Cvičení 18 – 19 - výsledky

16.12.2011 a 19.12.2011

Příklad 1. *Pomocí Heavisideovy funkce vyjádřete*

$$a)\ f(x) = \begin{cases} 1,\ x \in \langle 1; 4) \cup \langle 5; 10), \\ 0,\ jinde, \end{cases}$$

$$f(t) = H(t-1) - H(t-4) + H(t-5) - H(t-10),$$

$$b)\ f(x) = \begin{cases} 1,\ x \in \langle -2; -1), \\ 2,\ x \in \langle 0; 1), \\ -3,\ x \in \langle 1; 2), \\ 4,\ x \in \langle 3; 6), \\ 0,\ jinde, \end{cases}$$

$$f(t) = (H(t+2) - H(t+1)) + 2\,(H(t) - H(t-1)) - 3\,(H(t-1) - H(t-3)) +$$
$$+4\,(H(t-3) - H(t-6)) = H(t+2) - tH(t+1) + 2H(t) - 5H(t-1) + 7H(t-3) - 4H(t-6),$$

$$c)\ f(x) = \begin{cases} 0,\ x \in (-\infty; 0), \\ 1,\ x \in \langle 0; 1), \\ x,\ x \in \langle 1; 2), \\ 2,\ x \in \langle 2; \infty), \end{cases}$$

$$f(t) = H(t) - H(t-1) + t(H(t-1) - H(t-2)) + 2H(t-2) =$$
$$= H(t) + (t-1)H(t-1) + (2-t)H(t-2),$$

$$d)\ f(x) = \begin{cases} 1,\ x \in (-\infty; 0\rangle, \\ 0,\ x \in (0; 1), \\ 2,\ x \in \langle 1; \infty), \end{cases}$$

$$f(t) = H(-t) + 2H(t-1).$$

Příklad 2. *Vyjádřete následující funkce jako $f(t-c) \cdot H(t-c)$:*

$$a)\ g(t) = \begin{cases} e^{3t}; t \in \langle 3; 6) \\ 0,\ jinde, \end{cases}$$

$$g(t) = e^9 \cdot e^{3(t-3)} H(t-3) - e^{18} \cdot e^{3(t-6)} H(t-6),$$

$$b)\ g(t) = \begin{cases} 0, t \in (-\infty; \frac{\pi}{2}) \\ \sin t,\ t \in \langle \frac{\pi}{2}; \pi), \\ -2, jinde, \end{cases}$$

$$g(t) = \cos(t - \tfrac{\pi}{2}) H(t - \tfrac{\pi}{2}) + \sin(t - \pi) H(t - \pi) - 2H(t - \pi),$$

$$c)\ g(t)=\begin{cases}0, t\in(-\infty;-2),\\ t^2+8t+16,\ \ t\in\langle -2;5)\\ 12-t,\ \ t\in\langle 5;\infty),\end{cases}$$

$$g(t)=((t+2)+2)^2H(t+2)-((t-5)+9)^2H(t-5)+(7-(t-5))H(t-5),$$

d) $g(t)$ je funkce z cvičení 16 příklad 1,

$$g(t)=2tH(t)-2((t-1)+1)H(t-1)+2H(t-1)-2H(t-3)+(2-(t-3))H(t-3)+$$
$$-(-(t-5))H(t-5)=$$
$$=2tH(t)-2(t-1)H(t-1)+2H(t-3)-(t-3)H(t-3)+(t-5)H(t-5),$$

Příklad 3. *Určete $\mathcal{L}(f)$ pro následující funkce:*

$$a)\ f(t)=\begin{cases}(t-2)^2,\ \ t\in\langle 2;4)\\ 0,\ \ jinde,\end{cases}$$

$$f(t)=(t-2)^2H(t-2)-((t-4)+2)^2H(t-4),$$

$$\mathcal{L}\{f\}(p)=\mathcal{L}\{(t-2)^2H(t-2)-((t-4)^2+4(t-4)+4)H(t-4)\}(p)=\tfrac{2}{p^3}e^{-2p}-e^{-4p}(\tfrac{2}{p^3}+\tfrac{4}{p^2}+\tfrac{4}{p}),$$

$$b)\ f(t)=\begin{cases}\frac{3}{2}t,\ \ 0\le t<3,\\ -2t+8,\ \ 3\le t<4,\\ 0,\ \ 4\le t<5,\\ -t+5,\ \ 5\le t,\end{cases}$$

