Cvičení 16 - výsledky

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Příklad 1. $f(x) = \begin{cases} 2x, & x \in (0;1), \\ 2, & x \in \langle 1;3 \rangle, \\ 5-x, & x \in (3;5), \\ 0, & \textit{jinde.} \end{cases}$

$$\mathcal{L}\{f\}(p) = \int_0^\infty f(t)e^{-pt}\,\mathrm{d}t = \int_0^1 2te^{-pt}\,\mathrm{d}t + \int_1^3 2e^{-pt}\,\mathrm{d}t + \int_3^5 (5-t)e^{-pt}\,\mathrm{d}t =$$

$$= \frac{2}{p^2}\left(e^{-5p} - e^{-3p} - e^{-p} + 1\right), \quad p > 0,$$

$\int_0^1 2te^{-pt}\,\mathrm{d}t \overset{p.\ p.}{=} \begin{vmatrix} u = 2t, & u' = 2 \\ v' = e^{-pt}, & v = -\frac{e^{-pt}}{p} \end{vmatrix} = \left[-2t \cdot \frac{e^{-pt}}{p}\right]_0^1 + 2\int_0^1 \frac{e^{-pt}}{p}\,\mathrm{d}t = -\frac{2e^{-p}}{p} + \left[-\frac{2e^{-pt}}{p^2}\right]_0^1 =$
$= -\frac{2e^{-p}}{p} + (-\frac{2e^{-p}}{p^2} + \frac{2}{p^2}) = -e^{-p}(\frac{2}{p} + \frac{2}{p^2}) + \frac{2}{p^2},$

$\int_1^3 2e^{-pt}\,\mathrm{d}t = \left[-\frac{2e^{-pt}}{p}\right]_1^3 = \frac{2e^{-p} - 2e^{-3p}}{p},$

$\int_3^5 (5-t)e^{-pt}\,\mathrm{d}t = \left[-\frac{5e^{-pt}}{p}\right]_3^5 - \left[t \cdot -\frac{e^{-pt}}{p}\right]_3^5 - \left[-\frac{e^{-pt}}{p^2}\right]_3^5 =$
$= -\frac{5e^{-5p}}{p} + \frac{5e^{-3p}}{p} - \left(-\frac{5e^{-5p}}{p} + \frac{3e^{-3p}}{p}\right) - \left(-\frac{e^{-5p}}{p^2} + \frac{e^{-3p}}{p^2}\right) = 2e^{-3p}\left(\frac{1}{p} + \frac{e^{-2p}}{p^2} - \frac{1}{p^2}\right),$

$$\mathcal{L}\{f\}(0) = \int_0^1 2t\,\mathrm{d}t + \int_1^3 2\,\mathrm{d}t + \int_3^5 5-t\,\mathrm{d}t = \left[t^2\right]_0^1 + [2t]_1^3 + \left[5t - \frac{t^2}{2}\right]_3^5 = 1+2\cdot 2+(25-\frac{25}{2}-15+\frac{9}{2}) = 7.$$

Příklad 2. *Pomocí definice najděte* $\mathcal{L}(f)$*, jestliže* $f(x) = \begin{cases} \frac{2}{3}x, & x \in \langle 0;3) \\ -2x+8, & x \in \langle 3;4) \\ -x+3, & x \in \langle 5;6) \\ 0, & \textit{jinde.} \end{cases}$.

$$\mathcal{L}\{f\}(0) = \int_0^3 \frac{3}{2}t\,\mathrm{d}t + \int_3^4 -2t+8\,\mathrm{d}t + \int_5^6 -t+3\,\mathrm{d}t = \left[3t^2\right]_0^3 + \left[-t^2+8t\right]_3^4 + \left[-\frac{t^2}{2} + 3t\right]_5^6 = 9+1-\frac{5}{2} = 7+\frac{1}{2},$$

$$\mathcal{L}\{f\}(p) = \int_0^3 \frac{3}{2}te^{-pt}\,\mathrm{d}t + \int_3^4 (-2t+8)e^{-pt}\,\mathrm{d}t + \int_5^6 (-t+3)e^{-pt}\,\mathrm{d}t = \left[-\frac{3}{2}t \cdot \frac{e^{-pt}}{p}\right]_0^3 + \left[-\frac{3e^{-pt}}{2p^2}\right]_0^3 +$$

