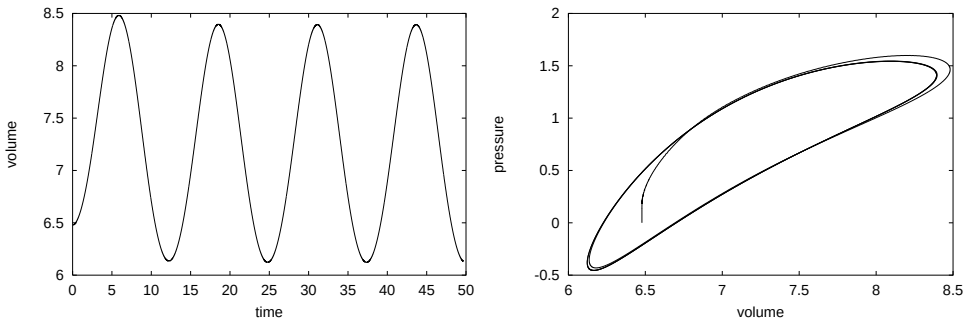

FIGURE 7. Volume of the fluid inside and the pressure-volume diagram

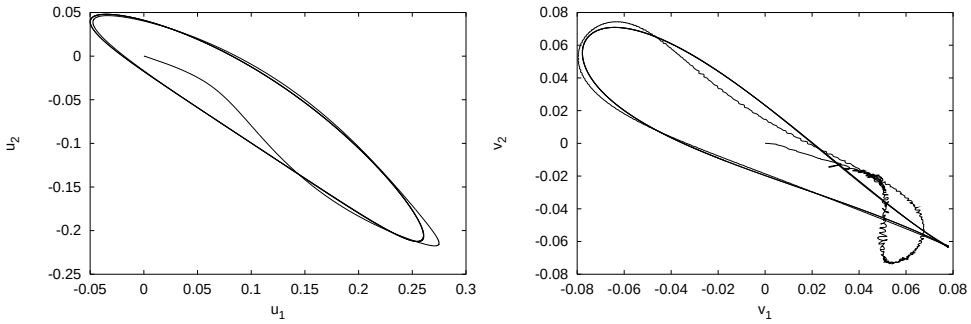

FIGURE 8. Displacement and velocity of the point at the fluid solid interface (inner side of the wall)

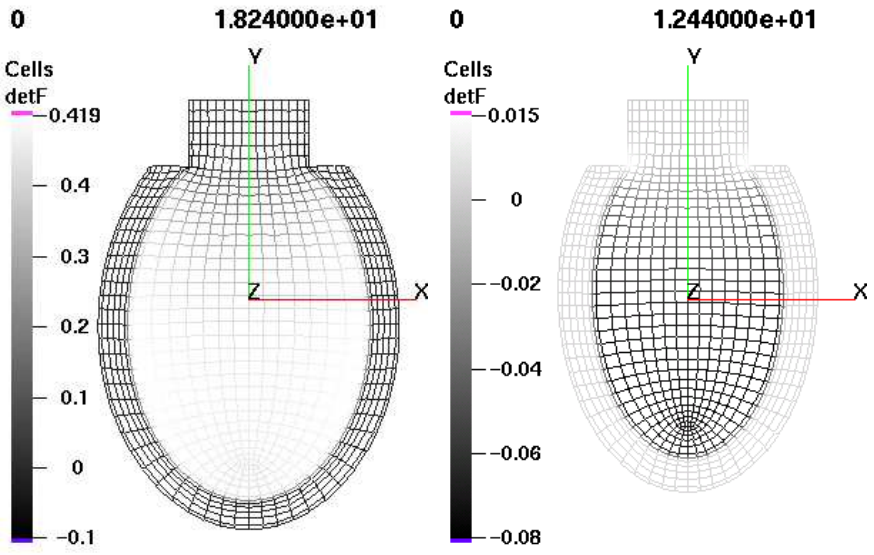

FIGURE 9. Maximum and minimum volume configuration

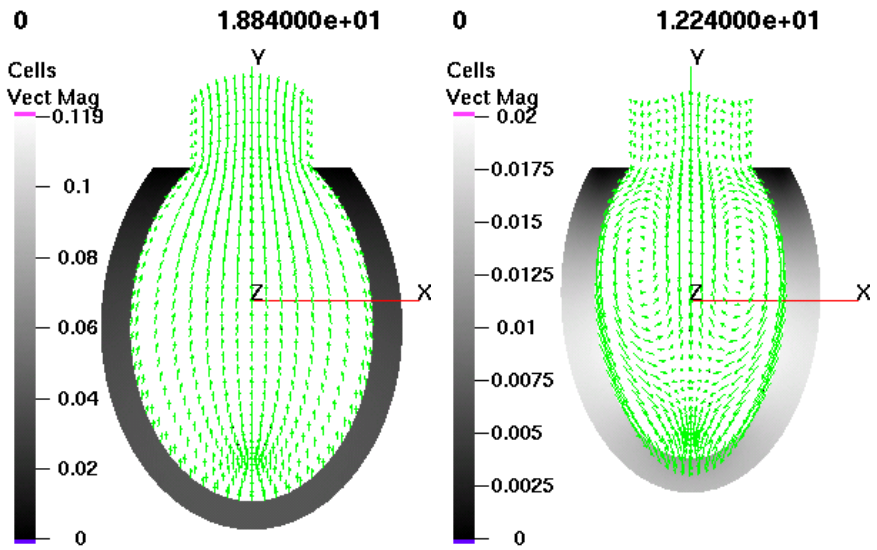

FIGURE 10. Maximum and minimum volume configuration with the fluid flow

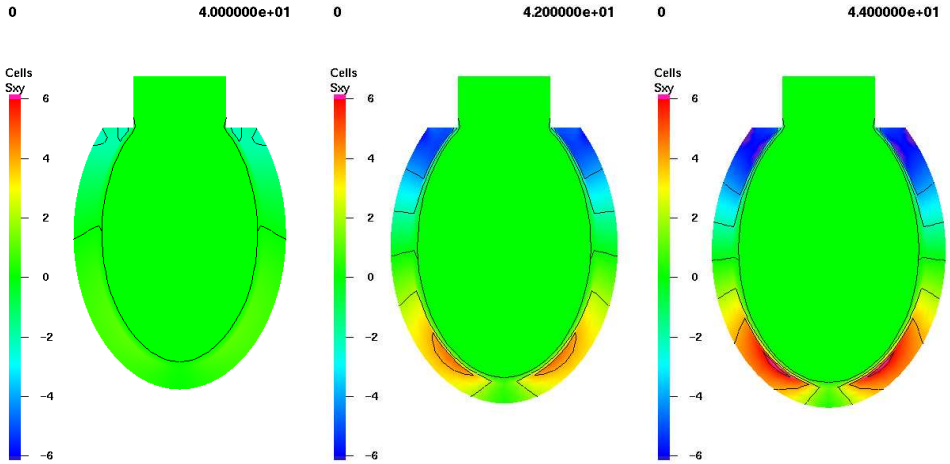

FIGURE 11. Shear stress distribution in the wall during one pulse

same material point. The velocity field is shown in figure 16 at different stages of the pulse. The figure 17 shows again a case leading to a claps of the tube at the outflow part which the present algorithm is not able to capture.

Finally in figure 18 the velocity field in the fluid and the pressure distribution throughout the wall is shown for the computation of the flow in a tube with elastic obstruction.

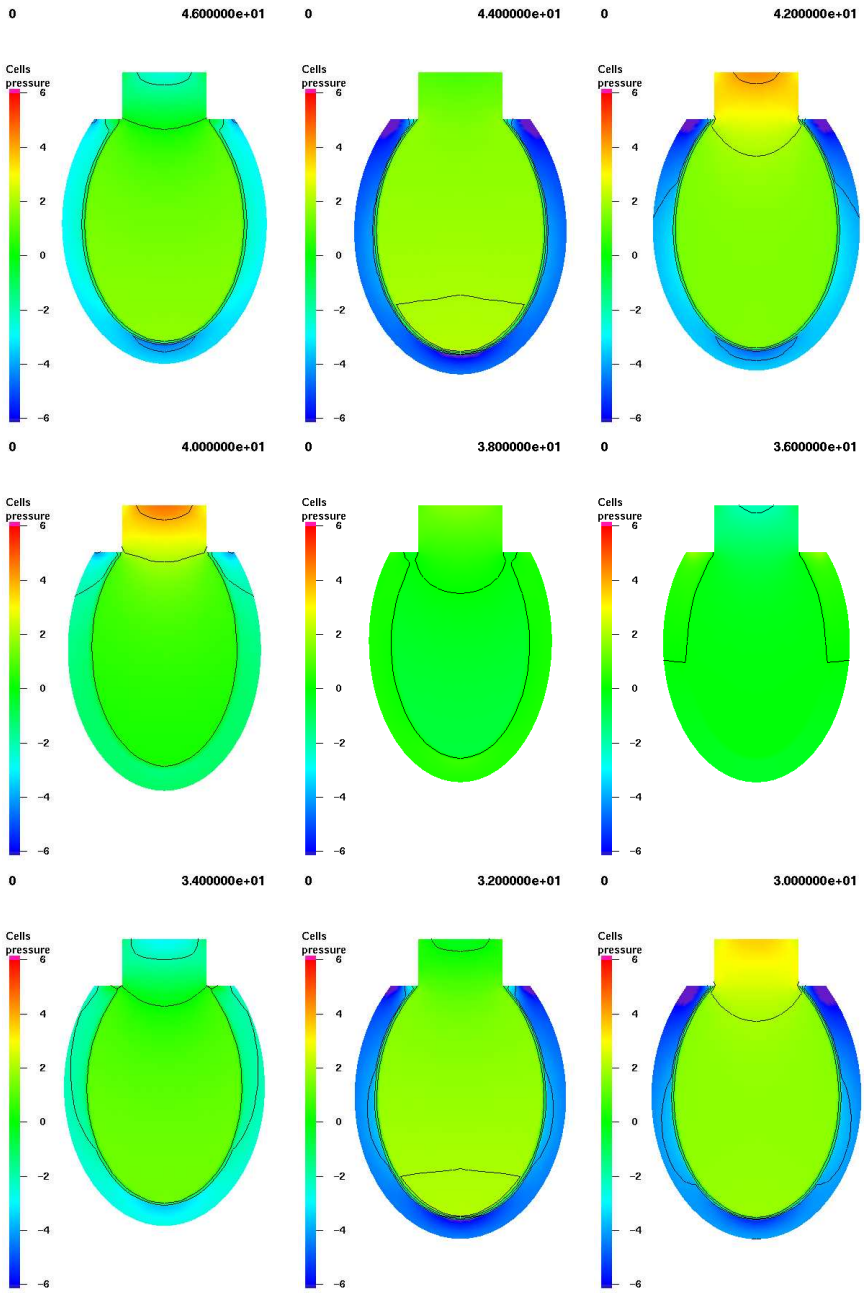

FIGURE 12. Pressure distribution in the wall during one pulse

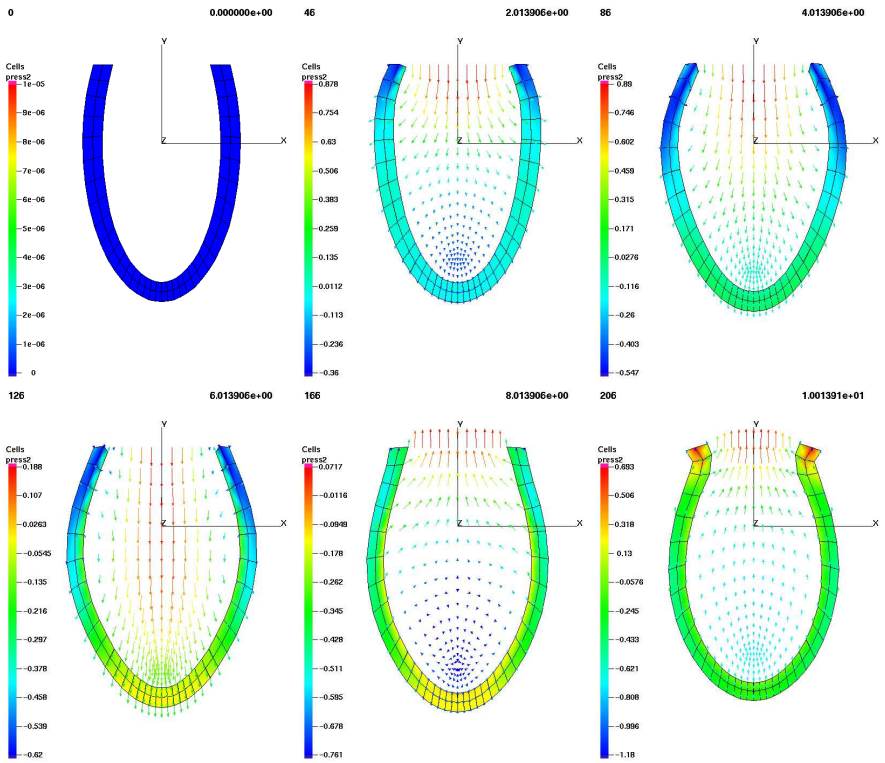

FIGURE 13. Computation leading to a collapse of the walls

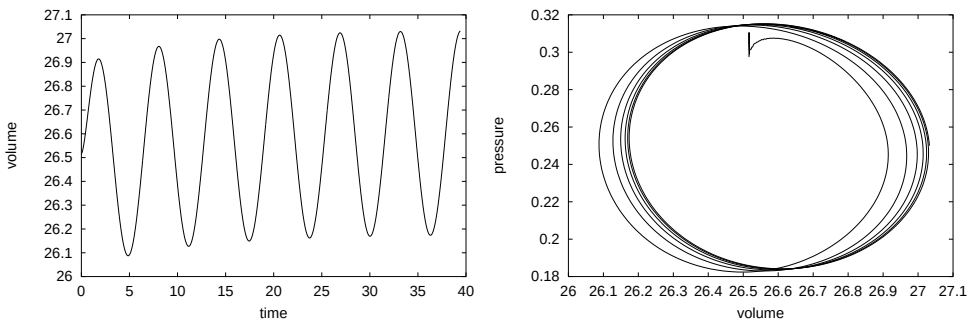

FIGURE 14. Volume of the fluid in the tube and the pressure-volume diagram

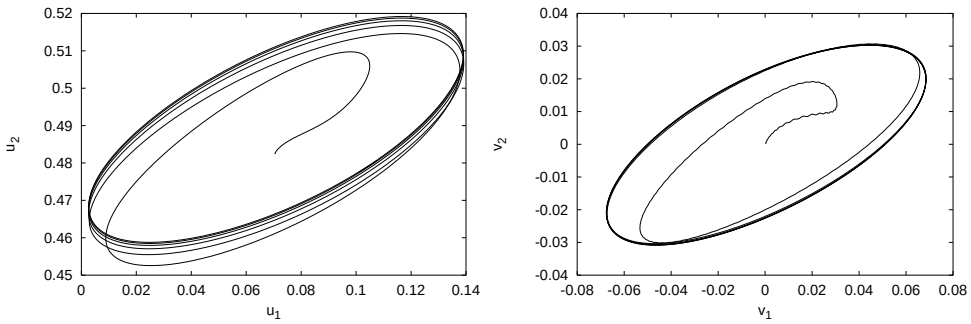

FIGURE 15. Displacement and velocity of the point at the fluid solid interface (inner side of the wall)

