Cvičení 17 - výsledky

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Příklad 1. *Určete $\mathcal{L}\{f\} = F$ pro následující funkce:*

a) $\mathcal{L}\{t^n\}(p) = \frac{n!}{p^{n+1}}$,

b) $\mathcal{L}\{\frac{\sin t + \cos t}{t}\}(p) = \int_p^\infty \frac{1}{q^2+1} + \frac{q}{q^2+1}\,\mathrm{d}q = \left[\arctan q + \frac{1}{2}\ln|q^2+1|\right]_p^\infty = \infty$ *to jest neexistuje,*

c) $\mathcal{L}\{t \cdot e^{5t}\}(p) = \left(\frac{1}{p-5}\right)' = \frac{1}{(p-5)^2}$,

d) $\mathcal{L}\{t^3 \cos 2t\}(p) = -\left(\frac{p}{p^2+4}\right)''' = \left(\frac{p^2-4}{(p^2+4)^2}\right)'' = \left(\frac{2p^3-24p}{(p^2+4)^3}\right)' = \frac{-6p^4+24p^2-96}{(p^2+1)^4}$,

e) $\mathcal{L}\{\frac{\sinh 6t}{t}\}(p) = \mathcal{L}\{\frac{e^{6t}}{2t} - \frac{e^{-6t}}{2t}\}(p) = \frac{1}{2}\int_p^\infty \frac{1}{q-6} - \frac{1}{q+6}\,\mathrm{d}q = [\ln|q-6| - \ln|q+6|]_p^\infty = \left[\ln\frac{|q-6|}{|q+6|}\right]_p^\infty =$
$= 0 - \frac{1}{2}\ln\frac{p-6}{p+6} = \frac{1}{2}\ln\frac{p+6}{p-6},\ p > 6$

f) $\mathcal{L}\{(1-4t+t^2)\sin\frac{t}{2}\}(p) = \frac{1}{p^2+\frac{1}{4}} + 4\cdot\left(\frac{1}{p^2+\frac{1}{4}}\right)' + \left(\frac{1}{p^2+\frac{1}{4}}\right)'' = \frac{1}{p^2+\frac{1}{4}} - \frac{8p}{(p^2+\frac{1}{4})^2} + \frac{3p^2-\frac{1}{4}}{(p^2+\frac{1}{4})^3}$.

Příklad 2. *Řešte následující diferenciální rovnice:*

a) $y'' + 3y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1,$

$p^2Y(p) - 0\cdot p - 1 + 3(pY(p) - 0) + 2Y(p) = 0, \quad (p^2+3p+2)Y(p) = 1,$
$Y(p) = \frac{1}{p^2+3p+2} = \frac{1}{(p+2)(p+1)} = \frac{1}{p+1} - \frac{1}{p+2},$
$y(t) = \mathcal{L}^{-1}\{\frac{1}{p+1} - \frac{1}{p+2}\}(t) = e^{-t} - e^{-2t};$

b) $y'' + 9y = 5\cos 3t, \quad y(0) = 2, \quad y'(0) = -1,$

$p^2Y(p) - 2p + 1 + 9Y(p) = \frac{5p}{p^2+9}, \quad (p^2+9)Y(p) = \frac{5p}{p^2+9} + 2p - 1,$
$Y(p) = \frac{5p}{(p^2+9)^2} + \frac{2p}{p^2+9} - \frac{1}{p^2+9},$
$y(t) = \mathcal{L}^{-1}\{\frac{5p}{(p^2+9)^2} + \frac{2p}{p^2+9} - \frac{1}{p^2+9}\}(t) = \frac{5}{6}t\sin t + 2\cos 3t - \frac{1}{3}\sin t;$

c) $y'' + 6y' + 9y = (2t+1)e^t, \quad y(0) = 0, \quad y'(0) = \frac{1}{8},$

$p^2Y - 0\cdot p - \frac{1}{8} + 6pY - 6\cdot 0 + 9Y = \frac{2}{(p-1)^2} + \frac{1}{p-1}, \quad (p^2+6p+9)Y = \frac{2}{(p-1)^2} + \frac{1}{p-1} + \frac{1}{8},$
$Y(p) = \frac{2}{(p-1)^2(p+3)^2} + \frac{1}{(p-1)(p+3)^2} + \frac{1}{8(p+3)^2} = -\frac{\frac{1}{16}}{p-1} + \frac{\frac{1}{8}}{(p-1)^2} + \frac{\frac{1}{16}}{p+3} + \frac{\frac{1}{8}}{(p+3)^2} + \frac{\frac{1}{16}}{p-1} - \frac{\frac{1}{16}}{p+3} - \frac{\frac{1}{4}}{(p+3)^2} + \frac{\frac{1}{8}}{(p+3)^2} =$
$= \frac{\frac{1}{8}}{(p-1)^2},$
$y(t) = \mathcal{L}^{-1}\{\frac{\frac{1}{8}}{(p-1)^2}\}(t) = \frac{1}{8}te^t;$

