Flow of Newtonian and Power-Law Fluids Past an Elliptical Cylinder: A Numerical Study

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Extensive numerical simulations of the 2-D laminar flow of power-law fluids over elliptical cylinders with different aspect ratios have been carried out to establish the conditions for the onset of wake formation and the onset of vortex shedding. The continuity and momentum equations were solved numerically using FLUENT (version 6.3.26). The influence of the power-law index ($0.3 \leq n \leq 1.8$) and the aspect ratio ($E = b/a; 0.2 \leq E \leq 5$) of the cylinder on the critical values of the Reynolds number denoting the onsets of flow separation and vortex shedding are presented. For shear-thinning ($n < 1$) fluid behavior, the onsets of wake formation and vortex shedding are both seen to be postponed to higher Reynolds numbers as compared to those in shear-thickening fluids ($n > 1$). Also, the values of the Strouhal number ($St_e$) and the time-average drag coefficient ($C_D$) corresponding to the cessation of the steady-flow regime are presented. Velocity vector plots denoting the flow separation and vorticity profiles showing the vortex shedding are also included. The delineation of different flow regimes also helps identify the range of validity of some of the results on flow and heat transfer available in the literature.

1. Introduction

The flow of fluids past cylinders of various cross sections represents an idealization of several industrially important applications. Typical examples include the flow in tubular and pin-type heat exchangers; filtration screens used for clarifying suspensions, sludges, and polymer melts; membrane-based separation modules; support structures exposed to flow streams of liquids; formation of weld lines in polymer processing; and thermal processing of variously shaped food particles. In addition to the foregoing pragmatic significance, there is an intrinsic theoretical interest in such model flows to further our understanding of the underlying physical processes. Therefore, bluff body flows constitute an important class of problems within the domain of fluid mechanics. Consequently, over the past 100 years or so, a voluminous body of information encompassing wide-ranging phenomena associated with flow past a cylinder has accrued; see, for example, refs 1–3. A quick inspection of these reviews clearly shows that the bulk of the literature relates to the flow of Newtonian fluids past a circular cylinder. Cylinders of noncircular cross sections, notably elliptical, are now receiving increasing attention because of the enhanced behavior index, the flow can remain attached to the surface of the unconfined circular cylinder, the transitions between different flow regimes are adequately described in terms of the value of the Reynolds number. The corresponding values of the critical Reynolds numbers for the flow of power-law fluids have been reported only recently.9 Depending on the values of the flow behavior index, the flow can remain attached to the surface of the cylinder; for example, it does not separate up to about $Re = 6$, and a visible wake appears at $Re = 6.5$, although the flow is still steady and laminar.9 With a gradual increase in the value of the Reynolds number, the wake region grows and ultimately becomes asymmetric and periodic in time at $Re \approx 45–50$, thereby leading to the onset of vortex shedding. Thus, for an unconfined circular cylinder, the transitions between different flow regimes are adequately described in terms of the value of the Reynolds number. The corresponding values of the critical Reynolds numbers for the flow of power-law fluids have been reported only recently.9 Depending on the values of the flow behavior index, the flow can remain attached to the surface of the cylinder; for example, it does not separate up to about $Re \approx 11–12$ in a highly shear-thinning fluid, whereas the flow transits to the periodic-flow regime at $Re \approx 33–34$ in a highly shear-thickening fluid. It is somewhat surprising that very little analogous information is available for elliptical cylinders, even in Newtonian fluids, let alone in power-law fluids.

This work thus aims to fill this gap in the current literature. The delineation of the different flow regimes directly influences the choice of numerics, such as whether to use a steady or an unsteady solver. In particular, reported herein are the values of the critical Reynolds numbers denoting the onset of the separation of flow and the onset of vortex shedding as functions of the power-law index and aspect ratio of the cylinder. However, prior to undertaking a detailed presentation of the new results, it is instructive to summarize the salient features of the scant literature available on the flow of Newtonian and...
power-law fluids over elliptical cylinders, which, in turn, facilitates the presentation of new results obtained in this work.

2. Previous Work

The flow of Newtonian fluids over elliptical cylinders has received much less attention than that accorded to the case of a circular cylinder. Early studies\textsuperscript{10–13} were based on the application of matched asymptotics to obtain analytical solution of Oseen’s linearized equation that led to the prediction of velocity and pressure fields close to the elliptical cylinder. Although it is impossible to delineate the range of validity of these solutions in a rigorous fashion, these analyses are believed to be useful up to about $Re \approx 1–2$. The next generation of solutions was based on numerical solutions of the Navier–Stokes equations. For instance, Epstein and Masliyah\textsuperscript{14} presented the steady-state results up to $Re = 90$ for elliptical cylinders with aspect ratios ($E = b/a$) ranging from 0.2 to 1. Meller\textsuperscript{15} numerically solved the elliptical transformation of the vorticity–stream function formulation of the Navier–Stokes equations using the finite difference method and matrix pivotal condensation method for two values of the Reynolds numbers (i.e., $Re = 20$ and 40) and for an elliptical cylinder with an aspect ratio of 10 at an angle of incidence of 30°. D’Alessio and Dennis\textsuperscript{16} presented drag and lift coefficients for $Re = 5$ and 20 for different inclinations of the elliptical cylinder ranging from 0° to 90°. Subsequently, Dennis and Young\textsuperscript{17} studied the impact of the angle of incidence (0°–90°) on the flow characteristics, but their results were limited to the so-called steady-flow regime, 1 ≤ $Re$ ≤ 40, and/or for a single value of the aspect ratio of cylinder, namely, 5. Notwithstanding the differences arising from the different numerics employed, broadly speaking, the values of the drag coefficient and pressure distribution on the surface of the cylinder are consistent with each other for Newtonian fluids.