$$f(t)=\tfrac{3}{2}t(H(t)-H(t-3))+(8-2t)(H(t-3)-H(t-4))+(-t+5)H(t-5)=$$
$$=\tfrac{3}{2}tH(t)-(\tfrac{3}{2}(t-3)-\tfrac{9}{2})H(t-3)+(-2(t-3)+2)H(t-3)-(-2(t-4))H(t-4)+$$
$$+(-(t-5))H(t-5)=\tfrac{3}{2}tH(t)+(-\tfrac{7}{2}(t-3)+\tfrac{13}{2})H(t-3)+2(t-4)h(t-4)-(t-5)H(t-5),$$

$$\mathcal{L}\{f\}(p)=\mathcal{L}\{\tfrac{3}{2}tH(t)+(-\tfrac{7}{2}(t-3)+\tfrac{13}{2})H(t-3)+2(t-4)H(t-4)-(t-5)H(t-5)\}(p)=$$
$$=\tfrac{3}{2p^2}-\tfrac{7}{2p^2}e^{-3p}+\tfrac{13}{2p}e^{-3p}+\tfrac{2}{p^2}e^{-4p}-\tfrac{1}{p^2}e^{-5p},$$

$$c)\ f(t)=\begin{cases}e^{3t}; t\in\langle 3;6)\\ 0,\ \ jinde.\end{cases}$$

$$f(t)=e^9\cdot e^{3(t-3)}H(t-3)-e^{18}\cdot e^{3(t-6)}H(t-6),$$

$$\mathcal{L}\{f\}(p)=\mathcal{L}\{e^9\cdot e^{3(t-3)}H(t-3)-e^{18}\cdot e^{3(t-6)}H(t-6)\}(p)=\tfrac{e^{9-3p}}{p-3}-\tfrac{e^{18-6p}}{p-6},$$

$$d)\ f(t)=\begin{cases} t^2-t-6, & t\in\langle 1;4),\\ 3t-1, & t\in\langle 4;6),\\ 0, & jinde,\end{cases}$$

$$f(t)=((t-1)^2+(t-1)-6)H(t-1)-((t-4)^2+7(t-4)+6)H(t-4)+(3(t-4)+11)H(t-4)+$$
$$-(3(t-6)+17)H(t-6)=$$
$$=((t-1)^2+(t-1)-6)H(t-1)+(-(t-4)^2-4(t-4)+5)H(t-4)-(3(t-6)+17)H(t-6),$$

$$\mathcal{L}\{f\}(p)=\mathcal{L}\{((t-1)^2+(t-1)-6)H(t-1)+(-(t-4)^2-4(t-4)+5)H(t-4)+$$
$$-(3(t-6)+17)H(t-6)\}(p)=e^{-p}(\tfrac{2}{p^3}+\tfrac{1}{p^2}-\tfrac{6}{p})-e^{-4p}(\tfrac{2}{p^3}+\tfrac{4}{p^2}-\tfrac{5}{p})-e^{-6p}(\tfrac{3}{p^2}+\tfrac{17}{p}).$$

Příklad 4. *Vyřešte diferenciální rovnici* $y''-y=f$ *s počátečními podmínkami* $y(0)=0, y'(0)=0$, *kde* f *je dána*

$$f(t)=\begin{cases} 1-t, & t\in\langle 0;1),\\ 0, & jinde\end{cases}=(1-t)H(t)+(t-1)H(t-1).$$

Použijeme Laplaceovu transformaci na rovnici:

$$p^2Y-0\cdot p-0-Y=\frac{1}{p}-\frac{1}{p^2}+\frac{e^{-p}}{p^2}.$$

Odtud

$$(p^2-1)Y=\frac{1}{p}-\frac{1}{p^2}+\frac{e^{-p}}{p^2},$$
$$Y(p)=\frac{1}{p(p-1)(p+1)}-\frac{1}{p^2(p-1)(p+1)}+\frac{e^{-p}}{p^2(p-1)(p+1)}=$$
$$=-\frac{1}{p}+\frac{\frac12}{p-1}+\frac{\frac12}{p+1}+(e^{-p}-1)(-\frac{1}{p^2}+\frac{\frac12}{p-1}-\frac{\frac12}{p+1})=$$
$$=-\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p+1}+e^{-p}(-\frac{1}{p^2}+\frac{\frac12}{p-1}-\frac{\frac12}{p+1}),$$
$$y(t)=\mathcal{L}^{-1}\{Y\}(t)=-1+t+e^{-t}+H(t-1)(-(t-1)+\frac{1}{2}e^{t-1}-\frac{1}{2}e^{-(t-1)}).$$