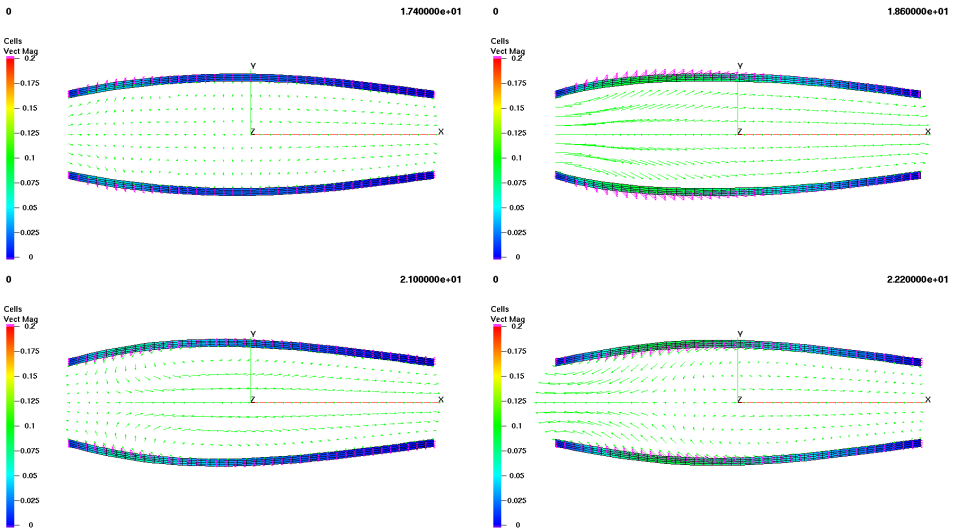

FIGURE 16. Velocity field during one pulse

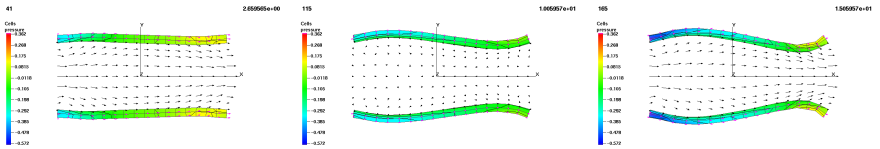

FIGURE 17. Computation leading to a collapse of the walls

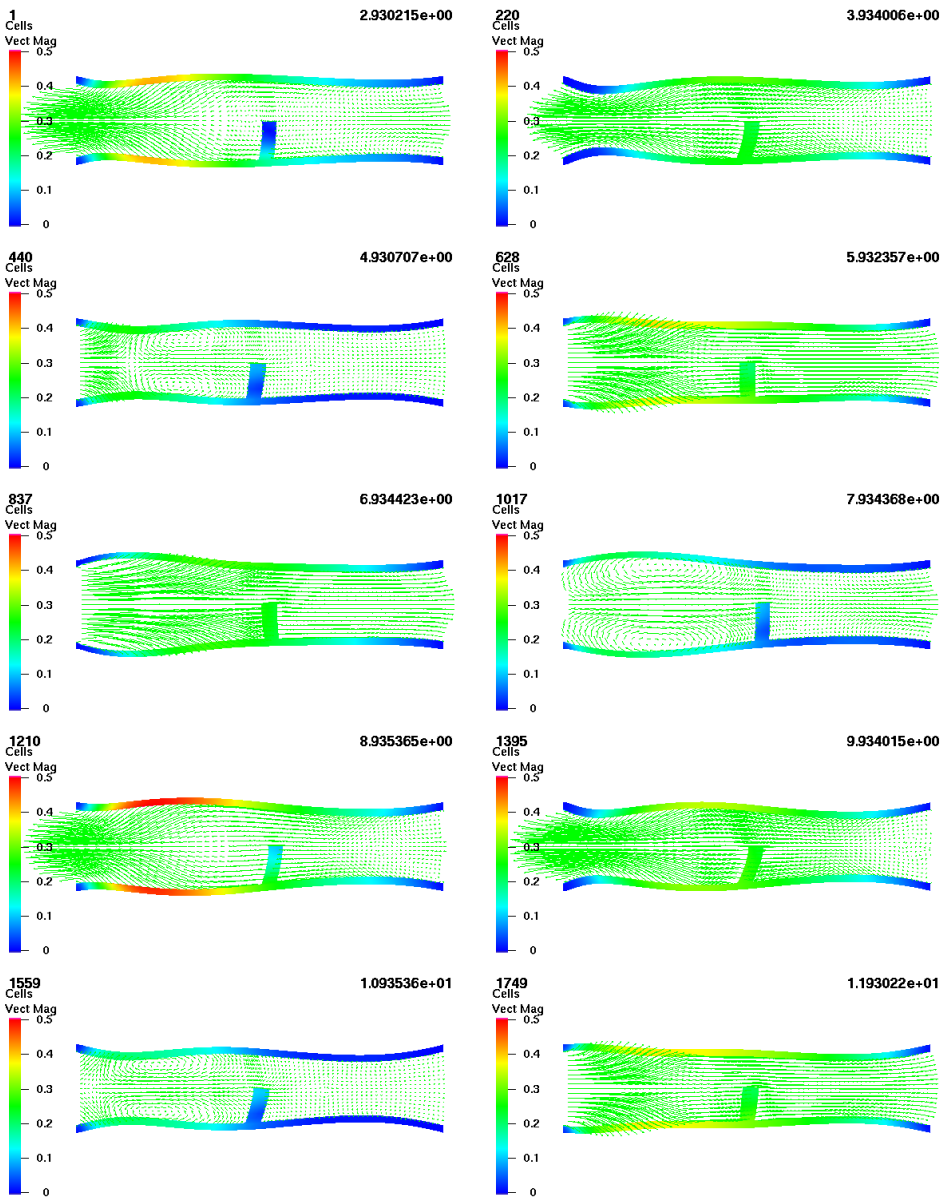

FIGURE 18. Fluid flow and pressure distribution in the wall during one pulse for the channel with constriction

CHAPTER 3

Soft tissue model

Muscle tissue consists of muscle fibers and capillary blood vessels aligned in one direction and amorphous extracellular matrix and interstitial fluid which fills the voids and is limited in its movement. The size of the capillaries is of the order of the diameter of the red blood cell and their distribution in the muscle tissue is regular. This fact allows us to use some kind of homogenization to describe the material. The uni-directional orientation of the muscle fibers also suggests that transversely isotropic symmetry is appropriate. An overview of models for soft tissues including models of active muscle contraction is presented in Fung [1993] and Maurel et al. [1998].

Since a mathematical description of such materials on a microscopical level would be too complicated, other means of description are used to describe mathematically the mechanics of such materials. One possible description is based on the mixture theory. The biphasic mixture model (see Holmes [1995], Wang and Mow [1998]), where the tissue is represented by mixture of solid and fluid components, has been successfully used to model compression of cartilage (see Spilker et al. [1988], Kwan et al. [1990]) and shear of skin (see Oomens and van Campen [1987]). In Vankan et al. [1996, 1997] a hierarchical mixture model is developed to analyze blood flow through biological tissues accounting for different levels of the arterial tree. In Miga et al. [1998], Paulsen et al. [1999] a real time prediction of soft tissue deformation during surgery is investigated using mixture theory and in Zoppou et al. [1997] mixture theory is applied to model milk expression during breast feeding.

Another important part of the description is the constitutive relations for the materials. In Humphrey et al. [1990a] and Humphrey et al. [1990b] a constitutive equation for passive myocardium is suggested and the material parameters are determined from experiments. Several simple constitutive relations for soft tissues are compared in Barry and Aldis [1990] and Holmes [1995]. In the framework of mixture theory, the constitutive relation for the interaction between the constituents has to be specified.

The main objective of this part is to develop and investigate a mathematical model which would make suitable basis for the description of the mechanical behavior of soft, hydrated tissues with significant perfusion. A typical example of interest is the case of muscle with significant blood perfusion through the capillary blood vessels. Such perfusion can significantly alter the overall mechanical response of the tissue.

The specific aims of this investigation are:

- To employ the continuum mixture theory to derive the appropriate balance relations.
- To investigate requirements on constitutive relations and appropriate boundary conditions in the context of the mixture theory.
- To solve illustrative steady, one and two dimensional problems of diffusion through a finite slab, and to develop and implement appropriate numerical schemes. The later will rely on finite element discretization for finding approximate solution to the continuous, nonlinear, steady, two dimensional problem.

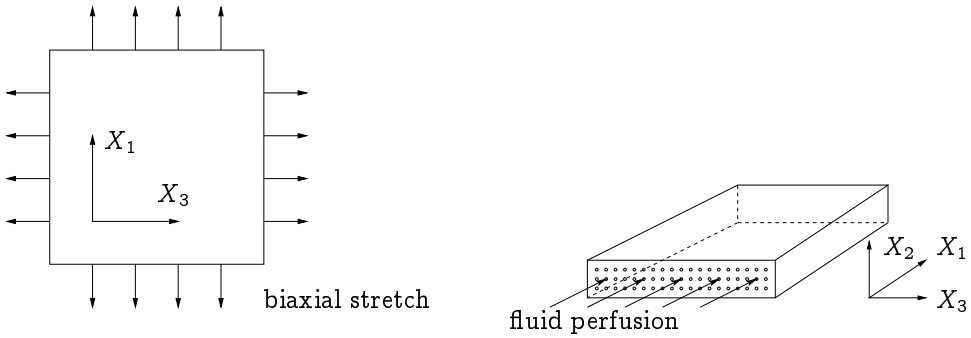

FIGURE 1. Schematic view of biaxial stretching experiment and muscle perfusion.

1. Overview

A typical experiment we would like to model is the biaxial stretching of a muscle specimen with simultaneous perfusion through the capillaries as schematically depicted in figure 1. In such a problem we consider following characteristics

- The tissue specimen in the experiment, and even in its natural environment, is subjected to large deformation.
- Muscle fibers in the specimen are oriented in one direction.
- The perfusion takes place through parallel capillaries.
- The perfusion flow can have significant velocity possibly varying in time.
- The diameter of the capillary is comparable with the diameter of the red blood cell. ($\approx 5\mu$m)
- The distribution of the capillaries is regular.

In section 2 we first introduce the basic notations of the continuum mixture theory and set up the balance laws. Then we focus on a solid-fluid mixture, formulate the balance laws for each constituent and its consequences for the mixture as a whole. Using standard thermodynamical arguments we define forms of the constitutive relations for the stresses and interactions so that the entropy inequality will be satisfied. In general, we will obtain a set of nonlinear partial differential equations governing the motion of the mixture.

In section 4.3 we investigate some applications of this description to simple problems. We restrict our considerations to isotropic binary mixture consisting of a solid and a fluid. A problem, in one spatial dimension, of a rectangular slab subjected to stretching with prescribed fluid flux is investigated in the first part. Allowing non-homogeneous deformation in one direction and prescribing homogeneous stretch in the other two directions we obtain a problem formulated as a set of nonlinear ordinary differential equations and solved numerically. The second part is devoted to a problem in two spatial dimensions. A weak formulation of the governing equations is introduced, with finite element discretization based on quadrilateral finite elements with continuous, piecewise biquadratic and bilinear shape functions. An iterative procedure based on the Newton method for solving the resulting set of nonlinear algebraic equations is used. This algorithm is applied to a problem of stretched isotropic slab with perfusion.

1.1. Continuum theory of mixtures. The problem of fluid saturated porous media was studied first by Fick [1855] and then by Darcy [1856]. These works gave

the basis for the development of mixture theory based on modern continuum mechanics in the works of Truesdell [1957, 1991] and Truesdell and Noll [1992]. A good overview of the historical development is given in Atkin and Craine [1976], Rajagopal and Tao [1995], and other important works on the development of mixture theories include Green and Naghdi [1965], Bowen [1971, 1980, 1982], Katsube and Carroll [1987], Ehlers and Kubik [1994], Green and Naghdi [1995] and Reid and Jafari [1995]. The important issue of constraints within the frame-work of mixture theory is discussed in Mills [1967], Svendsen [1994], whereas in Müller [1967] and Svendsen and Hutter [1995] thermodynamic ideas are applied to mixtures and the question of suitable boundary conditions is discussed in Rajagopal et al. [1986].

The basic idea of mixture theory is the assumption of co-occupancy, i.e., at each spatial point there is certain fraction of each constituent (with associated fields) and there are prescribed balance equations for each constituent of the mixture as is usual for a single continuum, with additional terms representing the interaction between constituents within the mixture.

Mixture theory was first applied to swelling and diffusion in rubber materials Dai and Rajagopal [1990], Rajagopal and Tao [1995], mechanics of skin Oomens and van Campen [1987], compression of cartilage Spilker et al. [1988], Kwan et al. [1990] and blood perfusion through biological tissues in Vankan et al. [1996, 1997].

1.2. Numerical solution. A numerical solution of the resulting equations of mixture theory poses great challenge since it includes the features of nonlinear elasticity and fluid mechanics coupled together and the volume additivity constraint corresponding to incompressibility constraint in a single continuum.

There have been several numerical studies of mixture models. One dimensional diffusion of fluid through an isotropic material is solved in Shi [1973], for transversely isotropic materials in Dai and Rajagopal [1990] and in Reynolds and Humphrey [1998] a one dimensional diffusion through isotropic stretched slab is solved using a velocity boundary condition.

Finite element solutions of mixture models for the small deformation, linear elastic case are presented in Kwan et al. [1990], Vankan et al. [1997] and for nonlinear large deformation description of various soft tissues in Spilker et al. [1988], Spilker and Suh [1990], Suh et al. [1991], Donzelli et al. [1992], Vermilyea and Spilker [1993], Almeida and Spilker [1998], Levenston et al. [1998].

2. Mixture theory

The continuum assumption has proven to be a very useful tool in fluid and solid mechanics. It idealizes the body of interest, consisting of individual particles (atoms or molecules), as a continuous region in space. There has to be certain conditions in order to use this approach, namely the characteristic size of the body and the disturbances imposed on the body have to be "much" greater than the size of the individual particles.

Similar ideas support the mixture theory. In the mixture, each of the constituents is treated as a continuum occupying the whole volume of the mixture. The connection with the real mixture can be established through a homogenization procedure. Clearly, the validity of such treatment can be expected only when the characteristic size of the structural constitution of the mixture (i.e. atomic or molecular level mixture, size of the particles of granular mixture, or size of the

pores in porous media) is "much" smaller than the characteristic size of the mixture body and the size of the disturbances imposed on the body.

2.1. Preliminary definitions. Let us consider a mixture of N constituents. For each constituent there is a reference configuration denoted by $\Omega^\alpha \subset \mathbb{R}^3, \alpha = 1, \dots, N$ and a sufficiently smooth, one to one mapping of the reference configuration Ω^α to the current configuration $\Omega_t^\alpha \subset \mathbb{R}^3$.

$$\vec{\chi}^\alpha : \Omega^\alpha \times [0,T] \mapsto \Omega_t^\alpha, \quad \alpha = 1, \dots, N. \tag{129}$$

The mixture occupies a region $\Omega \subseteq \bigcap_{\alpha=1}^N \Omega_t^\alpha$ in the three dimensional Euclidean space $\mathbb{R}^3$. Co-occupancy, the basic assumption of the mixture theory, requires that for any point in the mixture $\vec{x} \in \Omega$ and time $t \in [0,T]$ there exist material points $\vec{X}^\alpha \in \Omega^\alpha, \alpha = 1, \dots, N$ such that

$$\vec{x} = \vec{\chi}^\alpha(\vec{X}^\alpha, t), \quad \alpha = 1, \dots, N, \tag{130}$$

i.e. the particles of all constituents coexist at a given spatial point.

Let $\mathcal{P} \subset \Omega$ be an arbitrary part of the mixture and let $V(\mathcal{P}) = \int_{\mathcal{P}} dv$ denote the volume occupied by the part $\mathcal{P}$ in the Euclidean space $\mathbb{R}^3$ (i.e. Lebesgue measure). We assume that the mixture configuration is equipped with a mass measure $M(\mathcal{P})$ representing the mass of the part $\mathcal{P}$ and partial mass measures $M^\alpha(\mathcal{P})$ as mass of the α-th constituent in the part $\mathcal{P}$.

Further, we assume that there are measures $V^\alpha(\mathcal{P})$ associated with each constituent which represents the information about the distribution of the material α in the part $\mathcal{P}$. This information has to come from the homogenization process where it can be interpreted as the volume occupied by the α-th constituent in the volume occupied by the part $\mathcal{P}$.

These measures are assumed to be absolutely continuous with respect to the volume measure V, i.e.

$$M(\mathcal{P}) \ll V(\mathcal{P}), \tag{131}$$

$$M^\alpha(\mathcal{P}) \ll V(\mathcal{P}), \tag{132}$$

$$V^\alpha(\mathcal{P}) \ll V(\mathcal{P}), \tag{133}$$

and the partial mass measure M^α to be absolutely continuous with respect to the mixture mass and the partial volume measure V^α of the same constituent

$$M^\alpha(\mathcal{P}) \ll M(\mathcal{P}), \tag{134}$$

$$M^\alpha(\mathcal{P}) \ll V^\alpha(\mathcal{P}). \tag{135}$$

Then there exist densities corresponding to these measures by the Radon-Nikodým theorem. The mixture mass density ϱ and the constituent partial mass density ϱ^α satisfy

$$M(\mathcal{P}) = \int_{\mathcal{P}} \varrho \, dv, \tag{136}$$

$$M^\alpha(\mathcal{P}) = \int_{\mathcal{P}} \varrho^\alpha \, dv. \tag{137}$$

and the distribution density ϕ^α satisfies

$$V^\alpha(\mathcal{P}) = \int_{\mathcal{P}} \phi^\alpha \, dv. \tag{138}$$

In the mixture theory the quantity ϕ^α is usually called volume fraction.