The unsteady flow of an abruptly started elliptical cylinder at 45° incidence was studied by Lugt and Haussling.\textsuperscript{18} The major thrust of their study was the prediction of the time required for the flow to reach steady or quasisteady state by following the evolution of streamlines and drag and lift coefficients with time. Subsequently, Patel\textsuperscript{19} revisited this flow configuration for a range of angles of incidence (0°, 30°, 45°, 90°) for a fixed value of $Re = 200$. Using the spectral element method, Johnson et al.\textsuperscript{20} investigated the time-dependent flow over an elliptical cylinder over the range of conditions characterized by 30 ≤ $Re$ ≤ 200 and for aspect ratios varying from 0.01 (vertical thin plate) to 1 (circular cylinder). They reported the critical values of the Reynolds number denoting the cessation of the steady-flow regime to be only weakly dependent on the aspect ratio of the cylinder over this range of aspect ratios. For aspect ratios less than 1, the steady-flow regime was observed to persist up to about $Re \approx 35–45$, which is obviously due to the reduced streamlining of the cylinder, albeit this value is only slightly lower than the oft-quoted value of 46–48 for a circular cylinder. This clearly casts some doubts on the validity of the steady-flow assumption up to $Re = 90$ in an earlier study.\textsuperscript{14} Unsteady flow over an elliptical cylinder (aspect ratios 0.5 and 0.6) up to $Re \approx 500$ was investigated by Ahmad and Badr.\textsuperscript{21} They examined the unsteadiness arising from two sources, namely, the fluctuating imposed flow and the vortex-shedding phenomenon. Aside from the aforementioned studies, which utilized the complete Navier–Stokes equations, Khan et al.\textsuperscript{22} employed the boundary-layer approximation to develop closed-form analytic expressions for skin friction and heat transfer from an elliptical cylinder. It is fair to say that, with the exception of the work of Johnson et al.,\textsuperscript{20} none of the studies mentioned above addressed the delineation of flow regimes for the flow past an unconfined elliptical cylinder. However, there have been a few studies that have attempted to elucidate the influence of the aspect ratio of the cylinder on the transition from the steady-flow regime to a time-dependent periodic-flow regime. For instance, Jackson\textsuperscript{23} carried out a finite-element-based numerical study to delineate the transitional Reynolds number denoting the onset of the vortex-shedding flow regime. In particular, for a cylinder with an aspect ratio of 0.5, he reported the critical value of the Reynolds number to vary from 35.7 to 141 as the angle of orientation of cylinder to flow changed from 0° to 90°. Similarly, Johnson et al.\textsuperscript{20} reported on the various types of shedding patterns observed with elliptical cylinders with aspect ratios less than 1. More recently, Faruqueet al.\textsuperscript{24} investigated the first transition denoting the onset of flow separation for elliptical cylinders. They fixed the Reynolds number (based on the hydraulic diameter of the cylinder) at 40 and found that the first signature of flow separation occurred at about the aspect ratio of 0.34. Based on their\textsuperscript{25} numerical results, they also presented an empirical relation between the wake length and the aspect ratio at a fixed value of Reynolds number, and the wake size was seen to increase with increasing aspect ratio (≤ 1). This issue was investigated in a more detailed manner by Stack and Bravo,\textsuperscript{25} who studied the onset of flow separation for a flat plate (aspect ratio of 0), circular cylinder (aspect ratio of 1) using a cellular automata model and FLUENT. For fixed values of the Reynolds number, they varied the value of the aspect ratio to locate the onset of flow separation. All in all, it is thus fair to say that only limited information is available on the first two transitions, namely, flow separation/no separation and steady/time-dependent periodic-flow regimes for elliptical cylinders even in Newtonian fluids.

In contrast, the analogous body of knowledge for power-law fluids is not only very limited but also of recent vintage even for circular cylinders. In view of the available information on the critical Reynolds numbers corresponding to the first two transitions,\textsuperscript{9} suffice it to say here that reliable results on drag and wake characteristics are now available on the flow of power-law fluids past a circular cylinder in the steady-flow regime.\textsuperscript{26–30} The role of planar confining walls has been examined by Bharti et al.\textsuperscript{31} Limited results are also available on convective heat transfer from a circular cylinder submerged in power-law fluids in the steady-flow regime.\textsuperscript{32–34} Broadly, all else being equal, shear-thinning fluid behavior promotes heat transfer, and shear thickening has deleterious effect on it.\textsuperscript{32–34} Further enhancements in heat-transfer rates are possible in the presence of buoyancy effects.\textsuperscript{33,34} More recently, Patnana et al.\textsuperscript{35,36} carried out a numerical study in the periodic-flow regime to elucidate the role of power-law rheology in the vortex-shedding and heat-transfer characteristics of a cylinder in power-law fluids. In summary, whereas a reasonable body of knowledge is now available on the momentum and heat-transfer characteristics for a circular cylinder in power-law fluids in the steady-flow regime, only limited information is available in the time-dependent periodic-flow regime.

Finally, to the best of our knowledge, only two previous studies have been reported on the flow of power-law fluids over elliptical cylinders.\textsuperscript{37,38} Sivakumar et al.\textsuperscript{37} and Bharti et al.\textsuperscript{38} presented extensive results on flow (drag, wake size, surface pressure, and vorticity profiles) and heat-transfer (Nusselt number) over the range of parameters expressed as 0.01 ≤ $Re$ ≤ 40, 0.2 ≤ $E$ ≤ 5, and 0.2 ≤ $n$ ≤ 1.8, thereby covering both shear-thinning and shear-thickening fluid behaviors. The role
of the power-law index becomes accentuated depending on the value of the aspect ratio of the cylinder. However, the flow was assumed to be steady a priori over these ranges of conditions, and this is indeed the main weakness of their work. Therefore, the utility of their results rests on the validity of this assumption, which necessitates the delineation of the critical Reynolds numbers marking the limit of the steady-flow regime for various values of aspect ratio and power-law index.

Based on the preceding discussion, it is thus safe to conclude that only limited information is available on the transitional Reynolds numbers for elliptical cylinders even in Newtonian media and no prior results are available for power-law fluids. This work reports on the critical values of the Reynolds number denoting the onsets of flow separation and vortex shedding for an elliptical cylinder over a wide range of power-law indices (0.3 ≤ n ≤ 1.8) and cylinder aspect ratios (0.2 ≤ E ≤ 5). This information is useful not only in its own right, but also to ascertain the range of applicability of the currently available results in the literature.