$$+ \left[2t \cdot \frac{e^{-pt}}{p}\right]_3^4 + \left[\frac{2e^{-pt}}{p^2}\right]_3^4 + \left[-\frac{8e^{-pt}}{p}\right]_3^4 + \left[t \cdot \frac{e^{-pt}}{p}\right]_5^6 + \left[\frac{e^{-pt}}{p^2}\right]_5^6 + \left[-\frac{3e^{-pt}}{p}\right]_5^6 =$$

$$= \frac{1}{p}\left(-\frac{5}{2}e^{-3p} + 3e^{-6p} - 2e^{-5p}\right) + \frac{1}{p^2}\left(\frac{3 - 7e^{-3p}}{2} + 2e^{-4p} + e^{-6p} - e^{-5p}\right), \ p > 0.$$

Příklad 3. *Určete $\mathcal{L}(f) = F$ pro následující funkce:*

a) $\mathcal{L}\{\sinh ax\}(p) = \int_0^\infty \frac{e^{at}-e^{-at}}{2} e^{-pt}\,\mathrm{d}t = \frac{1}{2}\int_0^\infty e^{(a-p)t} - e^{-(a+p)t}\,\mathrm{d}t = \frac{1}{2}\left[\frac{e^{(a-p)t}}{a-p} + \frac{e^{-(a+p)t}}{a+p}\right]_0^\infty =$
$= 0 - \frac{1}{2(a-p)} - \frac{1}{2(a+p)} = \frac{a}{p^2-a^2},\ p > a$

b) $\mathcal{L}\{\cos x + e^{\frac{x}{2}}\}(p) = \int_0^\infty \cos t \cdot e^{-pt} + e^{(\frac{1}{2}-p)t}\,\mathrm{d}t = \frac{p}{p^2+1} + \frac{1}{p-\frac{1}{2}},$

$I = \int_0^\infty \cos t \cdot e^{-pt}\,\mathrm{d}t \overset{p.\ p.}{=} \left| \begin{matrix} u = e^{-pt}, & u' = -pe^{-pt} \\ v' = \cos t, & v = \sin t \end{matrix} \right| = [e^{-pt}\sin t]_0^\infty + p\int_0^\infty \sin t e^{-pt}\,\mathrm{d}t \overset{p.\ p.}{=}$
$= \left| \begin{matrix} u = e^{-pt}, & u' = -pe^{-pt} \\ v' = \sin t, & v = -\cos t \end{matrix} \right| = 0 + [-pe^{-pt}\cos t]_0^\infty - p^2\int \cos t e^{-pt}\,\mathrm{d}t = p + p^2 I,$
$(1+p^2)I = p, \quad I = \frac{p}{p^2+1},$
$\int_0^\infty e^{(\frac{1}{2}-p)t}\,\mathrm{d}t = \left[\frac{e^{(\frac{1}{2}-p)t}}{\frac{1}{2}-p}\right]_0^\infty = \frac{1}{p-\frac{1}{2}},$

c) $\mathcal{L}\{(1-x)\cos 2x\}(p) = \int_0^\infty \cos 2t e^{-pt} - t\cos 2t e^{-pt}\,\mathrm{d}t = \frac{p}{p^2+4} - \frac{p^2-4}{(p^2+4)^2},$