For a non-reacting mixture the assumption that the partial mass and distribution measures are additive is adopted. We require that the mass of the mixture in the part $\mathcal{P}$ is equal to the sum of the partial masses

$$\sum_{\alpha=1}^{N} M^{\alpha}(\mathcal{P}) = M(\mathcal{P}) \Rightarrow \sum_{\alpha=1}^{N} \varrho^{\alpha} = \varrho. \tag{139}$$

For the distribution measures we require the sum of the partial distribution measures to be equal to the mixture volume

$$\sum_{\alpha=1}^{N} V^{\alpha}(\mathcal{P}) = V(\mathcal{P}) \Rightarrow \sum_{\alpha=1}^{N} \phi^{\alpha} = 1. \tag{140}$$

Finally, the assumption that the partial mass measures M^{α} are absolutely continuous with respect to the corresponding partial distribution measure V^{α}, gives rise to the densities denoted by ϱ_t^{α} and defined by

$$M^{\alpha}(\mathcal{P}) = \int_{\mathcal{P}} \varrho_t^{\alpha} \, dV^{\alpha} = \int_{\mathcal{P}} \varrho_t^{\alpha} \phi^{\alpha} \, dv, \tag{141}$$

using (137) we obtain

$$\varrho^{\alpha} = \varrho_t^{\alpha} \phi^{\alpha}. \tag{142}$$

The quantity ϱ_t^{α} is usually called the true density of the constituent.

In the mixture theory the material distribution ϕ^{α} is additional independent field which have to be determined. One way to eliminate these additional variables is to make some additional assumption or impose an additional constraint. The simplest possibility available is to assume that the true densities are constant.

Now, we can define the mechanical fields as in classical continuum mechanics. For each constituent the velocity field is defined as

$$\vec{v}^{\alpha} = \frac{\partial \vec{\chi}^{\alpha}}{\partial t}, \tag{143}$$

and the deformation gradient for each constituent with respect to its reference configuration is defined as

$$\boldsymbol{F}^{\alpha} = \frac{\partial \vec{\chi}^{\alpha}}{\partial \vec{X}}. \tag{144}$$

Let us adopt following notations for the derivatives. Any field quantity φ with values in some space Y (i.e. scalar, vector or tensor valued) can be expressed in the Eulerian description as a function of the spatial position $\vec{x} \in \Omega$

$$\varphi = \tilde{\varphi}(\vec{x}, t) : \Omega \times [0, T] \mapsto Y.$$

Then we define following notations for the derivatives of the field φ

$$\frac{\partial \varphi}{\partial t} := \frac{\partial \tilde{\varphi}}{\partial t}, \tag{145}$$

$$\operatorname{grad} \varphi = \frac{\partial \varphi}{\partial \vec{x}} := \frac{\partial \tilde{\varphi}}{\partial \vec{x}}. \tag{146}$$

In the case of Lagrangian description we have to specify which material is taken as reference. Let α be the constituent we take as a reference then for any $\vec{X} \in \Omega^{\alpha}$ we can express the quantity φ as

$$\varphi = \overset{\alpha}{\varphi}(\vec{X}, t) : \Omega^{\alpha} \times [0, T] \mapsto Y,$$

and we define the derivatives of the field φ as

$$\frac{d^\alpha \varphi}{dt} := \frac{\partial^\alpha \varphi}{\partial t}, \tag{147}$$

$$\nabla^\alpha \varphi = \frac{\partial \varphi}{\partial \vec{X}^\alpha} := \frac{\partial^\alpha \varphi}{\partial \vec{X}}. \tag{148}$$

These two descriptions can be related to each other through following relations

$$\overset{\alpha}{\varphi}(\vec{X}, t) = \tilde{\varphi}(\vec{\chi}^\alpha(\vec{X}, t), t), \tag{149}$$

$$\frac{d^\alpha \varphi}{dt} = \frac{\partial \varphi}{\partial t} + (\operatorname{grad} \varphi)\vec{v}^\alpha, \tag{150}$$

$$\nabla^\alpha \varphi = (\operatorname{grad} \varphi)\boldsymbol{F}^\alpha. \tag{151}$$

3. Balance laws of general mixture

The balance laws for each constituent are postulated in a similar way as in the case of single continuum. Let $\mathcal{P} \subset \Omega$ be a fixed region occupied by the mixture and $\vec{n}$ denotes the unit, outward normal vector to the boundary $\partial\mathcal{P}$ of the part $\mathcal{P}$.

3.1. Balance of mass. In the case of multicomponent mixture we can include a mass inter-conversion between the constituents. Let m^α denote a mass source of the α-th constituent. Then the mass balance can be stated as

$$\frac{\partial}{\partial t}\int_{\mathcal{P}} \varrho^\alpha \, dv + \int_{\partial\mathcal{P}} \varrho^\alpha \vec{v}^\alpha \cdot \vec{n} \, da = \int_{\mathcal{P}} m^\alpha \, dv. \tag{152}$$

If all the fields are sufficiently smooth this equation can be written in local form as

$$\frac{\partial \varrho^\alpha}{\partial t} + \operatorname{div} \varrho^\alpha \vec{v}^\alpha = m^\alpha. \tag{153}$$

Further requirement that the mass conversion is only between the constituents but the overall mixture mass is conserved leads to

$$\sum_{\alpha=1}^{N} m^\alpha = 0. \tag{154}$$

It is useful to derive the mass balance equation from the Lagrangian point of view. Let $\mathcal{P}^\alpha \subset \Omega^\alpha$ be a fixed set of particles belonging to the material α. Then $\chi^\alpha(\mathcal{P}^\alpha) \subset \Omega$ is a region occupied by these particles at current time, and the balance of mass can be expressed as

$$\frac{d^\alpha}{dt}\int_{\chi^\alpha(\mathcal{P}^\alpha)} \varrho^\alpha \, dv = \int_{\chi^\alpha(\mathcal{P}^\alpha)} m^\alpha \, dv, \tag{155}$$

which in local form can be written as

$$\frac{d^\alpha}{dt}(\varrho^\alpha \det \boldsymbol{F}^\alpha) = m^\alpha \det \boldsymbol{F}^\alpha. \tag{156}$$

3.2. Balance of linear momentum. Let $\boldsymbol{\sigma}^\alpha$ denote the Cauchy stress tensor associated with the α-th constituent, representing the surface forces per mixture unit area, $\vec{f}^\alpha$ be the part of the body force acting on the α-th constituent per its unit mass, and $\vec{b}^\alpha$ denote the momentum source of the α-th component from the rest of the mixture. The balance of linear momentum is stated as

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathcal{P}} \varrho^\alpha \vec{v}^\alpha \, dv + \int_{\partial\mathcal{P}} \varrho^\alpha \vec{v}^\alpha \otimes \vec{v}^\alpha \vec{n} \, da = \int_{\partial\mathcal{P}} \left(\boldsymbol{\sigma}^\alpha\right)^T \vec{n} \, da \\ + \int_{\mathcal{P}} \left(\varrho^\alpha \vec{f}^\alpha + \vec{b}^\alpha\right) \, dv.\end{aligned} \tag{157}$$

The local form of the linear momentum balance is

$$\frac{\partial \varrho^\alpha \vec{v}^\alpha}{\partial t} + \operatorname{div} \varrho^\alpha \vec{v}^\alpha \otimes \vec{v}^\alpha = \operatorname{div} \left(\boldsymbol{\sigma}^\alpha\right)^T + \varrho^\alpha \vec{f}^\alpha + \vec{b}^\alpha. \tag{158}$$

For the whole mixture the action-reaction principle implies

$$\sum_{\alpha=1}^{N} \vec{b}^\alpha = 0. \tag{159}$$

3.3. Balance of angular momentum. We assume that the material is non-polar, i.e. there are no external sources of angular momentum. Let $\vec{p}^\alpha$ denote the angular momentum interaction between the α-th component and the rest of the mixture. The balance of angular momentum is

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathcal{P}} \vec{x} \times \varrho^\alpha \vec{v}^\alpha \, dv + \int_{\partial\mathcal{P}} \vec{x} \times \varrho^\alpha \vec{v}^\alpha \otimes \vec{v}^\alpha \vec{n} \, da = \int_{\partial\mathcal{P}} \vec{x} \times \left(\boldsymbol{\sigma}^\alpha\right)^T \vec{n} \, da \\ + \int_{\mathcal{P}} \vec{x} \times \varrho^\alpha \vec{f}^\alpha \, dv \\ + \int_{\mathcal{P}} \vec{p}^\alpha \, dv.\end{aligned} \tag{160}$$

Again expressed in local form it can be written as

$$\frac{\partial \vec{x} \times \varrho^\alpha \vec{v}^\alpha}{\partial t} + \operatorname{div} \left(\vec{x} \times \varrho^\alpha \vec{v}^\alpha \otimes \vec{v}^\alpha\right) = \operatorname{div} \left(\vec{x} \times \left(\boldsymbol{\sigma}^\alpha\right)^T\right) + \vec{x} \times \varrho^\alpha \vec{f}^\alpha + \vec{p}^\alpha. \tag{161}$$

For the mixture as a whole the interactions are assumed to be balanced, which implies

$$\sum_{\alpha=1}^{N} \vec{p}^\alpha = 0. \tag{162}$$

This also implies that the total stress of the mixture, defined as $\boldsymbol{\sigma} = \sum_{\alpha=1}^{N} \boldsymbol{\sigma}^\alpha$, is symmetric

$$\boldsymbol{\sigma} = \left(\boldsymbol{\sigma}\right)^T. \tag{163}$$

If the internal angular momentum interactions $\vec{p}^\alpha$ are assumed to be caused by the momentum interactions, i.e.

$$\vec{p}^\alpha = \vec{x} \times \vec{b}^\alpha, \tag{164}$$

then the partial stresses are also symmetric

$$\boldsymbol{\sigma}^\alpha = \left(\boldsymbol{\sigma}^\alpha\right)^T. \tag{165}$$

3.4. Balance of energy. Let e^α be the internal energy associated with the α-th constituent per its unit mass, $\vec{q}^\alpha$ be the heat flux per unit area of the constituent, r^α be the heat source per unit mass of the constituent and ε^α be the energy interaction among the constituent and the rest of the mixture. Denoting the total constituent energy per its mass as $E^\alpha = e^\alpha + \frac{1}{2}|\vec{v}^\alpha|^2$, the balance of energy is introduced as

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathcal{P}} \varrho^\alpha E^\alpha \, dv + \int_{\partial\mathcal{P}} \varrho^\alpha E^\alpha \vec{v}^\alpha \cdot \vec{n}\, da = &\int_{\partial\mathcal{P}} \left(\left(\boldsymbol{\sigma}^\alpha\right)^T \vec{v}^\alpha - \vec{q}^\alpha \right) \cdot \vec{n}\, da \\ &+ \int_{\mathcal{P}} \left(\varrho^\alpha \vec{f}^\alpha \cdot \vec{v}^\alpha + \varrho^\alpha r^\alpha \right) dv \\ &+ \int_{\mathcal{P}} \varepsilon^\alpha \, dv.\end{aligned} \tag{166}$$

Writing this in local form and using the mass and momentum balance equations we obtain

$$\begin{aligned}\frac{\partial \varrho^\alpha e^\alpha}{\partial t} + \operatorname{div} \varrho^\alpha e^\alpha \vec{v}^\alpha = &\operatorname{tr}\left(\left(\boldsymbol{\sigma}^\alpha\right)^T \operatorname{grad} \vec{v}^\alpha \right) - \operatorname{div} \vec{q}^\alpha + \varrho^\alpha r^\alpha \\ &- \vec{b}^\alpha \cdot \vec{v}^\alpha + \frac{1}{2} m^\alpha |\vec{v}^\alpha|^2 + \varepsilon^\alpha.\end{aligned} \tag{167}$$

Again, we require the energy interaction between the constituents to be in balance, i.e.

$$\sum_{\alpha=1}^{N} \varepsilon^\alpha = 0. \tag{168}$$

3.5. Balance of entropy. Finally, the balance of entropy is postulated. Let η^α be the constituent entropy per its unit mass, T^α be a temperature assigned to each constituent and s^α be the entropy exchange among the constituent and the rest of the mixture, then the entropy balance is

$$\frac{\partial}{\partial t}\int_{\mathcal{P}} \varrho^\alpha \eta^\alpha \, dv + \int_{\partial\mathcal{P}} \varrho^\alpha \eta^\alpha \vec{v}^\alpha \cdot \vec{n}\, da = -\int_{\partial\mathcal{P}} \frac{\vec{q}^\alpha}{T^\alpha} \cdot \vec{n}\, da + \int_{\mathcal{P}} \left(\frac{\varrho^\alpha r^\alpha}{T^\alpha} + s^\alpha \right) dv, \tag{169}$$

or in local form

$$\frac{\partial \varrho^\alpha \eta^\alpha}{\partial t} + \operatorname{div} \varrho^\alpha \eta^\alpha \vec{v}^\alpha = -\operatorname{div} \frac{\vec{q}^\alpha}{T^\alpha} + \frac{\varrho^\alpha r^\alpha}{T^\alpha} + s^\alpha. \tag{170}$$

Summing these equations together over all constituents we obtain

$$\frac{\partial \sum_\alpha \varrho^\alpha \eta^\alpha}{\partial t} + \operatorname{div} \sum_\alpha \varrho^\alpha \eta^\alpha \vec{v}^\alpha = \operatorname{div} \sum_\alpha \frac{\vec{q}^\alpha}{T^\alpha} + \sum_\alpha \frac{\varrho^\alpha r^\alpha}{T^\alpha} + \sum_\alpha s^\alpha. \tag{171}$$

The second law of thermodynamics says that the total entropy production is non-negative, implying

$$\sum_{\alpha=1}^{N} s^\alpha \geq 0. \tag{172}$$

In the rest of this work we assume that all the processes are isothermal, all the constituents are at the same temperature T and there is no mass conversion between the constituents, i.e. m^α is identically zero for all constituents. Introducing the

Helmholtz potential $\Psi^\alpha = e^\alpha - T\eta^\alpha$ and using (167), (171) and (172) we obtain the reduced entropy inequality in the following form

$$\frac{\partial \sum_\alpha \varrho^\alpha \Psi^\alpha}{\partial t} + \operatorname{div} \sum_\alpha \varrho^\alpha \Psi^\alpha \vec{v}^\alpha - \operatorname{tr}\left((\boldsymbol{\sigma}^\alpha)^T \operatorname{grad} \vec{v}^\alpha \right) + \vec{b}^\alpha \cdot \vec{v}^\alpha \leq 0. \tag{173}$$

This inequality is later used to narrow the class of constitutive relations for the stresses $\boldsymbol{\sigma}^\alpha$ and momentum interactions $\vec{b}^\alpha$.

4. Binary mixture of a solid and a fluid

Let us now focus on the steady state diffusion of one fluid through an incompressible homogeneous elastic solid subjected to a finite deformation, without external body forces $\vec{f}^\alpha = \vec{0}$. The motion of each constituent of the mixture is described by mappings

$$\vec{\chi}^s : \Omega^s \times [0,T] \mapsto \Omega \tag{174}$$

$$\vec{\chi}^f : \Omega^f \times [0,T] \mapsto \Omega \tag{175}$$

where the superscripts s and f refer to quantities associated with the solid and fluid, respectively. At each time $t \in [0,T]$, the mappings are one-to-one and sufficiently smooth to render the various mathematical operations meaningful.