3. Problem Statement and Governing Equations

Let us consider the two-dimensional (2-D) unconfined flow of an incompressible power-law fluid (uniform approach velocity \(U_\infty\)) over a long elliptical cylinder of aspect ratio \(E\) oriented normal to the oncoming uniform flow. Here, the aspect ratio of the cylinder is defined as the ratio of the length of the axis parallel to the flow to the length of the axis perpendicular to the flow. Thus, for cases with \(E > 1\), it becomes the major axis divided by the minor axis, whereas for cases with \(E < 1\), it is the minor axis divided by the major axis. The unconfined flow condition is approximated here by enclosing the elliptical cylinder in a large circular outer boundary of diameter \(D_\infty\), as shown schematically in Figure 1. A prudent choice of \(D_\infty\) is by necessity a tradeoff between the computational effort (which increases rapidly with increasing value of \(D_\infty\)) and the extent of boundary effects on the accuracy of the results (which deteriorates with decreasing value of \(D_\infty\)).

The continuity and momentum equations for this flow are written as

Continuity equation
\[
\nabla \cdot \mathbf{U} = 0
\]

Momentum equation
\[
\rho \left( \frac{D\mathbf{U}}{Dt} - \mathbf{f} \right) - \nabla \cdot \mathbf{\sigma} = 0
\]

where \(\rho\), \(\mathbf{U}\), \(\mathbf{f}\), and \(\mathbf{\sigma}\) are the density, velocity vector, body force, and stress tensor, respectively. For incompressible fluids, the stress tensor is made up of two components, the isotropic pressure and the extra stress tensor \(\mathbf{\tau}\), i.e.
\[ \sigma = -\rho I + \tau \] (3)

The extra stress tensor, \( \tau \), is, in turn, related to the rate of deformation tensor \( \epsilon(u) \) through the scalar viscosity of the fluid as

\[ \tau = 2\mu \epsilon(u) \] (4)

where \( \epsilon(u) \), the rate of deformation tensor, is related to the velocity field as

\[ \epsilon(u) = \frac{1}{2} \left[ \nabla U + (\nabla U)^T \right] \] (5)

and the power-law viscosity is given by

\[ \eta = m \left( \frac{I_2}{2} \right)^{(n-1)/2} \] (6)

where \( I_2 \) is the second invariant of the rate of deformation tensor and \( n \) is the flow behavior index. Evidently, \( n < 1 \) denotes shear-thinning behavior, \( n > 1 \) corresponds to shear-thickening behavior, and \( n = 1 \) represents the standard Newtonian fluid behavior. The expressions for \( I_2 \) in terms of velocity and velocity gradients are available in standard text books; for example, see Bird et al.39

To complete the problem statement, appropriate boundary conditions at the inlet and outlet boundaries, which are located at the front half and rear half, respectively, of the computational domain (Figure 1A), and at the cylinder wall must be specified. For this flow, these conditions are written as follows:

1. At the inlet boundary, the condition of uniform velocity in the x direction is prescribed by

\[ U_x = U_{\infty}; U_y = 0 \] (7)

2. On the surface of the cylinder, because it is a solid surface, the standard no-slip condition is used, that is

\[ U_x = 0; U_y = 0 \] (8)

3. At the exit boundary, the default outflow boundary condition in FLUENT is used, which assumes a zero diffusion flux for all flow variables (i.e., \( \partial \phi / \partial x = 0 \), where \( \phi \) is a scalar variable such as \( U_x \) or \( U_y \)), and this is similar to the fully developed flow condition. It is important to note here that gradients in the cross-stream direction might still exist at the outflow boundary. Because the flow becomes steady at the exit boundary as a result of the dissipation of vortices at the outer domain, this condition can be prescribed for both the steady and vortex-shedding flow regimes near the cylinder as

\[ \frac{\partial U_j}{\partial x} = 0 \quad \text{and} \quad \frac{\partial U_l}{\partial x} = 0 \] (9)

4. At the plane of symmetry \(( y = 0 \), prior to the onset of the vortex-shedding regime, the flow is symmetric about the midplane. Hence, it is sufficient to use half of the flow domain for steady simulations for estimating the critical Reynolds number for the onset of flow separation, namely

\[ \frac{\partial U_j}{\partial y} = 0 \quad \text{and} \quad U_j = 0 \] (10)

The numerical solution of eqs 1 and 2 subject to the boundary conditions in eqs 7–10 maps the flow domain in terms of the velocity and pressure fields, and these, in turn, are used to compute global characteristics such as the drag and lift coefficients and Strouhal number.

At low Reynolds numbers, the flow is expected to be steady and symmetric, and the scaling considerations suggest that the flow is governed by four dimensionless parameters, namely, aspect ratio, power-law index, Reynolds number, and drag coefficient, with both lift coefficients and Strouhal number being zero under these conditions. In the time-dependent periodic-flow regime, the Strouhal number and lift coefficient achieve nonzero values and are likely to be functions of the aspect ratio, power-law index, and Reynolds number. At this juncture, it is customary to introduce the definitions of the aforementioned dimensionless groups, which will be used extensively in the discussion of the results presented in the subsequent sections.

**Reynolds Number.**

\[ Re = \frac{\rho U_{\infty}^2 - n(2a)^2}{\eta m} \] (11)

The two critical Reynolds numbers denoting the onsets of flow separation and vortex shedding are denoted as \( Re^s \) and \( Re_c \), respectively. In the present work, the first transition (i.e., \( Re^s \)) was located by comparing the values of the dimensionless stream function, vorticity, and velocity vector near the surface of the bluff body to check for the onset of flow separation. Typical values of both the dimensionless stream function \([ \equiv \psi/\psi_{\infty}(2a) ] \) and vorticity \([ \equiv \omega/\psi_{\infty}(2a) ] \) near the cylinder are on the order of \( 10^{-3} \) to \( 10^{-8} \).

The other critical Reynolds number, \( Re_c \), denotes the cessation of the steady-flow regime, and this point was located by examining the lift coefficient–time history. For steady flow, the magnitude of the lift coefficient will gradually decrease with time, ultimately becoming zero or very small. In this work, the cutoff value of the lift coefficient was on the order of \( 10^{-4} \). This transition was further substantiated by examining the asymmetry of the wake about the center line, \( y = 0 \). Thus, in the range \( Re < Re_c \), the flow is steady with two symmetric vortices, and for \( Re < Re^s \), the flow remains attached to the surface of the bluff body.