$\int_0^\infty \cos 2t e^{-pt}\,\mathrm{d}t \overset{p.\ p.}{=} \left| \begin{matrix} u = cos2t, & u' = -2\sin 2t \\ v' = e^{-pt}, & v = -\frac{e^{-pt}}{p} \end{matrix} \right| = \left[\cos 2t \cdot -\frac{e^{-pt}}{p}\right]_0^\infty - \frac{2}{p}\int_0^\infty e^{-pt}\sin 2t\,\mathrm{d}t \overset{p.\ p.}{=}$
$= \left| \begin{matrix} u = \sin 2t, & u' = 2\cos 2t \\ v' = e^{-pt}, & v = -\frac{e^{-pt}}{p} \end{matrix} \right| = \frac{1}{p} - \frac{2}{p}\left[\sin 2t - \frac{e^{-pt}}{p}\right]_0^\infty - \frac{4}{p^2}\int_0^\infty \cos 2t e^{-pt}\,\mathrm{d}t$
$I(1+\frac{4}{p^2}) = \frac{1}{p}, \quad I = \frac{p}{p^2+4}$
$\int_0^\infty t\cos 2t e^{-pt}\,\mathrm{d}t \overset{p.\ p.}{=} \left| \begin{matrix} u = t, & u' = 1 \\ v' = \cos 2t e^{-pt}, & v = e^{-pt}(\frac{2\sin 2t}{p^2+4} + \frac{-p\cos 2t}{p^2+4}) \end{matrix} \right| =$
$= \left[te^{-pt}(\frac{2\sin 2t}{p^2+4} + \frac{-p\cos 2t}{p^2+4})\cdot\right]_0^\infty - \int_0^\infty e^{-pt}(\frac{2\sin 2t}{p^2+4} + \frac{-p\cos 2t}{p^2+4})\,\mathrm{d}t = 0 - \frac{2}{p^2+4}\cdot\frac{2}{p^2+4} + \frac{p}{p^2+4}\cdot\frac{p}{p^2+4} = \frac{p^2-4}{(p^2+4)^2}$

d) $\mathcal{L}\{(6x-5)^2 e^{-2x}\}(p) = \int_0^\infty (6t-5)^2 e^{-(2+p)t}\,\mathrm{d}t \overset{p.\ p.}{=} \left| \begin{matrix} u = 36t^2 - 60t + 25, & u' = 72t - 60 \\ v' = e^{-(2+p)t}, & v = -\frac{e^{-(2+p)t}}{p+2} \end{matrix} \right| =$
$= \left[-\frac{e^{-(p+2)t}}{p+2}(36t^2 - 60t + 25)\right]_0^\infty + \frac{1}{p+2}\int_0^\infty (72t-60)e^{-(2+p)t}\,\mathrm{d}t \overset{p.\ p.}{=} \left| \begin{matrix} u = 72t - 60, & u' = 72 \\ v' = e^{-(2+p)t}, & v = -\frac{e^{-(2+p)t}}{p+2} \end{matrix} \right| =$
$= \frac{25}{p+2} + \frac{1}{p+2}\left(\left[-\frac{e^{-(2+p)t}}{p+2}(72t-60)\right]_0^\infty + \frac{72}{p+2}\int_0^\infty e^{-(2+p)t}\,\mathrm{d}t\right) = \frac{25}{p+2} - \frac{60}{(p+2)^2} + \frac{72}{(p+2)^3},$

e) $\mathcal{L}\{(5x-2)\cdot e^{4x}\cdot\sin 2x\}(p) = \mathcal{L}\{(5x-2)\cdot\sin 2x\}(p-4) = \frac{20p-80}{(p^2-16p+20)^2} - \frac{4}{p^2-16p+20},$
$5\int_0^\infty t\sin 2t\cdot e^{-pt}\,\mathrm{d}t - 2\int_0^\infty \sin 2t\cdot e^{-pt}\,\mathrm{d}t = -5\left(\frac{2}{p^2+4}\right)' - 2\left(\frac{2}{p^2+4}\right) = \frac{20p}{(p^2+4)^2} - \frac{4}{p^2+4},$

f) $\mathcal{L}\{x + x^2\}(p) = \int_0^\infty (t+t^2)e^{-pt}\,\mathrm{d}t \overset{p.\ p.}{=} \left| \begin{matrix} u = t^2 + t, & u' = 2t + 1 \\ v' = e^{-pt}, & v = -\frac{e^{-pt}}{p} \end{matrix} \right| =$
$= \left[-\frac{e^{-pt}}{p}(t+t^2)\right]_0^\infty + \frac{1}{p}\int_0^\infty (2t+1)e^{-pt}\,\mathrm{d}t \overset{p.\ p.}{=} \left| \begin{matrix} u = 2t + 1, & u' = 2 \\ v' = e^{-pt}, & v = -\frac{e^{-pt}}{p} \end{matrix} \right| =$
$= 0 + \frac{1}{p}\left(\left[-\frac{e^{-pt}}{p}(2t+1)\right]_0^\infty + \frac{2}{p}\int_0^\infty e^{-pt}\,\mathrm{d}t\right) = \frac{1}{p^2} + \frac{2}{p^3},$