4.1. Balance equations. Following the definitions from the previous section we denote by ϱ^s and ϱ^f the partial densities of the solid and fluid, respectively. The total density of the mixture is given by $\varrho = \varrho^s + \varrho^f$. The deformation gradient for the solid is given by $\boldsymbol{F} = \frac{\partial \vec{x}^s}{\partial \vec{X}^s}$, and the velocity vector of the solid $\vec{v}^s = \frac{\partial \vec{x}^s}{\partial t}$. In the absence of inter conversion between the constituents, the equation for the conservation of mass for the solid is

$$\frac{\partial \varrho^s}{\partial t} + \operatorname{div}(\varrho^s \vec{v}^s) = 0. \tag{176}$$

The conservation of mass (176) can also be expressed in the Lagrangian form

$$\varrho^s \det \boldsymbol{F} = \varrho_0^s, \tag{177}$$

where ϱ_0^s is the density of the solid in the undeformed state. Let $\boldsymbol{\sigma}^s$ and $\boldsymbol{\sigma}^f$ denote the partial stresses associated with the solid and fluid constituents, respectively, and $\vec{b}^s$ and $\vec{b}^f$ denote the interactive forces acting on the solid and fluid. The balance of linear momentum for the solid is given by

$$\frac{\partial(\varrho^s \vec{v}^s)}{\partial t} + \operatorname{div}(\varrho^s \vec{v}^s \otimes \vec{v}^s) = \operatorname{div} \boldsymbol{\sigma}^s + \vec{b}^s. \tag{178}$$

For the fluid constituent we define the velocity $\vec{v}^f = \frac{\partial \vec{x}^f}{\partial t}$ and the conservation of mass and the balance of linear momentum for this constituent are

$$\frac{\partial \varrho^f}{\partial t} + \operatorname{div}(\varrho^f \vec{v}^f) = 0, \tag{179}$$

$$\frac{\partial(\varrho^f \vec{v}^f)}{\partial t} + \operatorname{div}(\varrho^f \vec{v}^f \otimes \vec{v}^f) = \operatorname{div} \boldsymbol{\sigma}^f + \vec{b}^f. \tag{180}$$

By the action-reaction principle we have $\vec{b} = \vec{b}^f = -\vec{b}^s$. For the mixture, we define the total stress $\boldsymbol{\sigma} = \boldsymbol{\sigma}^s + \boldsymbol{\sigma}^f$ which is symmetric as required by the balance of the angular momentum.

The entropy inequality for the mixture is used to restrict the class of constitutive forms for $\boldsymbol{\sigma}^s$, $\boldsymbol{\sigma}^f$ and $\vec{b}$. The reduced entropy inequality (173) for the binary mixture can be written as

$$\begin{aligned}\frac{\partial(\varrho\Psi)}{\partial t} &+ \operatorname{div}(\varrho^s\Psi^s\vec{v}^s + \varrho^f\Psi^f\vec{v}^f) \\ &- \operatorname{tr}\left(\boldsymbol{\sigma}^s \operatorname{grad}\vec{v}^s + \boldsymbol{\sigma}^f \operatorname{grad}\vec{v}^f\right) + \vec{b}\cdot(\vec{v}^s - \vec{v}^f) \leq 0,\end{aligned} \tag{181}$$

where $\varrho\Psi = \varrho^s\Psi^s + \varrho^f\Psi^f$. For a detailed discussion of the above derivations and related issues we refer to Green and Naghdi [1965], Müller [1967] and Rajagopal and Tao [1995].

4.2. Kinematical constraints. The distribution of each constituent is described by the quantities ϕ^s and ϕ^f and can be interpreted as the volumetric amount of the constituent per unit volume of the mixture. The volume additivity constraint for binary mixture

$$\phi^s + \phi^f = 1, \tag{182}$$

can be interpreted as requirement that the mixture consists only of these two components, not allowing voids. The true density of each constituent, denoted by ϱ^s_t and ϱ^f_t satisfy

$$\varrho^s = \phi^s\varrho^s_t, \tag{183}$$

$$\varrho^f = \phi^f\varrho^f_t. \tag{184}$$

So far the quantities $\varrho^s, \varrho^f, \phi^s, \phi^f$ are independent unknown fields and we have to add some more additional assumptions to be able to determine them.

The approach adopted in this work assumes that the true densities of both constituents are constant, and effectively eliminates for example the densities ϱ^s, ϱ^f by (183) and (184), leaving us with the volume fractions ϕ^s, ϕ^f as the unknowns. Then the volume additivity requirement (182) has to be considered as an additional kinematical constraint. It implies together with equations (176), (179) and (182) that

$$\operatorname{div}(\phi^s\vec{v}^s + \phi^f\vec{v}^f) = 0. \tag{185}$$

This constraint, written in integral form as

$$\int_{\partial\mathcal{P}} (\phi^s\vec{v}^s\cdot\vec{n} + \phi^f\vec{v}^f\cdot\vec{n})\,da = 0, \tag{186}$$

is equivalent to the requirement that the mixture as a whole preserves its volume or that the mixture can undergo only isochoric deformations in the sense that at any time instant the net volumetric flux through the boundary $\partial\mathcal{P}$ of the mixture part $\mathcal{P}$ is zero. Such mixture can be called an incompressible mixture. This constraint will be included into the formulation with corresponding Lagrange multiplier.

Other possible approach is to keep the quantities $\varrho^s, \varrho^f, \phi^s, \phi^f$ as independent and assume the two additional equations (182) and (185). In the case of binary mixture this seems to give sufficient number of equations. The resulting model would be volume preserving mixture of compressible components.

4.3. Constitutive equations. We will restrict our considerations to isothermal, non-reacting mixtures where the solid constituent is homogeneous and hyperelastic. For such mixtures it can be shown, using the second law of thermodynamics and the postulate of material frame indifference, that the Helmholtz potential can be expressed as a function of the deformation tensor $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ and the fluid volume fraction ϕ^f, i.e.,

$$\Psi = \hat{\Psi}(\boldsymbol{C}, \phi^f). \tag{187}$$

Using techniques that are standard in continuum mechanics (see Rajagopal and Tao [1995]) we obtain an expression for the partial stresses $\boldsymbol{\sigma}^s$ and $\boldsymbol{\sigma}^f$ and the interaction force $\vec{b}$ such that the entropy inequality is satisfied for all admissible processes. Using equation (181), the assumption that the temperature T is constant and incorporating the volume additivity constraint (185) by the method of Lagrange multiplier we obtain

$$\begin{aligned}\frac{\partial(\varrho\Psi)}{\partial t} + \operatorname{div}(\varrho^s\Psi^s\vec{v}^s + \varrho^f\Psi^f\vec{v}^f) - \operatorname{tr}\left(\boldsymbol{\sigma}^s \operatorname{grad}\vec{v}^s + \boldsymbol{\sigma}^f \operatorname{grad}\vec{v}^f\right)\\ + \vec{b}\cdot(\vec{v}^s - \vec{v}^f) + p\operatorname{div}(\phi^s\vec{v}^s + \phi^f\vec{v}^f) \leq 0,\end{aligned} \tag{188}$$

where the Lagrange multiplier is denoted by p. Inserting the function $\hat{\Psi}(\boldsymbol{C}, \phi^f)$ for Ψ and using the chain rule and the balance laws (177), (178), (179), (180) and (182) we obtain

$$\begin{aligned}&\operatorname{tr}\Bigg(\Big(\boldsymbol{\sigma}^s - \frac{\varrho^f\varrho^s}{\varrho}(\hat{\Psi}^s - \hat{\Psi}^f)\boldsymbol{I} + p\phi^s\boldsymbol{I} - 2\varrho\boldsymbol{F}\frac{\partial\hat{\Psi}}{\partial\boldsymbol{C}}\boldsymbol{F}^T\Big)\nabla\vec{v}^s\\ &\qquad + \Big(\boldsymbol{\sigma}^f + \frac{\varrho^f\varrho^s}{\varrho}(\hat{\Psi}^s - \hat{\Psi}^f)\boldsymbol{I} + p\phi^f\boldsymbol{I} + \varrho\phi^f\frac{\partial\hat{\Psi}}{\partial\phi^f}\boldsymbol{I}\Big)\nabla\vec{v}^f\Bigg)\\ &-\left(\vec{b} + \nabla\Big(\frac{\varrho^f\varrho^s}{\varrho}(\hat{\Psi}^s - \hat{\Psi}^f)\Big) + p\nabla\phi^s - \varrho\frac{\partial\hat{\Psi}}{\partial\phi^f}\nabla\phi^f + \varrho^f\nabla\hat{\Psi}\right)\cdot(\vec{v}^s - \vec{v}^f) \geq 0.\end{aligned} \tag{189}$$

Now if we choose the partial stresses to be given by

$$\boldsymbol{\sigma}^s = \frac{\varrho^s\varrho^f}{\varrho}(\hat{\Psi}^s - \hat{\Psi}^f)\boldsymbol{I} - \phi^s p\boldsymbol{I} + 2\varrho\boldsymbol{F}\frac{\partial\hat{\Psi}}{\partial\boldsymbol{C}}\boldsymbol{F}^T + \hat{\boldsymbol{\sigma}}^s \tag{190}$$

$$\boldsymbol{\sigma}^f = -\frac{\varrho^s\varrho^f}{\varrho}(\hat{\Psi}^s - \hat{\Psi}^f)\boldsymbol{I} - \phi^f p\boldsymbol{I} - \varrho\phi^f\frac{\partial\hat{\Psi}}{\partial\phi^f}\boldsymbol{I} + \hat{\boldsymbol{\sigma}}^f \tag{191}$$

and the interaction force to have the form

$$\vec{b} = -\nabla\left(\frac{\varrho^s\varrho^f}{\varrho}(\hat{\Psi}^s - \hat{\Psi}^f)\right) - p\nabla\phi^s + \varrho\frac{\partial\hat{\Psi}}{\partial\phi^f}\nabla\phi^f - \varrho^f\nabla\hat{\Psi} + \hat{\vec{b}}, \tag{192}$$

where $\hat{\boldsymbol{\sigma}}^f$, $\hat{\boldsymbol{\sigma}}^s$, $\hat{\vec{b}}$ are the parts of the partial stresses that depend only on the dynamical variables, i.e. $\vec{v}^s - \vec{v}^f$, $\nabla\vec{v}^s$, $\nabla\vec{v}^f$, then the entropy inequality reduces to

$$\operatorname{tr}\left(\hat{\boldsymbol{\sigma}}^s\boldsymbol{D}^s + \hat{\boldsymbol{\sigma}}^f\boldsymbol{D}^f + \hat{\boldsymbol{\sigma}}^s(\boldsymbol{W}^s - \boldsymbol{W}^f)\right) - \hat{\vec{b}}\cdot(\vec{v}^s - \vec{v}^f) \geq 0, \tag{193}$$

where $\boldsymbol{D}^s$, $\boldsymbol{W}^s$, $\boldsymbol{D}^f$, $\boldsymbol{W}^f$ denote the symmetric and skew-symmetric part of the velocity gradients.

Note that the terms containing the partial Helmholtz potentials $\hat{\Psi}^s$ and $\hat{\Psi}^f$ in the equations (190), (191) and (192) will cancel in the equations of motion and even in the expression for total stress $\boldsymbol{\sigma}$.

For example, assuming $\hat{\boldsymbol{\sigma}}^f$, $\hat{\boldsymbol{\sigma}}^s$, $\vec{\hat{b}}$ to be linear functions of their arguments we have (see. Dai and Rajagopal [1990])

$$\hat{\boldsymbol{\sigma}}^s = \gamma_{11} \operatorname{tr} \boldsymbol{D}^s \boldsymbol{I} + 2\mu_{11} \boldsymbol{D}^s + \gamma_{12} \operatorname{tr} \boldsymbol{D}^f \boldsymbol{I} + 2\mu_{12} \boldsymbol{D}^f - c(\boldsymbol{W}^s - \boldsymbol{W}^f) \tag{194}$$

$$\hat{\boldsymbol{\sigma}}^f = \gamma_{21} \operatorname{tr} \boldsymbol{D}^s \boldsymbol{I} + 2\mu_{21} \boldsymbol{D}^s + \gamma_{22} \operatorname{tr} \boldsymbol{D}^f \boldsymbol{I} + 2\mu_{22} \boldsymbol{D}^f + c(\boldsymbol{W}^s - \boldsymbol{W}^f) \tag{195}$$

$$\vec{\hat{b}} = \phi^s \phi^f \boldsymbol{\alpha}(\vec{v}^s - \vec{v}^f) \tag{196}$$

where γ and μ are restricted by equation (193) to be

$$\mu_{11} \geq 0, \qquad \mu_{22} \geq 0, \tag{197}$$

$$\gamma_{11} + \frac{2}{3}\mu_{11} \geq 0, \qquad \gamma_{22} + \frac{2}{3}\mu_{22} \geq 0, \tag{198}$$

$$4\mu_{11}\mu_{22} \geq (\mu_{12} + \mu_{21})^2, \tag{199}$$

$$4(\gamma_{11} + \frac{2}{3}\mu_{11})(\gamma_{22} + \frac{2}{3}\mu_{22}) \geq (\gamma_{12} + \frac{2}{3}\mu_{12} + \gamma_{21} + \frac{2}{3}\mu_{21})^2, \tag{200}$$

$$c \geq 0, \tag{201}$$

and $\boldsymbol{\alpha}$ is positive-definite tensor, in general (see Shi [1973] for the derivation of these results).

For isotropic materials the dependence of $\hat{\Psi}$ on $\boldsymbol{C}$ is only through the following three invariants

$$I_C = \operatorname{tr} \boldsymbol{C}, \qquad II_C = \frac{1}{2}(\operatorname{tr}^2 \boldsymbol{C} - \operatorname{tr} \boldsymbol{C}^2), \qquad III_C = \det \boldsymbol{C}. \tag{202}$$

Combining (182), (177), (183), (184) and using the definitions of III_C and $\boldsymbol{C}$, we can see that the fluid volume fraction ϕ^f can be expressed as a function of the third invariant III_C, i.e.,

$$\phi^f = 1 - \phi^s = 1 - \frac{\phi_0^s}{\sqrt{III_C}}. \tag{203}$$

This allows us to eliminate either ϕ^f or III_C in the constitutive relation for Ψ in favor of the other or ϕ^s. On the other hand, since we incorporated the constraint (182) in our formulation via a Lagrange multiplier, we can treat the quantities ϕ^f and ϕ^s as independent and the equation (182) is considered as an additional equation to be added to the balance equations.

For simplicity, let us set

$$\hat{\boldsymbol{\sigma}}^s = 0, \qquad \hat{\boldsymbol{\sigma}}^f = 0, \qquad \vec{\hat{b}} = \phi^s \phi^f \boldsymbol{\alpha}(\vec{v}^s - \vec{v}^f)$$

where $\boldsymbol{\alpha}$ is positive definite tensor, and denote

$$\bar{\boldsymbol{\sigma}}^s = \boldsymbol{\sigma}^s - \frac{\varrho^s \varrho^f}{\varrho}(\Psi^s - \Psi^f)\boldsymbol{I}, \tag{204}$$

$$\bar{\boldsymbol{\sigma}}^f = \boldsymbol{\sigma}^f + \frac{\varrho^s \varrho^f}{\varrho}(\Psi^s - \Psi^f)\boldsymbol{I}, \tag{205}$$

$$\bar{\vec{b}} = \vec{b} + \nabla\left(\frac{\varrho^s \varrho^f}{\varrho}(\Psi^s - \Psi^f)\right). \tag{206}$$

We shall assume that $\Psi = \bar{\Psi}(I_C, II_C, \phi^s, \phi^f)$. Then equations (190) and (191) yield

$$\bar{\boldsymbol{\sigma}}^s = -\phi^s(p + \varrho\frac{\partial\bar{\Psi}}{\partial\phi^s})\boldsymbol{I} + 2\varrho\left[\frac{\partial\bar{\Psi}}{\partial I_C}\boldsymbol{B} + \frac{\partial\bar{\Psi}}{\partial II_C}(I_C\boldsymbol{B} - \boldsymbol{B}^2)\right], \tag{207}$$

$$\bar{\boldsymbol{\sigma}}^f = -\phi^f(p + \varrho\frac{\partial\bar{\Psi}}{\partial\phi^f})\boldsymbol{I}, \tag{208}$$

$$\vec{\bar{b}} = -(p + \varrho\frac{\partial\bar{\Psi}}{\partial\phi^f})\nabla\phi^s - \varrho^f\nabla\bar{\Psi} + \phi^s\phi^f\boldsymbol{\alpha}(\vec{v}^s - \vec{v}^f). \tag{209}$$

Now, introducing $\bar{p} = p + \varrho\frac{\partial\bar{\Psi}}{\partial\phi^f}$ and $\tilde{\Psi}(I_C, II_C, \phi^s) = \bar{\Psi}(I_C, II_C, \phi^s, 1-\phi^s)$ we finally obtain

$$\bar{\boldsymbol{\sigma}}^s = -\phi^s(\bar{p} + \varrho\frac{\partial\tilde{\Psi}}{\partial\phi^s})\boldsymbol{I} + 2\varrho\left[\frac{\partial\tilde{\Psi}}{\partial I_C}\boldsymbol{B} + \frac{\partial\tilde{\Psi}}{\partial II_C}(I_C\boldsymbol{B} - \boldsymbol{B}^2)\right], \tag{210}$$

$$\bar{\boldsymbol{\sigma}}^f = -\phi^f\bar{p}\boldsymbol{I}, \tag{211}$$

$$\vec{\bar{b}} = -\bar{p}\nabla\phi^s - \varrho^f\nabla\tilde{\Psi} + \phi^s\phi^f\boldsymbol{\alpha}(\vec{v}^s - \vec{v}^f). \tag{212}$$

Substituting these constitutive equations into the balance of linear momentum for the constituents, we obtain

$$\frac{\partial(\varrho^s\vec{v}^s)}{\partial t} + \operatorname{div}(\varrho^s\vec{v}^s\otimes\vec{v}^s) = -\phi^s\nabla\bar{p} + \operatorname{div}\boldsymbol{\sigma}^E + \varrho^f\nabla\tilde{\Psi} - \phi^s\phi^f\boldsymbol{\alpha}(\vec{v}^s - \vec{v}^f), \tag{213}$$

$$\frac{\partial(\varrho^f\vec{v}^f)}{\partial t} + \operatorname{div}(\varrho^f\vec{v}^f\otimes\vec{v}^f) = -\phi^f\nabla\bar{p} - \varrho^f\nabla\tilde{\Psi} + \boldsymbol{\alpha}\phi^s\phi^f(\vec{v}^s - \vec{v}^f), \tag{214}$$

with $\boldsymbol{\sigma}^E = 2\varrho\left[\frac{\partial\tilde{\Psi}}{\partial I_C}\boldsymbol{B} + \frac{\partial\tilde{\Psi}}{\partial II_C}(I_C\boldsymbol{B} - \boldsymbol{B}^2)\right] - \varrho\phi^s\frac{\partial\tilde{\Psi}}{\partial\phi^s}\boldsymbol{I}$. These equations together with the constraint (182), mass balance equations (177) and (179), suitable boundary conditions and constitutive relation for the function $\tilde{\Psi}$ form complete problem to solve.