**Drag and Lift Coefficients.** The drag and lift coefficients are measures of the forces acting on the cylinder in the direction of flow (drag) and normal to flow (lift), respectively. They are defined as

\[ C_D = \frac{F_D}{(1/2)\rho U_{\infty}^2(2a)} \] (12)

\[ C_L = \frac{F_L}{(1/2)\rho U_{\infty}^2(2b)} \] (13)

where \( F_D \) and \( F_L \) are the drag and lift forces per unit length of the cylinder, respectively.

**Strouhal Number.** The Strouhal number is a measure of the frequency of vortex shedding and is defined as

\[ St = \frac{2af}{U_{\infty}} \] (14)

where \( f \) is the frequency of vortex shedding and its value was extracted from the lift coefficient–time behavior as a function of the power-law index.

**4. Numerical Solution Methodology.**

The present study was carried out using FLUENT (version 6.3.26). The non-orthogonal “quadrilateral” cells of nonuniform
grid spacing were generated using the commercial grid tool GAMBIT (version 2.3.16). At low Reynolds numbers, the flow is expected to be symmetric about the midplane, and the solution was sought only in a half-domain \((y \geq 0)\) for identifying the point of flow separation. On the other hand, full-domain computations were carried out to delineate the transition from the steady symmetric to time-periodic vortex-shedding regime. The two-dimensional, laminar, segregated solver was used to solve the incompressible flow on the collocated grid arrangement. The unsteady solver was used for the vortex-shedding case \((Re > Re_c)\), whereas the steady solver was used for locating the flow separation \((Re \approx Re^\infty)\) case. The second-order upwind scheme was used to discretize the convective terms in the momentum equations. The semi-implicit method for pressure-linked equations (SIMPLE) scheme was used for solving the pressure–velocity coupling. Second-order time discretization together with an implicit scheme was used for time integration in this work. The “constant-density” and “non-Newtonian power-law” viscosity models were used to input the physical properties of the fluid. This is, however, of no particular significance, as the final results are presented in a dimensionless form. FLUENT solves the system of algebraic equations using the Gauss–Siedel \((G–S)\) point-by-point iterative method in conjunction with the algebraic multigrid (AMG) method solver. The use of the AMG scheme can greatly reduce the number of iterations (thereby accelerating convergence) and thus reduces the CPU time required to obtain a converged solution, particularly when the model equations contain a large number of control volumes. The relative convergence criteria of \(10^{-10}\) for the continuity and \(x\) and \(y\) components of the momentum equations were prescribed in this work. Also, the solution was considered to have converged when there was no change (at least up to the fourth decimal place) in the total drag coefficient and the corresponding changes in the value of the lift coefficient (for the vortex-shedding case) were on the order of \(10^{-5}–10^{-6}\) for more than 1000 time steps, or when it showed more than 10 constant periodic cycles in the time history of the lift and drag coefficients.

5. Choice of Numerical Parameters

Undoubtedly, the accuracy and reliability of the numerical results is strongly influenced by the choice of numerical parameters, namely, grid, domain size and time step, convergence criterion, and so on. Although all these issues were addressed in detail in our previous studies, a short discussion of the salient features is included here.

5.1. Domain Size. It is well-known that the choices of domain size and grid size exert varying levels of influence on the accuracy of numerical results. Because unconfined flow was simulated in the current study, it was physically not possible to consider infinite dimensions. Therefore, computations were carried out by considering a domain of finite size, so it was necessary to obtain the size of the domain/grid that upon further increment/refinement produced negligible changes in the numerical results.

Intuitively, it appears that the smaller the value of the Reynolds number, the larger the value of the diameter of the circular domain, \(D_\infty\), required to minimize the boundary effects. This is obviously so because the boundary layer is very thick at low Reynolds numbers and thins with increasing Reynolds number. Bearing this fact in mind, several values of \(D_\infty/(2a)\) ranging from 150 to 1200 were used in this study to choose an optimal domain. A summary of representative results showing the effect of domain size (value of \(D_\infty\)) on the value of the drag coefficient is presented in Table 1 for a range of combinations of the values of the power-law index \((n)\), aspect ratio \((E)\), and Reynolds number \((Re)\). An inspection of this table and of the other results not shown here reveals that, for \(Re < 5\), the domain size of \(D_\infty/(2a) = 1200\) is quite adequate for the results to be free from boundary effects. Similarly, the value of \(D_\infty/(2a) = 300\) is believed to be satisfactory for \(Re \geq 5\). In fact, whereas the drag values for \(n = 0.3\) and \(n = 1.8\) changed by less than 1.5% upon increasing the value of \(D_\infty/(2a)\) from 1000 to 1200 for \(Re = 0.001\) and \(E = 0.2\), the corresponding change for \(n = 1\) was about 2.9%. These small changes undoubtedly are accompanied by a several-fold increase in CPU time. Whereas the choice of \(D_\infty/(2a) = 1200\) for \(Re < 5\) is in line with the study of Sivakumar et al., clearly for \(Re \geq 5\), the value of \(D_\infty/(2a) = 300\) used here is larger than the value of 200 used in previous studies. It is likely that a shorter domain \((for Re < 5)\) would have sufficed, but only along with a much finer grid than that used here. Therefore, the choice of \(D_\infty\) is somewhat linked to the choice of grid as well. Thus, \(D_\infty/(2a) = 1200\) used here permits the use of a slightly coarse mesh, thereby leading to some reduction in CPU time. Moreover, this choice also maintains consistency with our previous study.