g) $\mathcal{L}\{x^2\cdot\sin x\}(p) = (\mathcal{L}\{\sin x\}(p))'' = \left(\frac{1}{p^2+1}\right)'' = \left(\frac{-2p}{(p^2+1)^2}\right)' = \frac{6p^2-2}{(p^2+1)^3}.$

Příklad 4. *Určete $\mathcal{L}^{-1}(F)$ pro následující funkce:*

a) $\mathcal{L}^{-1}\{\frac{1}{p}+\frac{2}{(p-4)^3}\}(x)=1+\frac{1}{2}t^2e^{4x}$,

b) $\mathcal{L}^{-1}\{\frac{p}{p^2+9}+\frac{2}{p^2+4}\}(x)=\cos 3x+\sin 2x$,

c) $\mathcal{L}^{-1}\{\frac{p^4+3p^2-4p+1}{(p^2+1)^2(p-4)}\}(x)=\mathcal{L}^{-1}\{\frac{1}{p-4}+\frac{p}{(p^2+1)^2}\}(x)=e^{4x}+\frac{1}{2}x\sin x$,

d) $\mathcal{L}^{-1}\{\frac{p}{p^3+p^2+4p+4}\}(x)=\mathcal{L}^{-1}\{\frac{-\frac{1}{5}}{p+1}+\frac{\frac{1}{5}p}{p^2+4}+\frac{\frac{4}{5}}{p^2+4}\}(x)=\frac{1}{5}\left(-e^{-x}+\cos 2x+2\sin 2x\right)$,

e) $\mathcal{L}^{-1}\{\frac{p+2}{p^2-2p+3}\}(x)=\mathcal{L}^{-1}\{\frac{p+1}{(p+1)^2+2}+\frac{2-1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{(p+1)^2+2}\}(x)=e^{x}\cos\sqrt{2}x+\frac{\sqrt{2}}{2}e^{x}\sin\sqrt{2}x$,

f) $\mathcal{L}^{-1}\{\frac{p-3}{p^2+6p+10}\}(x)=\mathcal{L}^{-1}\{\frac{p+3}{(p+3)^2+1}-\frac{6}{(p+3)^2+1}\}(x)=e^{-3x}\cos x-6e^{3x}\sin x$,

g) $\mathcal{L}^{-1}\{\frac{p^5+50p^3-p^2+625p}{p^2(p^2+25)^2}\}(x)=\mathcal{L}^{-1}\{\frac{p^4+50p^2-p+625}{p(p^2+25)^2}\}(x)=\mathcal{L}^{-1}\{\frac{1}{p}-\frac{1}{(p^2+25)^2}\}(x)=$
$=1-\frac{1}{250}\sin 5x+\frac{1}{50}x\cos 5x$.

Příklad 5. *Víme, že* $\mathcal{L}\{\sin t\}(p)=\frac{1}{p^2+1}$. *Pomocí věty máme* $\mathcal{L}\{\sin 6t\}(p)=\frac{1}{6}\frac{1}{(\frac{p}{6}^2)+1}=\frac{1}{6(\frac{p^2+36}{36})}=\frac{6}{p^2+36}$.

Víme, že $\mathcal{L}\{t\}(p)=\frac{1}{p^2}$ *a* $\mathcal{L}\{t^2\}(p)=\frac{2}{p^3}$. *Odtud* $\mathcal{L}\{t+2t^2\}(p)=\frac{1}{p^2}+\frac{4}{p^3}$. *Díky větě o změně měřitka je* $\mathcal{L}\{2t+4t^2\}(p)=\frac{1}{2}\left(\frac{1}{(\frac{p}{2})^2}+\frac{4}{(\frac{p}{2})^3}\right)=\frac{2}{p^2}+\frac{16}{p^3}$.