$$y(t)=\begin{cases} e^{-t}+t-1, & t\in(0;1),\\ e^{-t}+\frac12 e^{t-1}-\frac12 e^{-(t-1)}, & t\in\langle 1;\infty).\end{cases}$$

Příklad 5. *Vyřešte diferenciální rovnici* $y''-4y'+4y=f$ *s počátečními podmínkami* $y(0+)=1, y'(0+)=-2$, *kde* f *je dána*

$$f(t)=\begin{cases} 8\cos 2t, & t\in\langle 0;\pi),\\ 0, & jinde,\end{cases}=8\cos 2t\cdot H(t)-8\cos 2(t-\pi)\cdot H(t-\pi).$$

Použijeme Laplaceovu transformaci na rovnici:

$$p^2Y - p + 2 - 4(pY - 1) + 4Y = \frac{8p}{p^2+4} - \frac{8p}{p^2+4} \cdot e^{-\pi p}.$$

Odtud

$$\begin{aligned}
(p^2 - 4p + 4)Y &= \frac{8p}{p^2+4} - \frac{8p}{p^2+4} \cdot e^{-\pi p} + p - 6,\\
Y(p) &= \frac{8p}{(p^2+4)(p-2)^2}(1 - e^{-\pi t}) + \frac{p-6}{(p-2)^2} =\\
&= (1 - e^{-\pi p})(\frac{2}{(p-2)^2} - \frac{2}{p^2+4}) + (\frac{1}{p-2} - \frac{4}{(p-2)^2}) =\\
&= \frac{1}{p-2} - \frac{2}{(p-2)^2} - \frac{2}{p^2+4} - e^{-\pi p}(\frac{2}{(p-2)^2} - \frac{2}{p^2+4}),\\
y(t) = \mathcal{L}^{-1}\{Y\}(t) &= e^{2t} - 2te^{2t} - \sin 2t - H(t-\pi)(2(t-\pi)e^{2(t-\pi)} - \sin 2(t-\pi)) =\\
&= e^{2t}(1-2t) - \sin 2t + H(t-\pi)((2\pi - 2t)\frac{e^{2t}}{e^{2\pi}} + \sin 2t).
\end{aligned}$$

$$y(t) = \begin{cases} e^{2t}(1-2t) - \sin 2t, & t \in \langle 0; \pi), \\ e^{2\pi}\left(1 + 2\pi e^{-2\pi} - 2t(1 + e^{-2\pi})\right), & t \in \langle \pi; \infty). \end{cases}$$

Příklad 6. *Vyřešte diferenciální rovnici* $2y'' - y' = f$ *s počátečními podmínkami* $y(0+) = 0, y'(0+) = -1$*, kde* f *je dána*

$$f(t) = \begin{cases} 1, & t \in \langle 0; 3), \\ 4 - t, & t \in \langle 3; \infty), \end{cases} \qquad = H(t) - H(t-3) + (1 - (t-3))H(t-1) = H(t) - (t-3)H(t-3).$$

Použijeme Laplaceovu transformaci na rovnici:

$$2(p^2Y - 0 \cdot p + 1) - pY + 0 \cdot p = \frac{1}{p} - \frac{e^{-3p}}{p^2}.$$

Odtud

$$\begin{aligned}
(2p^2 - p)Y &= \frac{1}{p} - \frac{e^{-3p}}{p^2} - 2,\\
Y(p) &= \frac{1}{p(2p^2-p)} - \frac{2}{(2p^2-p)} - \frac{e^{-3p}}{p^2(2p^2-p)} =\\
&= -\frac{2}{p} - \frac{1}{p^2} + \frac{2}{p-\frac{1}{2}} + \frac{2}{p} - \frac{2}{p-\frac{1}{2}} - e^{-3p}(-\frac{4}{p} - \frac{2}{p^2} - \frac{1}{p^3} + \frac{4}{p-\frac{1}{2}}) =\\
&= -\frac{1}{p^2} - e^{-3p}(-\frac{4}{p} - \frac{2}{p^2} - \frac{1}{p^3} + \frac{4}{p-\frac{1}{2}}),\\
y(t) = \mathcal{L}^{-1}\{Y\}(t) &= -t - H(t-3)(-4 - 2(t-3) - \frac{1}{2}(t-3)^2 + 4e^{\frac{t-3}{2}}).
\end{aligned}$$