5.2. Grid Independence Study. Having fixed the domain size, a grid independence study was carried out using three

| Table 1. Effect of Domain Size on Drag Coefficient for Elliptical Cylinders of Different Aspect Ratios |
|---|---|---|---|---|---|---|
| domain size \(D_\infty/(2a)\) | \(n = 0.3\) | \(n = 1.0\) | \(n = 1.8\) | \(n = 0.3\) | \(n = 1.0\) | \(n = 1.8\) |
| \(E = 0.2, Re = 0.001\) | | | | | | |
| 800 | 22316.427 | 3321.2450 | 135.92970 | 23800.961 | 4130.8649 | 180.71245 |
| 1000 | 22316.414 | 3445.1481 | 132.18359 | 23800.960 | 3982.3009 | 175.73176 |
| 1200 | 22228.392 | 3215.5381 | 129.72411 | 23788.840 | 3868.5038 | 172.94004 |
| \(E = 0.2, Re = 5\) | | | | | | |
| 100 | 4.95522 | 3.21107 | 2.47850 | 5.32413 | 3.84600 | 3.25199 |
| 200 | 4.95548 | 3.17827 | 2.44152 | 5.32439 | 3.80723 | 3.20535 |
| 300 | 4.95547 | 3.16763 | 2.43007 | 5.32438 | 3.79465 | 3.18853 |
| \(E = 2, Re = 1\) | | | | | | |
| \(E = 5,Re = 5\) | | | | | | |
| 150 | 2.31150 | 1.02009 | 0.73787 | 8.29242 | 5.12557 | 3.74132 |
| 200 | 2.31183 | 1.01478 | 0.73301 | 8.29299 | 5.10114 | 3.71728 |
| 300 | 2.31199 | 1.00964 | 0.72840 | 8.29329 | 5.07750 | 3.69467 |

nonuniform grids (G1, G2, and G3) for three values of power-law index ($n$) and a range of values of aspect ratio ($E$) and Reynolds number ($Re$). A schematic representation of the nonuniform grid structure used in the present work is shown in Figure 1B. The grid was divided into four separate zones, and nonuniform grid distributions were employed. In the first zone (radius of 20$a$), the grid distribution was made fine around the cylinder to adequately capture the flow separation and the vortex-shedding phenomenon. The other three zones were made up of relatively coarse cells, and the degree of coarseness increased toward the outer zone where the flow approached the free streamflow and gradients were therefore expected to be small as compared to that near the cylinder. Each grid was characterized in terms of the total number of cells ($N$) and the number of points on the surface of the cylinder ($N_p$). A summary of representative results is presented in Table 2. An inspection of Table 2 clearly shows that a single grid would not be satisfactory for all values of $E$ and/or Reynolds number. Based on a detailed analysis of these results, the following grids were used here to obtain the final results: For $E < 1$, grid G2 for $Re \geq 5$ and grid G3 for $Re < 5$; for $E = 2$, grid G2 for $Re \geq 1$; and for $E = 5$, grid G1 for $Re \geq 5$. These choices are also in line with the grids used in our previous studies. 

### 5.3. Effect of Time Step

Evidently, the solution of time-dependent equations is required to delineate the value of $Re_c$, denoting the onset of the time-dependent flow regime, and therefore, the choice of an appropriate time step ($\Delta T$) also influences the accuracy of numerical results. The characteristic time (time period of oscillations) of the primary flow is given by $T = (2a)/U_{\infty}$, and the time step is defined as $\Delta T = T/N_p$, where $N_p$ is the number of time steps per cycle. From Table 3, it can be seen that nearly identical results are obtained for the two vastly different time step values of 0.001 and 0.01 s. In this work, the choice of the time step was varied in this range depending on the level of convergence of the flow field. In this work, the time-marching numerical scheme was used with the initial velocity profile corresponding to higher Reynolds number and/or using a steady-flow field to initiate the calculations. In experimental work, the flow can be destabilized by several factors including nonuniform flow conditions, vibrations due to mechanical devices such as a pump or a compressor, irregularities of the boundary, surface roughness, and so on. Clearly, all of these factors are not present in numerical simulations. On the other hand, even when the initial and boundary conditions are symmetric, the truncation, roundoff, and discretization errors can act as destabilizing factors that ultimately destabilize the flow, thereby leading to the vortex-shedding regime under appropriate conditions of the pertinent governing factors such as power-law index, aspect ratio, and Reynolds number. Naturally, this approach requires a significantly large time for these errors to grow to the extent that will ultimately lead to the onset of time-dependent periodic-flow regime. To obviate this difficulty, the flow can be perturbed artificially, which accelerates the establishment of the time-dependent flow regime, without, of course, influencing the ultimate transitional values of the Reynolds and Strouhal number. This technique was used by Braza et al. and subsequently by others relating to the case of a circular cylinder. Although it is relatively straightforward to perturb the flow by rotating a circular cylinder for a short period, the relevance of this method is less clear for an elliptical cylinder. Therefore, in the present study, no external stimulus was used to accelerate the establishment of the fully periodic-flow regime. In other words, time-dependent calculations were continued for a long time until either the flow became steady or fully periodic in time. The time step was progressively increased from the initial value of 0.001 to 0.01 s as the solution approached the desired level of convergence. The simulations were carried out until the time history of lift coefficients either died out or reached steady periodic oscillations. The critical values of the parameters Reynolds number, $Re_c$; Strouhal number, $St_c$; and drag coefficients at the critical conditions are presented in the next section.

All computations were performed on Quad Core PCs (2.83 GHz, 3 GB RAM) operating in Windows. A typical run on a
Table 4. Steady Newtonian Flow over an Elliptical Cylinder

<table>
<thead>
<tr>
<th>source</th>
<th>$C_D$ at $Re = 0.01$</th>
<th>$C_D$ at $Re = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>present value</td>
<td>400.6309</td>
<td>3.7820</td>
</tr>
<tr>
<td>Sivakumar et al.</td>
<td>404.5256</td>
<td>3.7900</td>
</tr>
<tr>
<td>Dennis and Young</td>
<td>3.8540</td>
<td></td>
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<tr>
<td>D’Alessio and Dennis</td>
<td>3.8620</td>
<td></td>
</tr>
<tr>
<td>$E = 0.2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>present value</td>
<td>471.0733</td>
<td>5.0710</td>
</tr>
<tr>
<td>Sivakumar et al.</td>
<td>471.7567</td>
<td>5.0715</td>
</tr>
</tbody>
</table>

Table 5. Steady Power-Law Fluid Flow over an Elliptical Cylinder

<table>
<thead>
<tr>
<th>source</th>
<th>$C_D$ at $n = 0.6$</th>
<th>$C_D$ at $n = 1.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>present value</td>
<td>5.1008</td>
<td>3.3448</td>
</tr>
<tr>
<td>Sivakumar et al.</td>
<td>5.1173</td>
<td>3.3586</td>
</tr>
<tr>
<td>$E = 2$</td>
<td>1.3209</td>
<td>1.6507</td>
</tr>
<tr>
<td>present value</td>
<td>6.5487</td>
<td>3.7180</td>
</tr>
<tr>
<td>Sivakumar et al.</td>
<td>6.5021</td>
<td>3.6928</td>
</tr>
<tr>
<td>$E = 5$</td>
<td>1.3450</td>
<td>1.7061</td>
</tr>
</tbody>
</table>