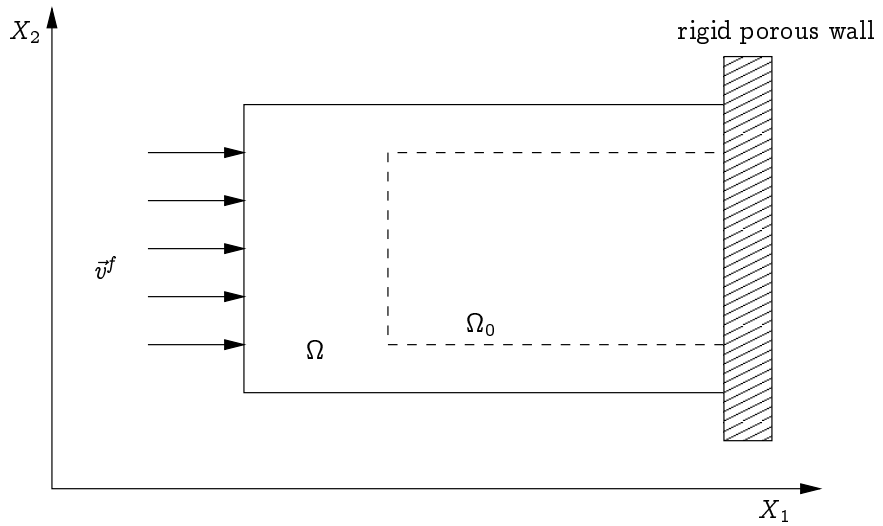

FIGURE 2. Undeformed and deformed configuration of the slab.

5. One dimensional problem

5.1. Determination of the response functions. In this section, we discuss a procedure for determining the response functions for a solid infused with a fluid. The analysis proceeds very much along the lines of the work of Rivlin and his co-workers Rivlin and Saunders [1951] for purely elastic solids. Our aim is to provide hints for experiments with biological tissues. In such cases the fluid part of the mixture is diffusing with speed given by biological conditions, e.g. blood flow in capillaries, and for which the inertial effects might be significant.

Let us consider one-dimensional steady diffusion through a biaxially stretched rectangular slab $\Omega_0 = [-L, L] \times [-H, H] \times [-L, L]$, see figure 2, with a motion of the form

$$x_1 = f(X_1), \qquad x_2 = \lambda_2 X_2, \qquad x_3 = \lambda_3 X_3, \tag{215}$$

where $f(X_1)$ is an unknown function that allows for a non-homogeneous deformation in the X_1 direction, and λ_2, λ_3 are prescribed positive constants. The fluid velocity is assumed to be $\vec{v}^f(x_1) = (v_1^f(x_1), 0, 0)$. The deformation gradient $\boldsymbol{F}$ and the deformation tensors $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ and $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$ have the forms

$$\boldsymbol{F} = \begin{pmatrix} f' & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \boldsymbol{C} = \boldsymbol{B} = \begin{pmatrix} f'^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}.$$

The invariants of $\boldsymbol{C}$ are

$$I_C = f'^2 + \lambda_2^2 + \lambda_3^2, \qquad II_C = f'^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + f'^2\lambda_3^2, \qquad III_C = f'^2\lambda_2^2\lambda_3^2.$$

Next, consider the following nondimensional quantities

$$x_1{}^\star = \frac{x_1}{H}, \qquad X_1{}^\star = \frac{X_1}{H}, \qquad f^\star = \frac{f}{H}, \qquad \varrho^\star = \frac{\varrho}{\varrho_t^f},$$
$$v_1^{f\star} = \frac{v_1^f}{V}, \qquad \Psi^\star = \frac{\Psi}{V^2}, \qquad p^\star = \frac{p}{\varrho_t^f V^2}, \qquad \alpha^\star = \frac{\alpha H}{V},$$

where H is the undeformed thickness of the slab and V is a characteristic fluid velocity. Nondimensional quantities are used in the rest of this study, thus we omit the superscript * below.

Suppose that the Helmholtz potential is a function of the first two invariants of the deformation tensor $\boldsymbol{C}$ and the volume fraction ϕ^s, i.e. $\Psi = \tilde{\Psi}(I_C, II_C, \phi^s)$. Thus, from (210) and (211) the total stress $\boldsymbol{\sigma}$ can be written as

$$\boldsymbol{\sigma} = -\big(p + \varrho\phi^s \frac{\partial\tilde{\Psi}}{\partial\phi^s}\big)\boldsymbol{I} + 2\varrho\left[\frac{\partial\tilde{\Psi}}{\partial I_C}\boldsymbol{B} + \frac{\partial\tilde{\Psi}}{\partial II_C}(I_C\boldsymbol{B} - \boldsymbol{B}^2)\right].$$

where the unknown response functions $\frac{\partial\tilde{\Psi}}{\partial I_C}$, $\frac{\partial\tilde{\Psi}}{\partial II_C}$ and $\frac{\partial\tilde{\Psi}}{\partial\phi^s}$ can depend on I_C, II_C, ϕ^s. In particular, for the deformation (215) we have

$$\sigma_{11} = -\tilde{p} + 2\varrho\left[\frac{\partial\tilde{\Psi}}{\partial I_C}f'^2 + \frac{\partial\tilde{\Psi}}{\partial II_C}(f'^2\lambda_2^2 + f'^2\lambda_3^2)\right], \tag{216}$$

$$\sigma_{22} = -\tilde{p} + 2\varrho\left[\frac{\partial\tilde{\Psi}}{\partial I_C}\lambda_2^2 + \frac{\partial\tilde{\Psi}}{\partial II_C}(f'^2\lambda_2^2 + \lambda_2^2\lambda_3^2)\right], \tag{217}$$

$$\sigma_{33} = -\tilde{p} + 2\varrho\left[\frac{\partial\tilde{\Psi}}{\partial I_C}\lambda_3^2 + \frac{\partial\tilde{\Psi}}{\partial II_C}(f'^2\lambda_3^2 + \lambda_2^2\lambda_3^2)\right], \tag{218}$$

where $\tilde{p} = p + \varrho\phi^s\frac{\partial\tilde{\Psi}}{\partial\phi^s}$. Now forming the equations for the stress differences we can eliminate the isotropic part of the stress

$$\sigma_{11} - \sigma_{22} = 2\varrho\left[\frac{\partial\tilde{\Psi}}{\partial I_C}(f'^2 - \lambda_2^2) + \frac{\partial\tilde{\Psi}}{\partial II_C}(f'^2\lambda_3^2 - \lambda_2^2\lambda_3^2)\right], \tag{219}$$

$$\sigma_{22} - \sigma_{33} = 2\varrho\left[\frac{\partial\tilde{\Psi}}{\partial I_C}(\lambda_2^2 - \lambda_3^2) + \frac{\partial\tilde{\Psi}}{\partial II_C}(f'^2\lambda_2^2 - f'^2\lambda_3^2)\right], \tag{220}$$

and thus solve for two of the response functions

$$\frac{\partial\tilde{\Psi}}{\partial I_C} = \frac{1}{2\varrho(f'^2 - \lambda_3^2)}\left[\frac{f'^2(\sigma_{11} - \sigma_{22})}{f'^2 - \lambda_2^2} - \frac{\lambda_3^2(\sigma_{22} - \sigma_{33})}{\lambda_2^2 - \lambda_3^2}\right], \tag{221}$$

$$\frac{\partial\tilde{\Psi}}{\partial II_C} = \frac{1}{2\varrho(f'^2 - \lambda_3^2)}\left[\frac{\sigma_{11} - \sigma_{22}}{f'^2 - \lambda_2^2} - \frac{\sigma_{22} - \sigma_{33}}{\lambda_2^2 - \lambda_3^2}\right]. \tag{222}$$

This is of the same form as for an elastic solid, but in the case of a mixture the quantities on the right hand side are functions of spatial position X_1. Of course, we seek the response functions as a function of I_C, II_C, ϕ^s.

Conservation of mass for the fluid yields (179)

$$\phi^f v_1^f = F_0, \tag{223}$$

where the constant F_0 can be measured as it is the mass flow rate of the fluid. Using this fact we can integrate the momentum equation for the mixture, obtained

by summing the constituents momentum balances (178) and (180)

$$\phi^f \frac{dv_1^f}{dx_1} v_1^f - \frac{d\sigma_{11}}{dx_1} = 0, \tag{224}$$

to get

$$F_0 v_1^f - \sigma_{11} = M_0, \tag{225}$$

where the constant M_0 can be determined by measuring the quantities on the left hand side at one point. Then if it is possible to measure the fluid velocity at any other point X_1 in the slab this equation allows us to determine the stress σ_{11} at the same point. We can see that neglecting the inertial term will result in constant stress σ_{11} throughout the specimen while in the case of significant inertial effects the stress σ_{11} will vary with X_1.

Since the values of response functions $\frac{\partial \tilde{\Psi}}{\partial I_C}$ and $\frac{\partial \tilde{\Psi}}{\partial II_C}$ are determined by equations (221) and (222) we can write (225) as

$$\tilde{p} = M_0 - F_0 v_1^f - \sigma_{11}^E, \tag{226}$$

where $\sigma_{11}^E = 2\varrho \left[\frac{\partial \tilde{\Psi}}{\partial I_C} f'^2 + \frac{\partial \tilde{\Psi}}{\partial II_C} (f'^2 \lambda_2^2 + f'^2 \lambda_3^2) \right]$ and the quantity $\tilde{p}$ can be determined as follows.

The balance of linear momentum for the fluid (180) is

$$\phi^f \frac{dv_1^f}{dx_1} v_1^f = -\phi^f \frac{dp}{dx_1} + \phi^f \frac{d\tilde{\Psi}}{dx_1} - \alpha \phi^s \phi^f v_1^f. \tag{227}$$

By using our definition of $\tilde{p} = p + \varrho \phi^s \frac{\partial \tilde{\Psi}}{\partial \phi^s}$ and expanding the derivative of the Helmholtz potential, this equation can be rewritten as

$$\frac{dv_1^f}{dx_1} v_1^f = -\frac{dp}{dx_1} - \frac{p - \tilde{p}}{\varrho \phi^s} \frac{d\phi^s}{dx_1} + \frac{\partial \tilde{\Psi}}{\partial I_C} \frac{dI_C}{dx_1} + \frac{\partial \tilde{\Psi}}{\partial II_C} \frac{dII_C}{dx_1} - \alpha \phi^s v_1^f. \tag{228}$$

Assuming that the drag coefficient α is known, this equation can be used to obtain a differential equation for $\frac{dp}{dx_1}$

$$\frac{dp}{dx_1} + \frac{p}{\varrho \phi^s} \frac{d\phi^s}{dx_1} = -\frac{dv_1^f}{dx_1} v_1^f + \frac{\tilde{p}}{\varrho \phi^s} \frac{d\phi^s}{dx_1} + \frac{\partial \tilde{\Psi}}{\partial I_C} \frac{dI_C}{dx_1} + \frac{\partial \tilde{\Psi}}{\partial II_C} \frac{dII_C}{dx_1} - \alpha \phi^s v_1^f. \tag{229}$$

Denoting the right hand side of (229) by $Q(x_1)$, which consists of already determined quantities, we have

$$\frac{dp}{dx_1} + p \frac{d}{dx_1} (\ln \phi^s - \ln \varrho) = Q(x_1), \tag{230}$$

which we can solve for p to obtain

$$p(x_1) = \frac{\varrho}{\phi^s} \left(\int \frac{\phi^s}{\varrho} Q(x_1)\, dx_1 + c \right), \tag{231}$$

the c being the integration constant and could be determined by specifying one of the partial stresses at any point. Consequently we can determine the third response function as

$$\frac{\partial \tilde{\Psi}}{\partial \phi^s} = \frac{\tilde{p} - p}{\varrho \phi^s}.$$

5.2. Illustrative example of swelling. The steady diffusion of a fluid through a non-linearly elastic slab has been studied by Shi et al. [1981] and Reynolds and Humphrey [1998] within the context of mixture theory. The study by Shi et al. ignores the inertial effects in the flow of the fluid and also assumes that the solid is in equilibrium. The interactive term includes the effects due to drag and also incorporates effects due to density gradients and gradients in the Helmholtz potential. The more recent study by Reynolds and Humphrey ignores inertial terms in the balance of linear momentum for the fluid and neglects the effects due to the gradients of the Helmholtz potential in the interaction term. The main difference between the two studies, however, is the manner in which the boundary conditions are enforced. Shi et. al. generate boundary conditions through a thermodynamic argument based on the notion of a saturated mixture, while Reynolds and Humphrey prescribe boundary conditions directly for the velocity of the fluid. Here, we re-visit the problem, including inertial effects and retaining the effects of the gradient of the Helmholtz potential in the interaction term.

We are looking for a steady state solution, i.e., $\vec{v}^s = 0$ and $\frac{\partial \vec{v}^f}{\partial t} = 0$, of the one-dimensional problem introduced in the previous section. We assume the following form for the Helmholtz potential

$$\tilde{\Psi}(I_C, \phi^s) = c_1(I_C - 3) + c_2 \ln(\phi^s). \tag{232}$$

Due to the choice of the deformation, the deformation gradient $\boldsymbol{F}$ and the right and left Cauchy-Green stretch tensors $\boldsymbol{B}$, $\boldsymbol{C}$ are diagonal and depend only on X_1 and consequently, the stress tensor $\boldsymbol{\sigma}^E$ is diagonal and depends on X_1. By (203) the mass density of the fluid ϱ^f also depends only on X_1. Assuming that the fluid velocity is only in the x_1 direction (which is the same as X_1), i.e. $\vec{v}^f(x_1) = (v_1^f(x_1), 0, 0)$, equation (179) yields

$$\frac{d(\phi^f v_1^f)}{dx_1} = 0, \tag{233}$$

which allows us to express the fluid velocity and its spatial derivative in terms of the volume fraction of the fluid.