$d)$ $y'' - 4y' + 4y = 8\cos 2t, \quad y(0) = 1, \quad y'(0) = -2;$

$$p^2Y - p + 2 - 4(pY - 1) + 4Y = \tfrac{8p}{p^2+4}, \quad (p-2)^2Y = \tfrac{8p}{p^2+4} + p - 6,$$
$$Y(p) = \tfrac{8p}{(p^2+4)(p-2)^2} + \tfrac{p-6}{(p-2)^2} = -\tfrac{2}{p^2+4} + \tfrac{2}{(p-2)^2} + \tfrac{1}{p-2} - \tfrac{4}{(p-2)^2},$$
$$y(t) = \mathcal{L}^{-1}\{-\tfrac{2}{p^2+4} + \tfrac{1}{p-2} - \tfrac{2}{(p-2)^2}\}(t) = -\sin 2t + e^{2t} - 2te^{2t};$$

$e)$ $y'' - y = \frac{3}{2}(e^t - t^2 - t) - 13, \quad y(0) = 3, \quad y'(0) = 1,$

$$p^2Y - 3p - 1 - Y = \tfrac{3}{2}\left(\tfrac{1}{p-1} - \tfrac{2}{p^3} - \tfrac{1}{p^2}\right) - \tfrac{13}{p}, \quad (p^2-1)Y = \tfrac{3}{2}\left(\tfrac{1}{p-1} - \tfrac{2}{p^3} - \tfrac{1}{p^2}\right) - \tfrac{13}{p} + 3p + 1,$$
$$Y(p) = \tfrac{3}{2}\left(\tfrac{1}{(p-1)^2(p+1)} - \tfrac{2}{p^3(p-1)(p+1)} - \tfrac{1}{p^2(p-1)(p+1)}\right) - \tfrac{13}{p(p-1)(p+1)} + \tfrac{3p+1}{(p-1)(p+1)},$$
$$Y(p) = \tfrac{3}{2}\left((-\tfrac{\frac{1}{4}}{p-1} + \tfrac{\frac{1}{2}}{(p-1)^2} + \tfrac{\frac{1}{4}}{p+1}) - (-\tfrac{2}{p} - \tfrac{2}{p^3} + \tfrac{1}{p-1} + \tfrac{1}{p+1}) - (-\tfrac{1}{p^2} + \tfrac{\frac{1}{2}}{p-1} - \tfrac{\frac{1}{2}}{p+1})\right) +$$
$$-(-\tfrac{13}{p} + \tfrac{\frac{13}{2}}{p-1} - \tfrac{\frac{13}{2}}{p+1}) + (\tfrac{2}{p-1} - \tfrac{1}{p+1}) = \tfrac{3}{2}\left(\tfrac{2}{p} + \tfrac{1}{p^2} + \tfrac{2}{p^3} + \tfrac{\frac{1-4+2}{4}}{p+1} - \tfrac{\frac{1-4-2}{4}}{p-1} + \tfrac{\frac{1}{2}}{(p-1)^2}\right) + \tfrac{13}{p} + \tfrac{\frac{-13+4}{2}}{p-1} + \tfrac{\frac{13-2}{2}}{p+1} =$$
$$= \tfrac{16}{p} + \tfrac{\frac{3}{2}}{p^2} + \tfrac{3}{p^3} + \tfrac{\frac{41}{8}}{p+1} - \tfrac{\frac{51}{8}}{p-1} + \tfrac{\frac{3}{4}}{(p-1)^2},$$
$$y(t) = \mathcal{L}^{-1}\{\tfrac{16}{p} + \tfrac{\frac{3}{2}}{p^2} + \tfrac{3}{p^3} + \tfrac{\frac{41}{8}}{p+1} - \tfrac{\frac{51}{8}}{p-1} + \tfrac{\frac{3}{4}}{(p-1)^2}\}(t) = 16 + \tfrac{3}{2}t + \tfrac{3}{2}t^2 + \tfrac{41}{8}e^{-t} - \tfrac{51}{8}e^t + \tfrac{3}{4}te^t.$$