Table 6. Critical Reynolds Number for Wake Formation for Newtonian Flow over an Elliptical Cylinder

<table>
<thead>
<tr>
<th>source</th>
<th>$E$</th>
<th>$Re_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>present value</td>
<td>0.2</td>
<td>0.85</td>
</tr>
<tr>
<td>Stack and Bravo</td>
<td>0.5</td>
<td>3.0</td>
</tr>
</tbody>
</table>

Table 7. Critical Reynolds Number for Onset of Vortex Shedding for Newtonian Flow over an Elliptical Cylinder

<table>
<thead>
<tr>
<th>source</th>
<th>$E$</th>
<th>$Re_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>present value</td>
<td>0.5</td>
<td>40</td>
</tr>
<tr>
<td>Jackson</td>
<td>0.5</td>
<td>35.704</td>
</tr>
<tr>
<td>Johnson et al.</td>
<td>0.5</td>
<td>42.5–45</td>
</tr>
<tr>
<td>present value</td>
<td>1</td>
<td>48</td>
</tr>
<tr>
<td>Sivakumar et al.</td>
<td>1</td>
<td>47</td>
</tr>
<tr>
<td>Jackson</td>
<td>1</td>
<td>45.403</td>
</tr>
<tr>
<td>Norberg</td>
<td>1</td>
<td>47.5(±0.5)</td>
</tr>
<tr>
<td>Williamson</td>
<td>1</td>
<td>47.9</td>
</tr>
<tr>
<td>Kumar and Mittal</td>
<td>1</td>
<td>47.336</td>
</tr>
<tr>
<td>Morzynski et al.</td>
<td>1</td>
<td>47.0</td>
</tr>
<tr>
<td>present value</td>
<td>2</td>
<td>78</td>
</tr>
<tr>
<td>Jackson</td>
<td>2</td>
<td>76.794</td>
</tr>
</tbody>
</table>

grid with 48750 cells for $n = 0.3$ and $E = 0.2$ required around 14 days of computing to obtain 10 stable cycles of vortex shedding.

6. Results and Discussion

6.1. Validation of Results. Prior to presenting the new results on the delineation of the critical Reynolds numbers, it is worthwhile to validate the choices of numerical parameters and numerical methodology employed here. Over the years, extensive numerical results have been reported in the literature on the values of drag coefficients for the steady 2-D flow of Newtonian fluids over circular and elliptical cylinders. Suffice to say here that the present values for a circular cylinder are within 3% of that reported previously by several authors34,37,41–45 for $Re = 1$, 5, and 40. Table 4 shows comparisons between the present and literature values for a range of values of $E$ and $Re$. Excellent correspondence is seen to exist between the present and literature values. Furthermore, for $E = 5$, the present results (not shown here) are within 1% of those reported by Dennis and Young17 and D’Alessio and Dennis16 for $Re = 5$ and 40. Table 5 presents a similar comparison for the steady flow of power-law fluids at $Re = 5$ and 40 where, once again, good match is seen to exist. Also, the present numerical solution procedure was validated by comparing the critical Reynolds numbers in Newtonian fluids for the onset of wake formation (Table 6) and the onset of vortex shedding (Table 7) for elliptical cylinders of different aspect ratios. Once again, the agreement with the literature values is seen to be good. Such a close correspondence between the present and literature values over wide ranges of conditions inspires confidence and lends credibility to the reliability and accuracy of the results reported herein. Suffice it to add here that the order of differences seen in Tables 4–7 are not at all uncommon in numerical studies and are often attributed to differences in grid, domain, and solution methods used in different studies.51

6.2. Onset of Wake Formation. The appearance of wakes was ascertained here by the inspection of the computed velocity vector field at the critical and nearby values of the Reynolds number. When the flow remains attached to the surface of the cylinder, velocity vectors in the proximity of the cylinder all point in the downstream direction; the onset of wake formation manifests itself by the fact that some velocity vectors near the cylinder point in the upstream direction and form a close recirculating flow region. Representative results on the streamline patterns and velocity vectors in the vicinity of cylinder for $n = 0.3$ and $n = 1.6$ at two closeby values of Reynolds number that bracket the transition point are shown in Figure 2A,B. It can be seen in this figure that, at the smaller value of the Reynolds number (left column), no wake forms, and all velocity vectors near the cylinder (not shown here) point in the downstream direction, whereas at the higher value of the Reynolds number (right column), some of the velocity vectors are oriented in the direction opposite to the main flow, which denotes flow separation and the formation of a wake. For instance, in Figure 2A, for $n = 0.3$ and $E = 0.2$, the flow remains attached to the surface of the cylinder at $Re = 1.4$, whereas at $Re = 1.5$, separation has occurred in the rear of the cylinder. Thus, for $n = 0.3$, the critical Reynolds number ($Re_c$) lies in the range $1.4 \leq Re \leq 1.5$ for an elliptical cylinder of aspect ratio $0.2$. This approach was used to ascertain the interval for flow separation for the other values of the power-law index and aspect ratio. The resulting values of the critical Reynolds number for $E = 0.5$, 2, and 5 are in the intervals $6.4 \leq Re \leq 6.6$, $32 \leq Re \leq 33$, and $98 \leq Re \leq 99$, respectively. As expected, the critical Reynolds number increases with increasing degree of streamlining of the bluff body (i.e., with increasing value of $E$). The ranges of critical Reynolds numbers for all values of power-law index, $n$, and aspect ratio, $E$, considered herein are summarized in Table 8 and shown in Figure 3. In this figure, the top and bottom lines represent the most ($E = 5$) and least ($E = 0.2$) streamlined shapes, respectively. As expected, bluntness of the cylinder causes wake formation at lower values of Reynolds number for both shear-thinning and shear-thickening fluids. The flow separation that causes wake formation behind the cylinder occurs at higher Reynolds number in shear-thinning fluids and continues to decrease as the power-law index increases for Newtonian and shear-thickening fluids.
fluids. This form of dependence of the critical Reynolds number on the power-law index for elliptical cylinders of aspect ratio \( E = 2 \) and 5 is similar to that seen for a circular cylinder. For a fixed shape of the cylinder (i.e., for a given value of the aspect ratio), the dependence of the critical Reynolds number on the flow behavior index can be explained as follows: A fluid element experiences the maximum rate of deformation near the bluff body, and it decays as one