The balance of linear momentum for the solid (178) implies that the Lagrange multiplier p is a function of x_1 alone and reduces to

$$0 = -\phi^s \frac{dp}{dx_1} + \frac{d\sigma_{11}^E}{dx_1} - \phi^f \frac{d\tilde{\Psi}}{dx_1} + \alpha \phi^s \phi^f v_1^f. \tag{234}$$

The balance of linear momentum for the mixture, obtained by summing the momentum balance equations of both constituents (178) and (180) together reduces to

$$\phi^f \frac{dv_1^f}{dx_1} v_1^f = -\frac{dp}{dx_1} + \frac{d\sigma_{11}^E}{dx_1}. \tag{235}$$

Now, multiplying (235) by ϕ^s and subtracting (234) we can eliminate the Lagrange multiplier p and obtain

$$\phi^s \frac{dv_1^f}{dx_1} v_1^f = -\frac{d\sigma_{11}^E}{dx_1} + \frac{d\tilde{\Psi}}{dx_1} - \alpha \phi^s v_1^f. \tag{236}$$

a	the full problem with inertia and $\nabla\Psi$ term
b	inertia neglected
c	$\nabla\Psi$ term neglected
d	inertia and $\nabla\Psi$ term neglected

TABLE 1. Figure captions explanation.

Next, subtracting (234) from (235) we obtain

$$\frac{dv_1^f}{dx_1}v_1^f = \frac{dp}{dx_1} + \frac{d\tilde{\Psi}}{dx_1} - \alpha\phi^s v_1^f. \tag{237}$$

The next step is to express all the terms in terms of the unknown functions f' and p in the reference coordinates. Using (233) the inertial term can be rewritten as

$$\frac{dv_1^f}{dx_1}v_1^f = -\frac{1}{f'}\frac{F_0^2}{{\phi^f}^3}\frac{d\phi^f}{dX_1} = -\frac{f''}{{f'}^2}\frac{F_0^2}{{\phi^f}^2}\frac{\phi^s}{\phi^f}, \tag{238}$$

and using (232) to express the derivative of the σ_{11}^E and $\tilde{\Psi}$ we have

$$\frac{d\sigma_{11}^E}{dx_1} = \frac{1}{f'}\frac{d}{dX_1}(c_1\varrho{f'}^2 - c_2\varrho) = \frac{f''}{{f'}^2}\left[c_1{f'}^2(1+\varrho) - c_2(1-\varrho)\right], \tag{239}$$

$$\frac{d\tilde{\Psi}}{dx_1} = \frac{1}{f'}\frac{d}{dX_1}\left[c_1(I_C - 3) + c_2\ln(\phi^s)\right] = \frac{f''}{{f'}^2}(c_1{f'}^2 - c_2), \tag{240}$$

where $\varrho = \phi^f + \frac{\varrho_t^s}{\varrho_t^f}\phi^s$, $\phi^f = 1 - \phi^s$ and $\phi^s = \frac{\phi_0^s}{f'\lambda_2\lambda_3}$. Substituting this into the (236) we have a first order ordinary differential equation for the unknown function $f'(X_1)$ and then (237) gives us a first order ordinary differential equation for p. The resulting system can be written as

$$f'' = G_1(f')$$
$$p' = G_2(f').$$

To obtain the solution for f we need two boundary conditions:

- the boundary displacement $f(X_1 = -1) = -1$,
- the inlet fluid velocity $v_1^f(X_1 = -1)$, which is measurable.

If prescribed it gives us the value of $f'(X_1 = -1)$ through (223) and (203) via

$$f' = \frac{\phi_0^s}{\lambda_2\lambda_3}\left(\frac{1}{1 - \frac{F_0}{v_1^f}}\right)$$

To find the corresponding p we need one additional boundary condition on stress

- the total stress value $\sigma_{11}(X_1 = -1)$ can be prescribed.

This choice of the boundary conditions allows us to solve the resulting ordinary differential equations as an initial value problem by a standard method such as Runge-Kutta.

For illustration, the material parameters used in our computations are

$$\phi_0^s = 0.85, \quad \frac{\varrho_t^s}{\varrho_t^f} = 1.1, \quad \alpha = 1, \quad c_1 = 1, \quad c_2 = 1$$

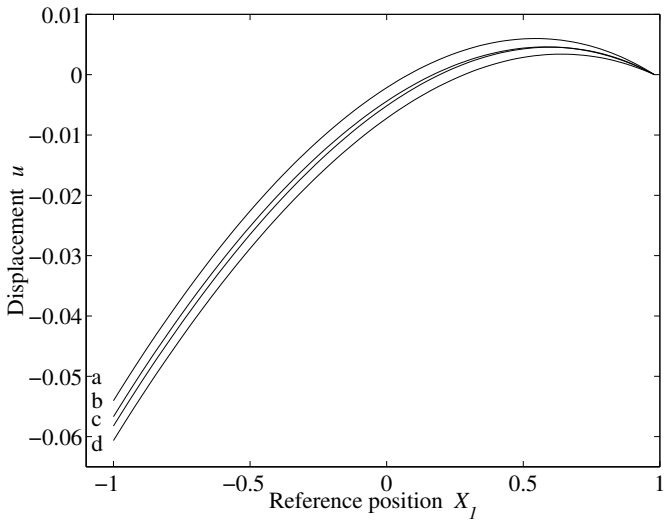

FIGURE 3. Displacement as a function of reference position X_1.

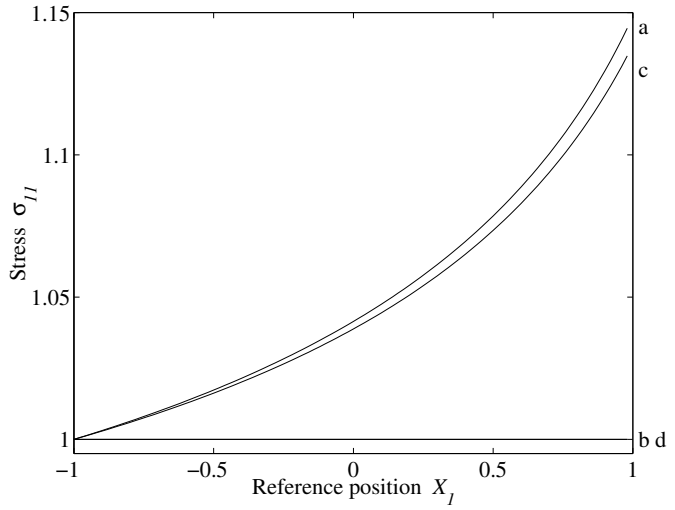

FIGURE 4. Total stress σ_{11} as a function of reference position X_1.

and the prescribed stretch ratios and boundary conditions are

$$\lambda_2 = 1.1, \quad \lambda_3 = 0.9, \quad \phi^f(-1) = 0.2, \quad v_1^f(-1) = 1, \quad \sigma_{11}(-1) = 1.$$

We have compared the solutions of the problem with neglected inertia or $\nabla\Psi$ terms to the solution of the complete problem. The captions used in all figures for each solution are explained in table 1.

As expected, the influence of the fluid inertia is insignificant for small prescribed inflow velocities, but for higher fluid velocities the effects can be significant. In figures 3, 4, 5 and 6 we show displacement of the solid u_1^s, total stress σ_{11}, fluid volume fraction ϕ^f and fluid velocity v_1^f throughout the slab. First, we can see in

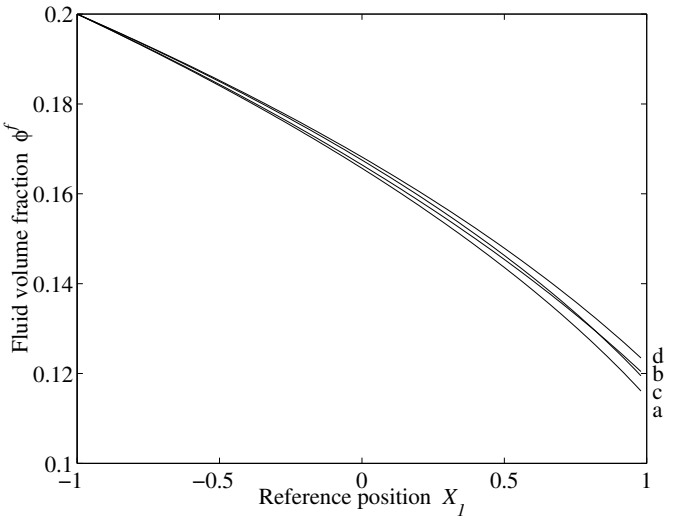

FIGURE 5. Volume fraction as a function of reference position X_1.

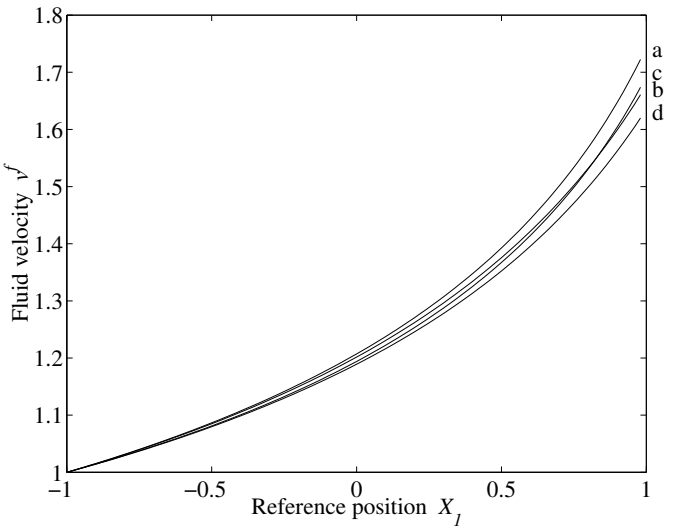

FIGURE 6. Fluid velocity as a function of reference position X_1.

the figure 4 that the stress σ_{11} varies with the position X_1 when the inertial term is included (cases **a**, **c**) but the relative difference compared to its value is quite small (0.1%). In the cases **b** and **d** the stress σ_{11} is constant trough the domain. The relative difference for the displacement of the end point, fluid outflow velocity, and volume fraction are of order of 10%.

In figure 7 we depict the computed values of outflow fluid velocity at the end-point $X_1 = 1$ while varying the stretch ratio λ_2 in the range from 0.5 to 2. In this case we can see that neglecting the term $\nabla\Psi$ in the interaction force (cases **c** and **d**) can significantly change the trend of the dependence, especially for the fluid

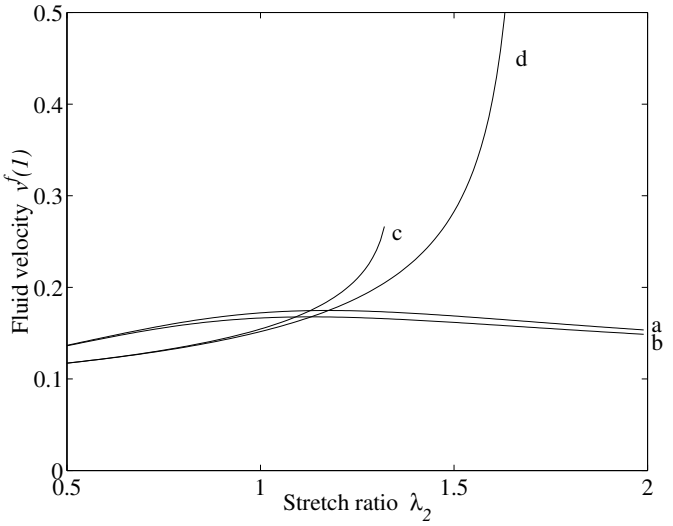

FIGURE 7. Outflow velocity $v_1^f(1)$ as a function of stretch ratio λ_2.

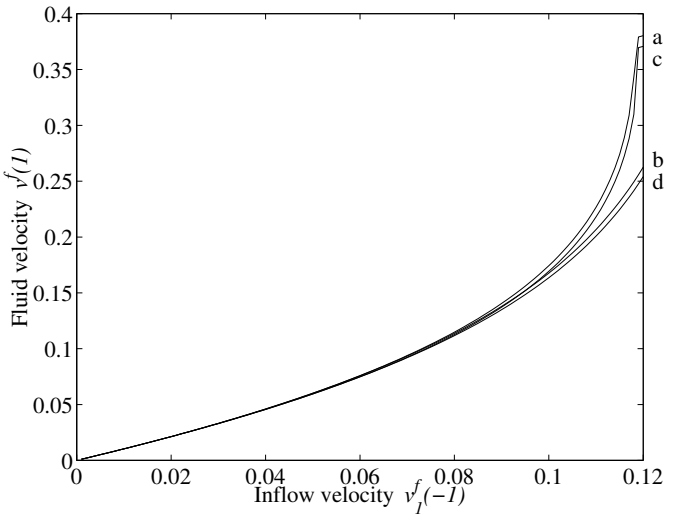

FIGURE 8. Outflow velocity $v_1^f(1)$ as a function of inflow velocity $v_1^f(-1)$.

velocity and volume fraction, while the influence on displacement and stress values is negligible.

In figure 8 we compare the computed values of outflow fluid velocity at the endpoint $X_1 = 1$ as a function of the prescribed inflow velocity $v_1^f(-1)$ which is varied from 0 up to 0.12. When including the inertia term, then for certain interval of values of the inflow velocity the solution does not exists on the whole spatial interval $(-1, 1)$. There is a solution for which the fluid volume fraction decreases to zero inside the slab, which is a singular point for the fluid mass conservation equation (223). In figures 9 and 10 we apply the boundary conditions at the point

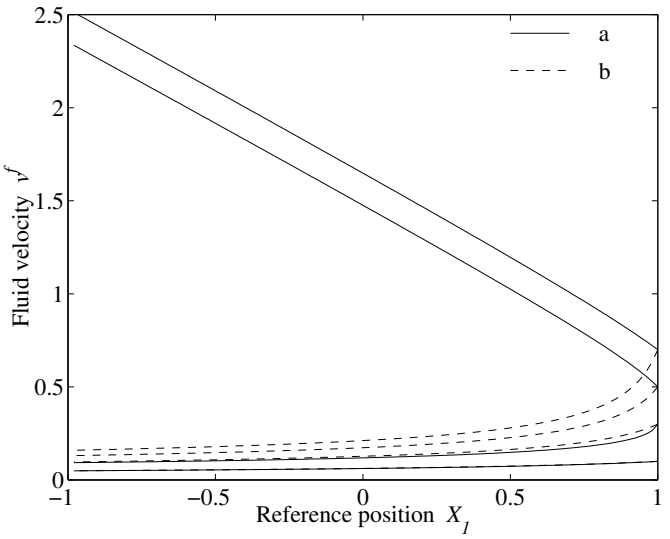

FIGURE 9. Fluid velocity as a function of reference position X_1.

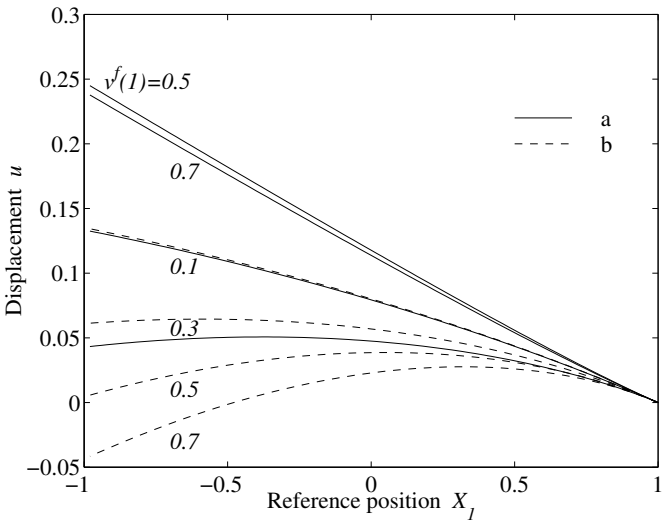

FIGURE 10. Displacement as a function of reference position X_1.

$X_1 = 1$, i.e. the outflow velocity is prescribed. If the inertia term is included we can observe in figure 9 a jump in the inflow velocity as the outflow velocity is increased. This jump occurs for higher value of the velocity boundary condition when the drag coefficient α is decreased, i.e. weaker solid fluid interaction. In figure 10 the displacement u_1^s of the solid is show for the same values of the outflow velocity boundary condition as in figure 9.