$$y(t)=\begin{cases}-t, & t\in\langle 0;3),\\ \frac{1}{2}\left(t^2-4t-5+8e^{\frac{t-3}{2}}\right), & t\in\langle 3;\infty).\end{cases}$$

Příklad 7. $f(t)=\begin{cases}-\cos t, & t\in\langle\frac{\pi}{2};\pi),\\ 0, & jinde\end{cases}$ *, g je π-periodická funkce a* $g(t)=f(t)$ *na* $\langle 0;\pi)$. *Odtud* $T=\pi$.

$$\mathcal{L}\{g\}(p)=\frac{\int_0^\pi f(t)e^{-pt}\,\mathrm{d}t}{1-e^{-\pi p}}=\frac{-pe^{-\pi p}+e^{-\frac{\pi}{2}p}}{(p^2+1)(1-e^{-\pi p})}.$$

Neboť

$$I=\int_0^\pi f(t)e^{-pt}\,\mathrm{d}t=\int_{\frac{\pi}{2}}^{\pi}-\cos te^{-pt}\,\mathrm{d}t\overset{p.\ p.}{=}\left|\begin{array}{ll}u=-\cos t, & u'=\sin t\\ v'=e^{-pt}, & v=-\frac{e^{-pt}}{p}\end{array}\right|=$$

$$=\left[\frac{e^{-pt}}{p}\cos t\right]_{\frac{\pi}{2}}^{\pi}+\frac{1}{p}\int_{\frac{\pi}{2}}^{\pi}\sin te^{-pt}\,\mathrm{d}t\overset{p.\ p.}{=}\left|\begin{array}{ll}u=\sin t, & u'=\cos t\\ v'=e^{-pt}, & v=-\frac{e^{-pt}}{p}\end{array}\right|=$$

$$=-\frac{e^{-\pi p}}{p}+\frac{1}{p^2}\left(\left[-e^{-pt}\sin t\right]_{\frac{\pi}{2}}^{\pi}-\int-\cos te^{-pt}\,\mathrm{d}t\right)=-\frac{e^{-\pi p}}{p}+\frac{e^{-\frac{\pi}{2}p}}{p^2}-\frac{1}{p^2}I,$$

tedy

$$I(1+\frac{1}{p^2})=\frac{p^2+1}{p^2}I=-\frac{e^{-\pi p}}{p}+\frac{e^{-\frac{\pi}{2}p}}{p^2}.$$

Celkem $I=\frac{p^2}{p^2+1}(-\frac{e^{-\pi p}}{p}+\frac{e^{-\frac{\pi}{2}p}}{p^2})=\frac{-pe^{-\pi p}+e^{-\frac{\pi}{2}p}}{p^2+1}$.

Příklad 8. $f(t)=\begin{cases}1-t, & t\in\langle 0;1\rangle,\\ 0, & jinde\end{cases}$ *, g je 2-periodická funkce a* $g(t)=f(t)$ *na* $\langle 0;2\rangle$. *Odtud* $T=2$.

$$\mathcal{L}\{g\}(p)=\frac{\int_0^2 f(t)e^{-pt}\,\mathrm{d}t}{1-e^{-2p}}=(\frac{1}{p}-\frac{1}{p^2}+\frac{e^{-p}}{p^2})\cdot\frac{1}{1-e^{-2p}}.$$