Příklad 3. *Řešte následující integrodiferenciální rovnice:*

$a)$ $y' - \int_0^t y(r)\,\mathrm{d}r = \sin t, \quad y(0) = -1;$
$$pY + 1 - \tfrac{Y}{p} = \tfrac{1}{p^2+1}, \quad (p^2-1)Y = \tfrac{p}{p^2+1} - p,$$
$$Y(p) = \tfrac{p}{(p^2+1)(p+1)(p-1)} - \tfrac{p}{(p+1)(p-1)} = \tfrac{\frac{1}{4}}{p+1} + \tfrac{\frac{1}{4}}{p-1} - \tfrac{\frac{p}{2}}{p^2+1} - (\tfrac{\frac{1}{2}}{p+1} + \tfrac{\frac{1}{2}}{p-1}) = \tfrac{-\frac{1}{4}}{p+1} - \tfrac{\frac{1}{4}}{p-1} - \tfrac{\frac{p}{2}}{p^2+1},$$
$$y(t) = \mathcal{L}^{-1}\{\tfrac{-\frac{1}{4}}{p+1} - \tfrac{\frac{1}{4}}{p-1} - \tfrac{\frac{p}{2}}{p^2+1}\}(t) = -\tfrac{1}{4}e^{-t} - \tfrac{1}{4}e^t - \tfrac{1}{2}\cos t;$$

$b)$ $y - \int_0^t y(r)\,\mathrm{d}r = \cosh t = \frac{1}{2}(e^t + e^{-t}),$

$$Y - \tfrac{Y}{p} = \tfrac{1}{2}(\tfrac{1}{p-1} + \tfrac{1}{p+1}), \quad (p-1)Y = \tfrac{1}{2}(\tfrac{p}{p-1} + \tfrac{p}{p+1}),$$
$$Y(p) = \tfrac{1}{2}(\tfrac{p}{(p-1)^2} + \tfrac{p}{(p+1)(p-1)}) = \tfrac{1}{2}(\tfrac{1}{p-1} + \tfrac{1}{(p-1)^2} + \tfrac{\frac{1}{2}}{p-1} + \tfrac{\frac{1}{2}}{p+1}) = \tfrac{1}{4}(\tfrac{1}{p+1} + \tfrac{3}{p-1} + \tfrac{2}{(p-1)^2})$$
$$y(t) = \mathcal{L}^{-1}\{\tfrac{1}{4}(\tfrac{1}{p+1} + \tfrac{3}{p-1} + \tfrac{2}{(p-1)^2})\}(t) = \tfrac{1}{4}(e^{-t} + 3e^t + 2te^t),$$

$c)$ $y' + y + \int_0^t y(r)\,\mathrm{d}r = e^{-t}, \quad y(0+) = 2$

$$pY - 2 + Y + \tfrac{Y}{p} = \tfrac{1}{p+1}, \quad (p^2+p+1)Y = \tfrac{p}{p+1} + 2,$$
$$Y(p) = \tfrac{p}{(p+1)(p^2+p+1)} + \tfrac{2}{p^2+p+1} = -\tfrac{1}{p+1} + \tfrac{p+1+2}{p^2+p+1},$$
$$y(t) = \mathcal{L}^{-1}\{-\tfrac{1}{p+1} + \tfrac{p+3}{p^2+p+1}\}(t) = -e^{-t} + \mathcal{L}^{-1}\{\tfrac{p+\frac{1}{2}}{(p+\frac{1}{2})^2+\frac{3}{4}} + \tfrac{\frac{5}{2}}{\frac{\sqrt{3}}{2}} \cdot \tfrac{\frac{\sqrt{3}}{2}}{(p+\frac{1}{2})^2+\frac{3}{4}}\} =$$
$$= -e^{-t} + e^{-\frac{t}{2}}\left(\cos(\tfrac{\sqrt{3}}{2}t) + \tfrac{5\sqrt{3}}{3}\sin(\tfrac{\sqrt{3}}{2}t)\right),$$

d) $y' + 2y + 2\int_0^t e^{t-r}y(r)\,\mathrm{d}r = e^t, \qquad y(0) = 1,$

(Věta o konvoluci:)$\mathcal{L}\{f * g\}(t) = F(t)\cdot G(t),$
$\mathcal{L}\{\int_0^t e^{t-r}y(r)\,\mathrm{d}r\}(p) = \mathcal{L}\{e^t\}(p)\cdot Y(p) = \frac{Y(p)}{p-1},$
$pY - 1 + 2Y + 2\frac{Y}{p-1} = \frac{1}{p-1}, \quad (p^2+p)Y = p,$
$Y(p) = \frac{p}{p(p+1)} = \frac{1}{p+1},$
$y(t) = \mathcal{L}^{-1}\{\frac{1}{p+1}\}(t) = e^{-t}.$