Figure 2. Streamline patterns near elliptical cylinders with aspect ratios of 0.2, 0.5, 2, and 5 at low Reynolds number values for (A) \( n = 0.3 \) and (B) \( n = 1.6 \).
Table 8. Effect of Power-Law Index on the Critical Reynolds Number ($Re^c$) for Flow with and without Wake Formation

<table>
<thead>
<tr>
<th>$E$</th>
<th>$Re$ (no wake)</th>
<th>$Re^c$ (wake)</th>
<th>$n$</th>
<th>$Re$ (no wake)</th>
<th>$Re^c$ (wake)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.5</td>
<td>0.3</td>
<td>0.3</td>
<td>6.4</td>
<td>6.6</td>
</tr>
<tr>
<td>0.4</td>
<td>2.2</td>
<td>0.4</td>
<td>0.4</td>
<td>6.6</td>
<td>6.7</td>
</tr>
<tr>
<td>0.6</td>
<td>2.3</td>
<td>0.6</td>
<td>0.6</td>
<td>5.8</td>
<td>5.9</td>
</tr>
<tr>
<td>0.8</td>
<td>1.7</td>
<td>0.8</td>
<td>0.8</td>
<td>4.5</td>
<td>4.6</td>
</tr>
<tr>
<td>1</td>
<td>0.8</td>
<td>1</td>
<td>1</td>
<td>2.9</td>
<td>3</td>
</tr>
<tr>
<td>1.2</td>
<td>0.4</td>
<td>1.2</td>
<td>1.2</td>
<td>1.9</td>
<td>2</td>
</tr>
<tr>
<td>1.4</td>
<td>0.2</td>
<td>1.4</td>
<td>1.4</td>
<td>1.1</td>
<td>1.2</td>
</tr>
<tr>
<td>1.6</td>
<td>0.006</td>
<td>0.01</td>
<td>1.6</td>
<td>0.4</td>
<td>0.6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$E$</th>
<th>$Re$ (no wake)</th>
<th>$Re^c$ (wake)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>32</td>
<td>33</td>
</tr>
<tr>
<td>0.3</td>
<td>32</td>
<td>33</td>
</tr>
<tr>
<td>0.4</td>
<td>30</td>
<td>31</td>
</tr>
<tr>
<td>0.5</td>
<td>24</td>
<td>25</td>
</tr>
<tr>
<td>0.8</td>
<td>19</td>
<td>20</td>
</tr>
<tr>
<td>1</td>
<td>14</td>
<td>15</td>
</tr>
<tr>
<td>1.2</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>1.4</td>
<td>6</td>
<td>6.4</td>
</tr>
<tr>
<td>1.6</td>
<td>4</td>
<td>4.5</td>
</tr>
<tr>
<td>1.8</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

6.3. Onset of Vortex Shedding. Once wake formation has set in, the size of this region grows as the Reynolds number is progressively increased. The flow is steady and symmetric about the midplane, however. Finally, a point (value of $Re$ depending on the value of $n$) is reached where the standing vortices lose symmetry and begin to flap. Only upon a slight further increase in the value of the Reynolds number are the standing vortices convected by the bulk flow in the downstream direction. In the flapping zone, simulations were run for a sufficiently long time to ensure that the vortices flap only, without exhibiting vortex shedding. The transition from the flapping to shedding of vortices occurs over a very narrow range of Reynolds number, and this range depends primarily on the values of the aspect ratio of the cylinder and the power-law index. Also, this range of Reynolds number where only flapping of vortices occurred was found to be a bit wider in the case of elliptical cylinders with aspect ratios greater than 1. The onset of vortex shedding was judged by examining the values of the vorticity and stream function contours near the surface of the cylinder along with the evolution of the lift coefficient with time. The vorticity contours in the vicinity of elliptical cylinders of aspect ratio 0.2, 0.5, 2, and 5 for two values of the Reynolds number bracketing the transition and for extreme values of the power-law index 0.3 and 1.8 are shown in Figure 4A and B, respectively. In these figures, depending on the values of the aspect ratio and power-law index, vortices are seen to be either standing or flapping at the lower value of the Reynolds number (left column), whereas at the higher Reynolds number (right column), shedding of vortices has initiated. For instance, in Figure 4A, for $n = 0.3$ and $E = 0.2$, the flow is seen to be asymmetric because of the flapping of the vortices at $Re = 32$, whereas at $Re = 33$, vortex shedding is clearly seen to have started. Because the flow is asymmetric at $Re = 32$, the cylinder experiences a lift force and the lift coefficient oscillates between $+0.019$ and $-0.019$ with a constant frequency. The lift coefficient at $Re = 32$ is nearly as that at $Re = 33$. However, for the cases where the flow is steady and symmetric at lower Reynolds number (in Figure 4B for $n = 1.8$ and $E = 0.2$), the lift coefficient ultimately approaches zero as the time progresses. Therefore, for the elliptical cylinder of aspect ratio 0.2, the critical Reynolds number for power-law index of 0.3 lies in the range $32 \leq Re \leq 33$. Qualitatively, the same procedure is followed to infer the value of $Re^c$ for the other values of the power-law index and the aspect ratio. Figure 5 shows the dependence of the critical Reynolds number ($Re^c$) on the power-law index and aspect ratio. It can be seen that the critical Reynolds number is higher for shear-thinning fluids than for shear-thickening fluids only for $n \geq 0.6$. As the power-law index increases from 0.3 to 1.0, the critical Reynolds number goes through a maximum value at about $n = 0.6$, which is similar to that seen for a circular cylinder. On the other hand, for shear-thickening fluids, the decrease in the critical Reynolds number is rather steep as the fluid behavior transits from Newtonian ($n = 1$) to shear-thickening ($n > 1$). The complex dependence of the critical Reynolds number on the flow behavior index seen in Figure 5 stems from the interaction between the two nonlinear terms (inertial and viscous) present in the momentum equations. These two nonlinear terms scale differently with velocity. The viscous term scales as $U_n^2$ whereas the inertial term scales as $U_0^2$. Thus, at low Reynolds numbers, the viscous term diminishes with decreasing value of a power-law index for shear-thinning fluids, but it increases...
with velocity in shear-thickening fluids. Also, the viscosity of a shear-thinning fluid becomes very large as the shear rate decreases, and it thus tends to an infinite value far from the cylinder where the shear rate tends to zero. Therefore, the viscous effects always dominate the flow characteristics even far from the cylinder under these conditions. Conversely, the viscosity of shear-thickening fluids is maximum near the obstacle and gradually decreases away from the object. For a fixed value of the power-law index, as the aspect ratio of the cylinder is increased from 0.2 to 5, shedding is delayed.
to higher Reynolds numbers. This is due to the increasing streamlining of the body, which delays the boundary layer separation.