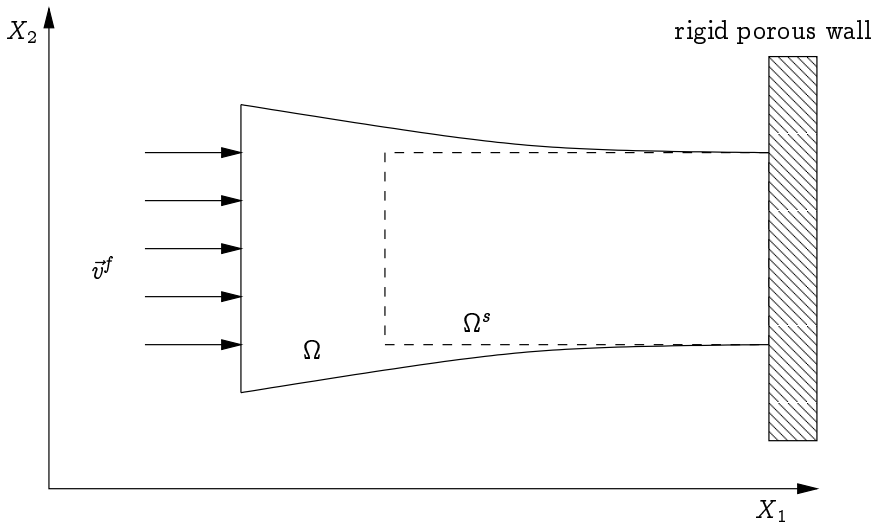

FIGURE 11. Undeformed and deformed configuration in two dimensional problem.

6. Two dimensional problem

In this section we formulate and propose a solution algorithm for a steady state, two-dimensional problem of stretched rectangular slab of solid-fluid mixture. Two or three dimensional problems for solid fluid mixtures in connection with hydrated biological tissue were recently presented in Spilker et al. [1988], Spilker and Suh [1990], Suh et al. [1991], Donzelli et al. [1992], Vermilyea and Spilker [1993], Almeida and Spilker [1998] for biphasic mixture to model behavior of articular cartilage under compression. In all these works the nonlinear inertial of the fluid is neglected.

Let $\vec{u}$ denotes the solid displacement, $\vec{v}$ denotes the fluid velocity, ϕ is the fluid volume fraction and p is the Lagrange multiplier associated with the kinematical constraint (185). Let $\Omega^s = [-L, L] \times [-H, H] \times [-L, L]$ be the reference domain occupied by the solid. The deformed domain, shown in figure 11, is

$$\Omega = \{x = X + \vec{u}(X), \forall X \in \Omega^s\}. \tag{241}$$

The deformation is assumed to be of a form

$$x_1 = X_1 + u_1(X_1, X_2), \qquad x_2 = X_2 + u_2(X_1, X_2), \qquad x_3 = \lambda_3 X_3, \tag{242}$$

where λ_3 is prescribed positive constant. The fluid velocity is assumed to be

$$\vec{v}(x_1) = (v_1(x_1, x_2), v_2(x_1, x_2), 0). \tag{243}$$

Let the symbol ∇ stands for ∇^s, then $\boldsymbol{F} = \boldsymbol{I} + \nabla\vec{u}$ and our task is to find $(\vec{u}, \vec{v}, \phi, p)$ such that

$$\phi(\operatorname{grad}\vec{v})\vec{v} + \operatorname{grad} p - \operatorname{div}\boldsymbol{\sigma}^E = 0 \tag{244}$$

$$(\operatorname{grad}\vec{v})\vec{v} + \operatorname{grad}(p+\Psi) + (1-\phi)\boldsymbol{\alpha}\vec{v} = 0 \tag{245}$$

$$(1-\phi)\det\boldsymbol{F} = \phi_0^s \tag{246}$$

$$\operatorname{div}(\phi\vec{v}) = 0 \tag{247}$$

holds in domain Ω, where ϕ_0^s is the volume fraction of the solid in the reference state and the Helmholtz potential Ψ is given by constitutive equation as a function of ϕ and $\boldsymbol{F}$. The Cauchy stress tensor $\boldsymbol{\sigma}$ is then given by

$$\boldsymbol{\sigma}^E = (\phi + \beta(1-\phi))\frac{\partial\Psi}{\partial\boldsymbol{F}}\boldsymbol{F}^T, \tag{248}$$

where $\beta = \frac{\varrho_t^s}{\varrho_t^f}$ is the true mass ratio. By $\boldsymbol{S}$ we will denote the Piola-Kirchoff stress tensor

$$\boldsymbol{S} = -p(\det\boldsymbol{F})\boldsymbol{F}^{-T} + \boldsymbol{S}^E, \tag{249}$$

$$\boldsymbol{S}^E = (\phi + \beta(1-\phi))(\det\boldsymbol{F})\frac{\partial\Psi}{\partial\boldsymbol{F}}. \tag{250}$$

Let the boundary $\partial\Omega$ be divided into three disjoint parts $\partial\Omega = \bigcup_{i=1}^3 \Gamma_i$. Let $\vec{n}$ be the unit normal vector to the boundary $\partial\Omega$. Then if $\vec{t}$ is given traction on the boundary, $\vec{u}_B$ is given solid boundary displacement and $\vec{v}_B$ is given fluid velocity at the boundary the possible boundary conditions are

$$\frac{1}{3}\operatorname{tr}\boldsymbol{\sigma} = p_B, \quad \vec{u} = \vec{u}_B \quad \text{on } \Gamma_1, \tag{251}$$

$$-p\vec{n} + \boldsymbol{\sigma}^E\vec{n} = \vec{t}, \quad \vec{v} = \vec{v}_B \quad \text{on } \Gamma_2, \tag{252}$$

$$\vec{u} = \vec{u}_B, \quad \vec{v} = \vec{v}_B \quad \text{on } \Gamma_3. \tag{253}$$

We avoid prescribing partial stresses on the boundary since it is not clear weather such values can be obtained by measurements or how it should be partitioned into the partial stresses. Instead, we focus on the prescription of the fluid velocity at the boundary. There are other possible boundary conditions, apart of prescribing the fluid velocity. We may prescribe

$$\phi\vec{v} = \vec{f}_B, \tag{254}$$

or the total fluid flux through certain part of the boundary

$$\int_\Gamma \phi\vec{v}\cdot\vec{n}\,da = f_B, \tag{255}$$

which would be convenient as an outflow boundary condition. On the other hand the implementation of these conditions in the solution process is more complicated.

6.1. Weak formulation. In order to apply the finite element method we formulate our problem in a weak sense. Let us define spaces U, V, M and P as

follows

$$U = \{\vec{u} \in [W^{1,2}(\Omega^s)]^2, \vec{u} = \vec{0} \text{ on } \Gamma_1 \cup \Gamma_3\} \tag{256}$$

$$V = \{\vec{v} \in [W^{1,2}(\Omega^s)]^2, \vec{v} = 0 \text{ on } \Gamma_2 \cup \Gamma_3\} \tag{257}$$

$$M = \{\phi \in W^{1,2}(\Omega^s), 0 \leq \phi \leq 1 \text{ in } \Omega\} \tag{258}$$

$$P = \{p \in L^2(\Omega^s)\} \tag{259}$$

Multiplying equations (244)–(247) by a test functions $(\vec{\zeta}, \vec{\xi}, \eta, \gamma)$, integrating over the domain Ω and using per-partes integration we obtain

$$\begin{aligned} \int_\Omega \phi \frac{\partial \vec{v}}{\partial \vec{x}} \vec{v} \cdot \vec{\zeta}\, dx - \int_\Omega p \operatorname{div}(\phi \vec{\zeta})\, dx + \int_\Omega \operatorname{tr}\left(\boldsymbol{\sigma}^E \frac{\partial \vec{\zeta}}{\partial \vec{x}}\right) dx \\ - \int_{\Gamma_2} \vec{t} \cdot \vec{\zeta}\, da = 0, \end{aligned} \tag{260}$$

$$\begin{aligned} \int_\Omega \frac{\partial \vec{v}}{\partial \vec{x}} \vec{v} \cdot \vec{\xi}\, dx - \int_\Omega (p + \Psi) \operatorname{div} \vec{\xi}\, dx + \int_\Omega (1-\phi) \boldsymbol{\alpha} \vec{v} \cdot \vec{\xi}\, dx \\ + \int_{\Gamma_1} (p_B - \frac{1}{3} \operatorname{tr} \boldsymbol{\sigma} + \Psi) \vec{n} \cdot \vec{\xi}\, da = 0, \end{aligned} \tag{261}$$

$$\int_\Omega \left(1 - \phi - \frac{\phi_0^s}{\det \boldsymbol{F}}\right) \eta\, dx = 0, \tag{262}$$

$$\int_\Omega \operatorname{div}(\phi \vec{v}) \gamma\, dx = 0. \tag{263}$$

In these equation, the integration is done over the yet unknown domain Ω so it is convenient to transform the integration to the reference domain Ω^s which yields

$$\begin{aligned} \int_{\Omega^s} \phi \nabla \vec{v} \operatorname{cof} \boldsymbol{F}^T \vec{v} \cdot \vec{\zeta}\, dX - \int_{\Omega^s} p \operatorname{cof} \boldsymbol{F} \cdot \nabla \vec{\zeta}\, dX + \int_{\Omega^s} \boldsymbol{S}^E \cdot \nabla \vec{\zeta}\, dX \\ - \int_{\Gamma_2} \vec{t} \operatorname{cof} \boldsymbol{F} \vec{N} \cdot \vec{\zeta}\, dA = 0, \end{aligned} \tag{264}$$

$$\begin{aligned} \int_{\Omega^s} \nabla \vec{v} \operatorname{cof} \boldsymbol{F} \vec{v} \cdot \vec{\xi}\, dX - \int_{\Omega^s} (p + \Psi) \operatorname{cof} \boldsymbol{F} \cdot \nabla \vec{\xi}\, dX \\ + \int_{\Omega^s} (1-\phi) \boldsymbol{\alpha} \vec{v} \cdot \vec{\xi} \det \boldsymbol{F}\, dX \\ + \int_{\Gamma_1} (p_B - \frac{1}{3} \operatorname{tr} \boldsymbol{\sigma} + \Psi) \operatorname{cof} \boldsymbol{F} \vec{N} \cdot \vec{\xi}\, dA = 0, \end{aligned} \tag{265}$$

$$\int_{\Omega^s} ((1-\phi) \det \boldsymbol{F} - \phi_0^s)\, \eta\, dX = 0, \tag{266}$$

$$\int_{\Omega^s} \nabla(\phi \vec{v}) \cdot \operatorname{cof} \boldsymbol{F} \gamma\, dX = 0, \tag{267}$$

where $\operatorname{cof} \boldsymbol{F} = \det \boldsymbol{F} \boldsymbol{F}^{-T}$ and the inner product for two tensors is defined as $\boldsymbol{F} \cdot \boldsymbol{G} = \operatorname{tr}(\boldsymbol{F}^T \boldsymbol{G})$. Then our task is to find $(\vec{u}, \vec{v}, \phi, p)$ such that $(\vec{u} - \vec{u}_B, \vec{v} - \vec{v}_B, \phi, p) \in U \times V \times M \times P$ and equations (264)–(267) are satisfied for all $(\vec{\zeta}, \vec{\xi}, \eta, \gamma) \in U \times V \times M \times P$.

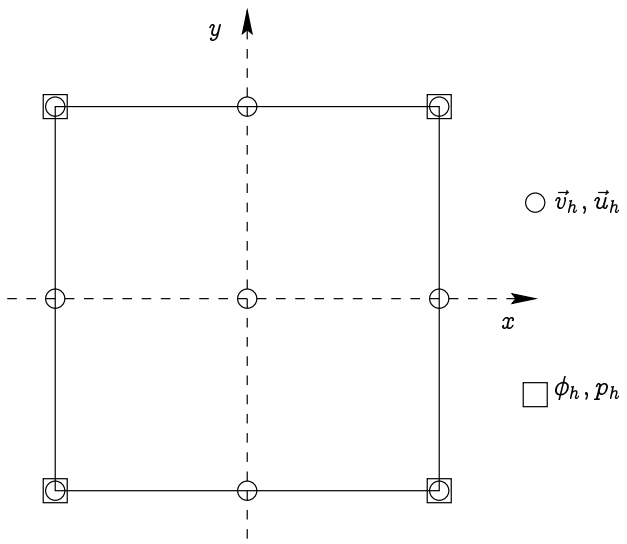

FIGURE 12. Degrees of freedom on the reference element and local coordinate system.

6.2. Finite element discretization. The reference domain Ω^s is approximated by a domain Ω_h with piecewise linear boundary. The interior is divided by regular quadrilateral mesh into convex quadrilateral elements. Any two elements have in common either whole side, vertex or are disjoint. Let the set of all quadrilaterals in Ω_h be denoted by $\mathcal{T}_h$ and let $\tilde{T} = (-1,1)^2$ be the reference quadrilateral. For each element $T \in \mathcal{T}_h$ there is a bilinear one to one mapping on to the reference element $\tilde{T}$. The spaces U, V, M resp. P are approximated by finite element spaces

$$U_h = \{\vec{u}_h \in [C(\Omega_h)]^2, \vec{u}_h/T \in [Q_2(T)]^2 \quad \forall T \in \mathcal{T}_h, \vec{u}_h = \vec{0} \text{ on } \Gamma_1 \cup \Gamma_3\}, \tag{268}$$

$$V_h = \{\vec{v}_h \in [C(\Omega_h)]^2, \vec{v}_h/T \in [Q_2(T)]^2 \quad \forall T \in \mathcal{T}_h, \vec{v}_h = 0 \text{ on } \Gamma_1 \cup \Gamma_3\}, \tag{269}$$

$$M_h = \{\phi_h \in C(\Omega_h), \phi_h/T \in Q_1(T) \quad \forall T \in \mathcal{T}_h, 0 \le \phi_h \le 1\}, \tag{270}$$

$$P_h = \{p_h \in C(\Omega_h), p_h/T \in Q_1(T) \quad \forall T \in \mathcal{T}_h\}. \tag{271}$$

Where $Q_n(T)$ denotes the space of polynomial functions, defined on the quadrilateral T, which transformed to the reference element $\tilde{T}$ are polynomials of order at most n in each reference coordinate. The location of local degrees of freedom on the reference element is shown in figure 12. The resulting discrete problem is obtained by taking usual nodal basis of the space $U_h \times V_h \times M_h \times P_h$ and using the elements of this basis as test functions $(\vec{\zeta}, \vec{\xi}, \eta, \gamma)$ in (264–267). This set of non-linear algebraic equations can be written as

$$\vec{\mathcal{R}}(\vec{X}) = \vec{0}, \tag{272}$$

where $\vec{X} = (\vec{u}_h, \vec{v}_h, \phi_h, p_h)$ is the vector of unknown components.