Příklad 4. *Řešte následující soustavy diferenciálních rovnic:*

a) $x' = -2x + y + t, \qquad x(0) = 0,$
$y' = 3x - 4y, \qquad y(0) = 0;$

$pX = -2X + Y + \frac{1}{p^2}, \qquad (p+2)X = Y + \frac{1}{p^2},$
$pY = \frac{17}{4}X - 4Y, \qquad (p+4)Y = \frac{17}{4}X,$
$(p^2 + 8p + \frac{15}{4})X = \frac{p+4}{p^2},$
$X(p) = \frac{p+4}{p^2(p+\frac{1}{2})(p+\frac{15}{2})} = -\frac{\frac{196}{225}}{p} + \frac{\frac{8}{15}}{p^2} + \frac{\frac{24}{7}}{p+\frac{1}{2}} + \frac{\frac{88}{1575}}{p+\frac{15}{2}},$
$Y(p) = \frac{(p+4)(p+2)}{p^2(p+\frac{1}{2})(p+\frac{15}{2})} - \frac{1}{p^2} = -\frac{\frac{664}{225}}{p} + \frac{\frac{32}{15}}{p^2} + \frac{12}{p+\frac{1}{2}} - \frac{\frac{44}{225}}{p+\frac{15}{2}} - \frac{1}{p^2},$
$x(t) = \mathcal{L}^{-1}\{-\frac{\frac{196}{225}}{p} + \frac{\frac{8}{15}}{p^2} + \frac{\frac{24}{7}}{p+\frac{1}{2}} + \frac{\frac{88}{1575}}{p+\frac{15}{2}}\}(t) = -\frac{196}{225} + \frac{8}{15}t + \frac{24}{7}e^{-\frac{t}{2}} + \frac{88}{1575}e^{-\frac{15}{2}t},$
$y(t) = \mathcal{L}^{-1}\{-\frac{\frac{664}{225}}{p} + \frac{\frac{17}{15}}{p^2} + \frac{12}{p+\frac{1}{2}} - \frac{\frac{44}{225}}{p+\frac{15}{2}}\}(t) = -\frac{664}{225} + \frac{17}{15}t + 12e^{-\frac{t}{2}} - \frac{44}{225}e^{-\frac{15}{2}t}.$

b) $x_1' = -2x_2 - 8\sin t, \qquad x_1(0+) = 0,$
$x_2' = x_1 + x_3, \qquad x_2(0+) = 0,$
$x_3' = 2x_2, \qquad x_3(0+) = 0.$

$(i)\;\; pX_1 - 0 = -2X_2 - \frac{8}{p^2+1},$
$(ii)\; pX_2 - 0 = X_1 + X_3,$
$(iii)\; pX_3 - 0 = 2X_2,$
$(i) + (iii) \Rightarrow (iv)\; p(X_1 + X_2) = -\frac{8}{p^2+1},$
$(ii) + (iv) \Rightarrow (v) p^2X_2(p) = p(X_1 + X_2) = -\frac{8}{p^2+1}, \quad X_2(p) = -\frac{8}{p^2(p^2+1)} = -\frac{8}{p^2} + \frac{8}{p^2+1},$
$(iii) + (v) \Rightarrow (vi) pX_3 = -\frac{16}{p^2(p^2+1)}, \quad X_3 = -\frac{16}{p^3(p^2+1)} = \frac{16}{p} - \frac{16}{p^3} - \frac{16}{p^2+1},$
$(ii) + (vi) \Rightarrow X_1 = -\frac{8}{p(p^2+1)} + \frac{16}{p^2(p^2+1)} = -\frac{8}{p} + \frac{8p}{p^2+1} - \frac{16}{p} + \frac{16}{p^3} + \frac{16}{p^2+1},$
$x_1(t) = \mathcal{L}^{-1}\{-\frac{24}{p} + \frac{8p}{p^2+1} + \frac{16}{p^3} + \frac{16}{p^2+1}\}(t) = -24 + 8\cos t + 8t^2 + 16\sin t,$
$x_2(t) = \mathcal{L}^{-1}\{-\frac{8}{p^2} + \frac{8}{p^2+1}\}(t) = -8t + 8\sin t,$
$x_3(t) = \mathcal{L}^{-1}\{\frac{16}{p} - \frac{16}{p^3} - \frac{16}{p^2+1}\}(t) = 16 - 8t^2 - 16\sin t.$