To proto, že

$$\int_0^2 f(t)e^{-pt}\,\mathrm{d}t=\int_0^1(1-t)e^{-pt}\,\mathrm{d}t=\left[-\frac{e^{-pt}}{p}\right]_0^1-\int_0^1 te^{-pt}\,\mathrm{d}t=-\frac{e^{-p}}{p}+\frac{1}{p}+\frac{e^{-p}}{p}+\frac{e^{-p}}{p^2}-\frac{1}{p^2}=$$

$$=\frac{1}{p}-\frac{1}{p^2}+\frac{e^{-p}}{p^2}$$

a

$$\int_0^1 te^{-pt}\,\mathrm{d}t\overset{p.\ p.}{=}\left|\begin{array}{ll}u=t, & u'=1\\ v'=e^{-pt}, & v=-\frac{e^{-pt}}{p}\end{array}\right|=\left[-t\frac{e^{-pt}}{p}\right]_0^1+\frac{1}{p}\int_0^1 e^{-pt}\,\mathrm{d}t=-\frac{e^{-p}}{p}+\left[-\frac{e^{-pt}}{p^2}\right]_0^1=$$

$$=-\frac{e^{-p}}{p}-\frac{e^{-p}}{p^2}+\frac{1}{p^2}.$$

Příklad 9. *Pomocí věty o konvoluci najděte*

a) $\mathcal{L}^{-1}\left\{\frac{1}{p}\cdot\frac{2}{p^2+4}\right\}(t) = (1*\sin(2x))(t) = \int_0^t 1\cdot\sin 2q \,\mathrm{d}q = \left[-\frac{\cos 2x}{2}\right]_0^t = -\frac{\cos 2t}{2}+\frac{1}{2}$,

b) $\mathcal{L}^{-1}\left\{\frac{1}{p^2+1}\cdot\frac{p}{p^2+9}\right\}(t) = (\sin x)*(\cos 3x)(t) = \int_0^t \sin(t-q)\cdot\cos 3q \,\mathrm{d}q =$
$= \int_0^t (\sin t\cos q - \cos t\sin q)\cos 3q \,\mathrm{d}q = \int_0^t (\sin t\cos q - \cos t \sin q)(\cos^3 q - 3\cos q\sin^2 q)\,\mathrm{d}q =$
$= \sin t\int_0^t \cos^4 q - 3\cos^2 q\sin^2 q \,\mathrm{d}q + \cos t\int_0^t 3\cos q\cdot\sin^3 q - \sin q\cos^3 q \,\mathrm{d}q =$
$= \sin t(\left[\frac{3}{8}q+\frac{1}{4}\sin 2q+\frac{1}{32}\sin 4x\right]_0^t - \left[\frac{3}{8}q - \frac{3}{32}\sin 4x\right]_0^t) + \cos t\left[\frac{\cos^4 q}{4}-\frac{3\sin^4 q}{4}\right]_0^t =$
$= \frac{1}{4}\left(\sin t\sin 2t - \frac{1}{4}\sin t\sin 4t + \cos^5 t - 3\cos t\sin^4 t + \cos t\right)$,

c) $\mathcal{L}^{-1}\left\{\frac{p}{(p+3)(p^2+4)}\right\}(t) = (e^{-3x}*\cos 2x)(t) = \int_0^t e^{-3(t-x)}\cos 2x \,\mathrm{d}x = e^{-3t}\int_0^t e^{3x}\cos 2x\,\mathrm{d}x =$
$= \frac{1}{13}e^{-3t}\left(3e^3 t\cos 2t - 1 + 2e^3 t\sin 2t\right) = 3\cos 2t + 2\sin 2t - e^{-3t}$,

d) $\mathcal{L}^{-1}\left\{\frac{1}{p^4-2p^3+9p^2-18p}\right\}(t) = \mathcal{L}^{-1}\left(\frac{1}{p(p-2)(p^2+9)}\right)(t) = (1*e^{2x}*\sin 3x)(t) = \int_0^t 1\cdot e^{2(t-x)}\sin 3x\,\mathrm{d}x =$
$= e^{2t}\int_0^t e^{-2x}\sin 3x\,\mathrm{d}x = -\frac{1}{13}(3\cos 3t + 2\sin 3t - 3e^{2t})$,

e) $\mathcal{L}^{-1}\left\{\frac{-2p}{(p-2)(p^2+1)^2}\right\}(t) = \left(e^{2x}*(x\cdot\sin x)\right)(t) = e^{2t}\int_0^t e^{-2x}x\sin x\,\mathrm{d}x =$
$= e^{2t}\left[-\frac{1}{25}e^{-2x}((4+5x)\cos x + (3+10x)\sin x)\right]_0^t =$
$= -\frac{1}{25}\left((4+5t)\cos t + (3+10t)\sin t - 4e^{2t}\right)$.