Next, we investigate the linearization of (272) around some point $\vec{X} = (\vec{u}, \vec{v}, \phi, p)$. Let us use the following notation

$$(f, g) = \int_{\Omega^s} f \cdot g \, dX,$$

for f and g being vector or tensor quantities. We define the following bilinear forms as variations of the momentum equation (264) with respect to each of the unknowns

$$\begin{aligned} a_{u,u}(\vec{w},\vec{\zeta}) = &\left(\phi\nabla\vec{v}(\frac{\partial \operatorname{cof} \boldsymbol{F}^T}{\partial \boldsymbol{F}}\nabla\vec{w})\vec{v},\vec{\zeta}\right) - \left(p\frac{\partial \operatorname{cof} \boldsymbol{F}}{\partial \boldsymbol{F}}\nabla\vec{w},\nabla\vec{\zeta}\right) \\ &+ \left(\varrho\frac{\partial^2\Psi}{\partial \boldsymbol{F}^2}\nabla\vec{w},\nabla\vec{\zeta}\right) - \int_{\Gamma_2}\vec{t}\frac{\partial \operatorname{cof} \boldsymbol{F}}{\partial \boldsymbol{F}}\nabla\vec{w}\vec{N}\cdot\vec{\zeta}\,dA, \end{aligned} \tag{273}$$

$$a_{u,v}(\vec{w},\vec{\zeta}) = \left(\phi\nabla\vec{w}\operatorname{cof}\boldsymbol{F}^T\vec{v},\vec{\zeta}\right) + \left(\phi\nabla\vec{v}\operatorname{cof}\boldsymbol{F}^T\vec{w},\vec{\zeta}\right), \tag{274}$$

$$b_{u,\phi}(w,\vec{\zeta}) = \left(w\nabla\vec{v}\operatorname{cof}\boldsymbol{F}^T\vec{v},\vec{\zeta}\right) + \left((1-\beta)w\frac{\partial\Psi}{\partial \boldsymbol{F}},\vec{\zeta}\right), \tag{275}$$

$$c_{u,p}(w,\vec{\zeta}) = \left(w\operatorname{cof}\boldsymbol{F},\nabla\vec{\zeta}\right), \tag{276}$$

variations of the momentum equation (265) with respect to each of the unknowns

$$\begin{aligned} a_{v,u}(\vec{w},\vec{\xi}) = &\left(\nabla\vec{v}(\frac{\partial \operatorname{cof} \boldsymbol{F}^T}{\partial \boldsymbol{F}}\nabla\vec{w})\vec{v},\vec{\xi}\right) - \left((p+\Psi)\frac{\partial \operatorname{cof} \boldsymbol{F}}{\partial \boldsymbol{F}}\nabla\vec{w},\nabla\vec{\xi}\right) \\ &- \left((\frac{\partial\Psi}{\partial \boldsymbol{F}}\cdot\nabla\vec{w})\operatorname{cof}\boldsymbol{F},\nabla\vec{\xi}\right) + \left((\operatorname{cof}F\cdot\nabla\vec{w})(1-\phi)\boldsymbol{\alpha}\vec{v},\vec{\xi}\right) \\ &+ \int_{\Gamma_1}(p_B - \frac{1}{3}\operatorname{tr}\boldsymbol{\sigma} + \Psi)\frac{\partial \operatorname{cof} \boldsymbol{F}}{\partial \boldsymbol{F}}\nabla\vec{w}\vec{N}\cdot\vec{\xi}\,dA \\ &+ \int_{\Gamma_1}(p_B - \frac{1}{3}\operatorname{tr}(\frac{\partial\boldsymbol{\sigma}}{\partial \boldsymbol{F}}\nabla\vec{w}) + \frac{\partial\Psi}{\partial \boldsymbol{F}}\cdot\nabla\vec{w})\operatorname{cof}\boldsymbol{F}\vec{N}\cdot\vec{\xi}\,dA, \end{aligned} \tag{277}$$

$$\begin{aligned} a_{v,v}(\vec{w},\vec{\xi}) = &\left(\nabla\vec{w}\operatorname{cof}\boldsymbol{F}^T\vec{v},\vec{\xi}\right) + \left(\nabla\vec{v}\operatorname{cof}\boldsymbol{F}^T\vec{w},\vec{\xi}\right) \\ &+ \left(\det\boldsymbol{F}(1-\phi)\boldsymbol{\alpha}\vec{w},\vec{\xi}\right), \end{aligned} \tag{278}$$

$$b_{v,\phi}(w,\vec{\xi}) = -\left(w\det\boldsymbol{F}\boldsymbol{\alpha}\vec{v},\vec{\xi}\right), \tag{279}$$

$$c_{v,p}(w,\vec{\xi}) = \left(w\operatorname{cof}\boldsymbol{F},\nabla\vec{\xi}\right), \tag{280}$$

variations of (266) with respect to each of the unknowns

$$a_{\phi,u}(\vec{w},\eta) = ((1-\phi)\operatorname{cof}\boldsymbol{F},\nabla\vec{w}\eta), \tag{281}$$

$$b_{\phi,\phi}(w,\eta) = -(\det\boldsymbol{F}w,\eta), \tag{282}$$

and variations of (267) with respect to each of the unknowns

$$a_{p,u}(\vec{w},\gamma) = \left(\nabla(\phi\vec{v}),\frac{\partial \operatorname{cof} \boldsymbol{F}}{\partial \boldsymbol{F}}\nabla\vec{w}\gamma\right), \tag{283}$$

$$a_{p,v}(\vec{w},\gamma) = (\nabla(\phi\vec{w}),\operatorname{cof}\boldsymbol{F}\gamma), \tag{284}$$

$$b_{p,\phi}(w,\eta) = (\nabla(w\vec{v}),\operatorname{cof}\boldsymbol{F}\gamma). \tag{285}$$

Then the linearization of (272) around solution $\vec{X} = (\vec{u}_h, \vec{v}_h, \phi_h, p_h)$ is

$$a_{u,u}(\delta\vec{u},\vec{\zeta}) + a_{u,v}(\delta\vec{v},\vec{\zeta}) + b_{u,\phi}(\delta\phi,\vec{\zeta}) + c_{u,p}(\delta p,\vec{\zeta}) = 0, \tag{286}$$

$$a_{v,u}(\delta\vec{u},\vec{\xi}) + a_{v,v}(\delta\vec{v},\vec{\xi}) + b_{v,\phi}(\delta\phi,\vec{\xi}) + c_{v,p}(\delta p,\vec{\xi}) = 0, \tag{287}$$

$$a_{\phi,u}(\delta\vec{u},\eta) + b_{\phi,\phi}(\delta\phi,\eta) = 0, \tag{288}$$

$$a_{p,u}(\delta\vec{u},\gamma) + a_{p,v}(\delta\vec{v},\gamma) + b_{p,\phi}(\delta\phi,\gamma) = 0, \tag{289}$$

for all $(\vec{\zeta},\vec{\xi},\eta,\gamma) \in U_h \times V_h \times M_h \times P_h$. This can be rewritten in the corresponding matrix notation. Find $\delta\vec{X}$ such that

$$\left[\frac{\partial\vec{\mathcal{R}}}{\partial\vec{X}}(\vec{X})\right]\delta\vec{X} = 0, \tag{290}$$

where the Jacobian matrix $\left[\frac{\partial\vec{\mathcal{R}}}{\partial\vec{X}}(\vec{X})\right]$ has following structure

$$\left[\frac{\partial\vec{\mathcal{R}}}{\partial\vec{X}}(\vec{X})\right] = \begin{pmatrix} \boldsymbol{A}_{u,u} & \boldsymbol{A}_{u,v} & \boldsymbol{B}_{u,\phi} & \boldsymbol{C} \\ \boldsymbol{A}_{v,u} & \boldsymbol{A}_{v,v} & \boldsymbol{B}_{v,\phi} & \boldsymbol{C} \\ (1-\phi)\boldsymbol{C}^T & \boldsymbol{0} & \boldsymbol{B}_{\phi,\phi} & \boldsymbol{0} \\ \boldsymbol{A}_{p,u} & \phi\boldsymbol{C}^T & \boldsymbol{B}_{p,\phi} & \boldsymbol{0} \end{pmatrix} \tag{291}$$

where each block of the matrix corresponds to one of the bilinear forms defined above, for example $\boldsymbol{A}_{u,u}$ is such that $\boldsymbol{A}_{u,u}\delta\vec{u}\cdot\vec{\zeta} = a_{u,u}(\delta\vec{u},\vec{\zeta})$, etc. This matrix has typical structure of constraint system with zero diagonal block.

6.3. Solution algorithm. The aproach is the same as in the previous problem. Additionally, the continuation method is employed in order to have the starting approximation in the Newton iteration in the range of convergence in this steady computation. In the continuation method the problem $\vec{F}(\vec{X}) = 0$ is replaced by

$$\vec{G}(\vec{X},\lambda) = 0 \tag{292}$$

where λ is a parameter such that for $\vec{G}(\vec{X},0) = 0$ we know the solution, while for $\lambda = 1$ the original problem is recovered

$$\vec{G}(\vec{X},1) \equiv \vec{F}(\vec{X}). \tag{293}$$

For example making the boundary conditions to depend on the parameter λ in such a way that for $\lambda = 0$ we have the undeformed, stress free state, and for $\lambda = 1$ we have the original boundary conditions.

In the process of the continuation method, we follow the solution curve given by the initial value problem

$$\frac{d}{ds}\vec{G}(\vec{X}(s),\lambda(s)) = 0, \tag{294}$$

$$(\vec{X}(0),\lambda(0)) = (\vec{X}_0,0), \tag{295}$$

until the point $\lambda(s) = 1$. The basic method used to solve this problem is the Euler-Newton iteration, outlined in figure 13, where the explicit Euler method is applied to (294) as predictor and then the solution is corrected by the Newton method. This step is repeated until $\lambda^n = 1$. The parameter ω in the update step can be fixed or can be chosen adaptively, for example depending on the number of Newton iterations in the correction step needed to correct the solution.

1. Let $\vec{X}^n$ be given starting approximation and λ^n the value of the continuation parameter.
2. Predictor step. Solve for $(\dot{\vec{X}}^n, \dot{\lambda}^n)$

$$\left[\frac{\partial \vec{G}}{\partial \vec{X}}(\vec{X}^n, \lambda^n)\right] \dot{\vec{X}}^n + \left[\frac{\partial \vec{G}}{\partial \lambda}(\vec{X}^n, \lambda^n)\right] \dot{\lambda}^n = 0, \tag{296}$$

$$\|(\dot{\vec{X}}^n, \dot{\lambda}^n)\| = 1. \tag{297}$$

3. Update the solution $(\vec{X}^{n+\frac{1}{2}}, \lambda^{n+1}) = (\vec{X}^n, \lambda^n) + \omega(\dot{\vec{X}}^n, \dot{\lambda}^n)$.
4. Correction step. Solve for $\vec{X}^{n+1}$ by Newton iteration with $\vec{X}^{n+\frac{1}{2}}$ as starting guess

$$\vec{G}(\vec{X}^{n+1}, \lambda^{n+1}) = 0. \tag{298}$$

FIGURE 13. One step of Euler-Newton algorithm.

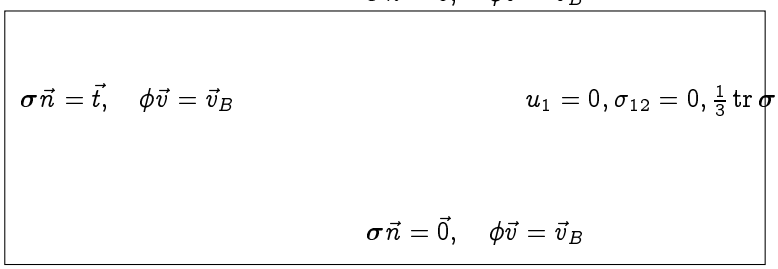

FIGURE 14. Applied boundary condition in the two dimensional problem.

The jacobian matrix is computed via finite differences. To invert the matrices in the most inner loops, BiCGStab or GMRES methods are used with suitable preconditioning. See Barrett et al. [1994], Bramley and Wang [1997] for further details on these methods. The incomplete LU decomposition is used for preconditioning with suitable ordering of unknowns and with allowed fill-in for certain pattern in the zero diagonal block of the jacobian matrix.

6.4. Illustrative example of perfusion. We take two dimensional crossection of the slab along the direction of the perfusion and assume the constitutive relation for the Helmholtz potential

$$\Psi = c_1(I_C - 3) + c_2 \ln(\phi). \tag{299}$$

The boundary conditions applied are shown in figure 14.

In figure 15 is shown the finite element grid on the reference configuration of the solid. The initial solution is taken to be zero displacement and velocity, given constant volume fraction and lagrange multiplier p such that the solution is stress free. The the solution for required values of the boundary conditions is computed by continuation.

In figure 16 we show the finite element grid on the deformed configuration of the solid. We can see that the slab becomes thicker in the X_2 direction at the left end, and thinner at the fluid outflow end. This variation in the thickness is caused by the gradual decrease in the pressure along the fluid flow. Figure 17 shows the

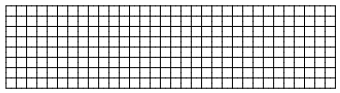

FIGURE 15. Finite element grid on the reference configuration of the solid.

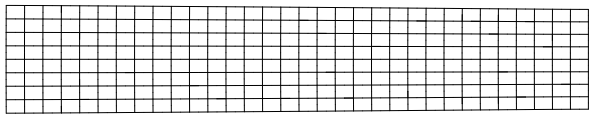

FIGURE 16. Finite element grid on the deformed configuration of the solid.

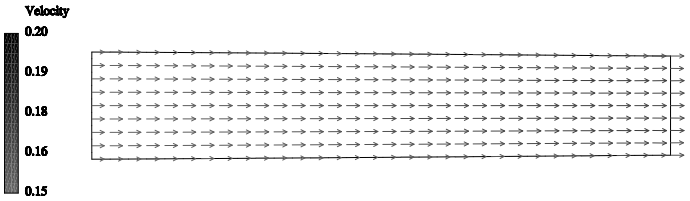

FIGURE 17. Fluid velocity.

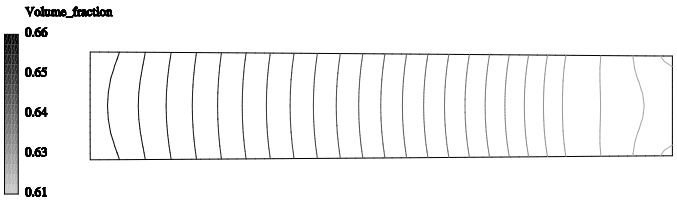

FIGURE 18. Isolines of the fluid volume fraction.

velocity field of the perfusion and 18 shows the fluid volume fraction throughout the slab. As in the problem studied in previous section, the fluid velocity increases toward the end of the slab while the volume fraction decreases. In figure 19 the pressure field is shown and in figure 20 the components of the stress tensor are depicted. We can notice the presence of stress concentration around the corners of the slab where the type of the boundary condition changes.

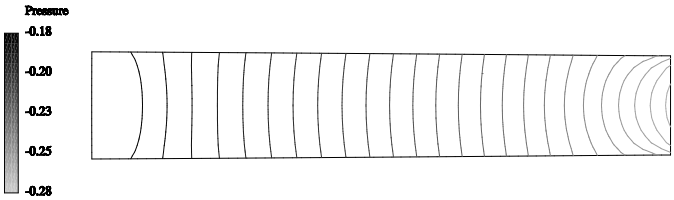

FIGURE 19. Isolines of the pressure field.

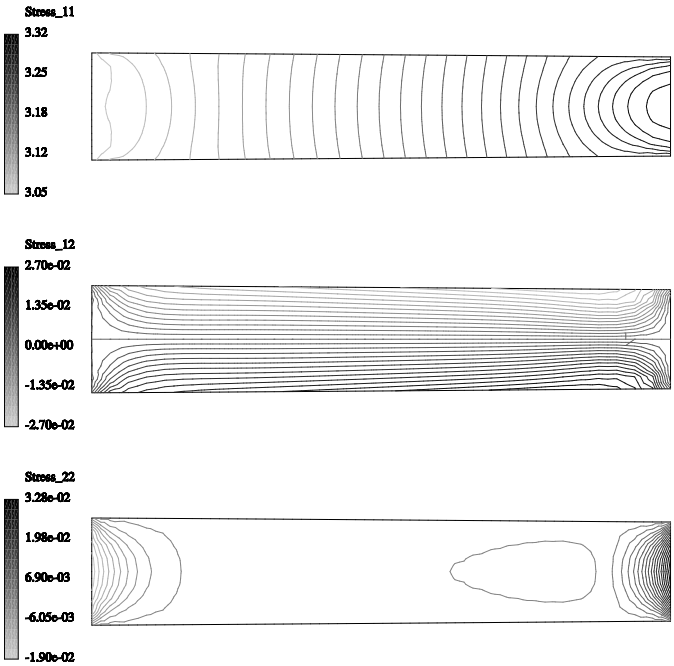

FIGURE 20. Isolines of the stress components $\sigma_{11}, \sigma_{12}, \sigma_{22}$.

CHAPTER 4

Conclusions

In this work we have investigate two applications in biomechanics based on the fluid structure interaction. The main goal was to develop such models which could be solved numerically and suitable for further extension.

In the first part a classical problem of flow in a region surrounded by elastic material. A time dependent problem is solved for three examples. Further development in the formulation part is needed if we wish to be able to capture a collapse of the elastic structure, which is not possible with the present formulation. Another area is the efficiency of the numerical solution which is necessary if we want to extend the model to the three dimensions.

The second part investigates a possibility to use the mixture theory to model hydrated soft tissues with perfusion. The mixture theory was shown to be capable to describe simple finite deformation with simultaneous perfusion of quadrilateral slab. There are still open questions of appropriate constitutive equations and boundary conditions for the constituents and the mixture. One and two dimensional illustrative examples were solve, but further development has to be done to be able to solve more general setting.

Such model of a soft tissue could be used as a part of more complex model of the type discussed in the first part of this work. In both cases a further optimization of the numerical algorithm is needed to be able to solve more complex or three dimensional real-life applications.

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