Figure 6 shows the variation of the critical Strouhal number ($St_c$) at the corresponding critical Reynolds number with power-law index ($0.3 \leq n \leq 1.8$) and aspect ratio ($0.2 \leq E \leq 5$). An increase in power-law index results in a decrease in the critical Strouhal number, and the trend is seen to be similar for all values of aspect ratio. This can be explained as follows: All else being equal, the effective viscosity in the vicinity of the cylinder gradually increases with increasing value of the power-law index, which tends to suppress vortex shedding. The corresponding values of the time-average drag coefficient are shown in Figure 7. It can be seen that the time-average drag coefficient increases with increasing power-law index ($n$), which is in line with the trend seen in the steady-flow regime.\textsuperscript{37} For aspect ratios less than 1, the drag coefficient decreases as the value of the power-law index is increased from 0.3 to 0.4, and with a further increase in the power-law index from 0.4 to 1.8, the drag coefficient continuously increases. For aspect ratios greater than 1, as the power-law index is increased from 0.3 to 1.8, the drag coefficient rises continuously. Thus, highly shear-thinning fluids exhibit different behavior than mildly shear-thinning and shear-thickening fluids. In view of these results, it is fair to say that the literature values of drag coefficient and Nusselt number based on the assumption of steady flow for $Re > 30$, $E = 0.2-5$ and/or $n \geq 0.2$ are likely to be rather inaccurate.\textsuperscript{37,38}

7. Conclusions

Extensive numerical results on the critical Reynolds numbers denoting wake formation and the onset of vortex shedding for the flow of Newtonian and power-law fluids have been obtained over wide ranges of conditions: $0.2 \leq E \leq 5$ and $0.3 \leq n \leq 1.8$. For elliptical cylinders with aspect ratios varying from 0.2 to 5, wake formation is seen to be delayed as the fluid behavior changes from shear-thickening to Newtonian and finally to shear-thinning. In the case of shear-thinning fluids, for aspect ratios less than 1, the critical Reynolds number goes through a maximum value as the fluid nature changes from highly shear-thinning to mildly shear-thinning. For shear-thickening fluids, the critical Reynolds number decreases monotonically with power-law index for all values of aspect ratio of elliptical cylinders considered here. Furthermore, the critical Reynolds number curve for the onset of vortex shedding for shear-thinning fluids also goes through a maximum at $n = 0.6$, whereas it decreases continually with power-law index for shear-thickening fluids for all values of aspect ratio. It is also seen that, irrespective of the value of aspect ratio ($0.2 \leq E \leq 5$), vortex shedding occurs at lower Reynolds number in shear-thickening fluids than in Newtonian fluids. Therefore, shear-thinning behavior seems to have a stabilizing influence, and shear-thickening behavior tends to induce instability in the flow at lower and lower values of Reynolds number. In addition, the onsets of wake formation and vortex shedding in both shear-thinning and shear-thickening fluids are seen to be delayed to higher Reynolds number as the aspect ratio of the elliptical cylinder increases from 0.2 to 5 because of the increasing streamlining of the shape of the body.
Notation

\(a\) = semi-axis of the elliptical cylinder normal to the direction of flow, m
\(b\) = semi-axis of the elliptical cylinder along the direction of flow, m
\(C_D\) = drag coefficient \(= \frac{F_D}{(1/2)\rho U^2(2a)}\)
\(C_t\) = time-average drag coefficient
\(C_f\) = friction component of drag coefficient \(= \frac{F_{DF}}{(1/2)\rho U^2(2a)}\)
\(C_p\) = pressure component of drag coefficient \(= \frac{F_{DP}}{(1/2)\rho U^2(2a)}\)
\(C_L\) = lift coefficient \(= \frac{F_L}{(1/2)\rho U^2(2b)}\)
\(D_e\) = diameter of the outer boundary of the circular domain, m
\(E\) = aspect ratio \(= b/a\)
\(f\) = frequency of vortex shedding, \(s^{-1}\)
\(F_D\) = drag force per unit length of cylinder, N·m\(^{-1}\)
\(F_{DP}\) = pressure force per unit length of cylinder, N·m\(^{-1}\)
\(F_L\) = lift force per unit length of cylinder, N·m\(^{-1}\)
\(I_2\) = second invariant of the rate of the strain tensor, \(s^{-2}\)
\(m\) = power-law consistency index, Pa·s\(^n\)
\(n\) = power-law flow behavior index
\(N_p\) = number of grid points on the surface of the cylinder
\(p\) = pressure, Pa
\(Re\) = Reynolds number \(= \frac{\rho U m}{\mu}\)
\(St\) = Strouhal number \(= (2\pi a) f U_m\)
\(U_x, U_y\) = \(x\) and \(y\) components of the velocity, m·s\(^{-1}\)
\(U_m\) = free stream velocity, m·s\(^{-1}\)
\(x\) = stream-wise coordinates, m
\(y\) = transverse coordinates, m

Greek Symbols

\(\Delta t\) = time step, s
\(\varepsilon\) = component of the rate of the strain tensor, \(s^{-1}\)
\(\eta\) = viscosity, Pa·s
\(\rho\) = density of the fluid, kg·m\(^{-3}\)
\(\tau\) = extra stress, Pa
\(\phi\) = scalar variable \((U_x, U_y)\), m·s\(^{-1}\)
\(\psi\) = stream function, m\(^2\)·s\(^{-1}\)
\(\omega\) = vorticity, \(s^{-1}\)

Superscript/Subscript

\(c\) = critical

Literature Cited


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