Cvičení 15

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Příklad 1. *Vypočítejte pomocí vhodné substituce:*

a) $\int \frac{-x}{1+\sqrt[3]{x+1}}\,\mathrm{d}x = \left| \begin{array}{l} y = \sqrt[3]{x+1} \\ \mathrm{d}y = \frac{1}{3\sqrt[3]{(x+1)^2}}\,\mathrm{d}x \end{array} \right| = \int \frac{-x}{1+\sqrt[3]{x+1}} \cdot \frac{3\sqrt[3]{(x+1)^2}}{3\sqrt[3]{(x+1)^2}}\,\mathrm{d}x = 3\int \frac{1-y^3}{1+y} \cdot y^2\,\mathrm{d}y =$
$= 3\int -y^4 + y^3 - y^2 + 2y - 2 + \frac{2}{y+1}\,\mathrm{d}y =$
$= 3(-\frac{\sqrt[3]{(x+1)^5}}{5} + \frac{\sqrt[3]{(x+1)^4}}{4} - \frac{x+1}{3} + \sqrt[3]{(x+1)^2} - 2\sqrt[3]{x+1} + 2\ln|1+\sqrt[3]{x+1}|) + c, x \in \mathbb{R} \setminus \{-2\},$

b) $\int \frac{1}{\sqrt{1+x+x^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(x-\frac{1}{2})^2+\frac{3}{4}}}\,\mathrm{d}x = \int \frac{1}{\frac{\sqrt{3}}{2}\sqrt{(\frac{2\sqrt{3}}{3}x+\frac{\sqrt{3}}{3})^2+1}}\,\mathrm{d}x = \left| \begin{array}{l} y = \frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3} \\ \mathrm{d}y = \frac{2\sqrt{3}}{3}\,\mathrm{d}x \end{array} \right| =$
$= \int \frac{1}{\sqrt{y^2+1}}\,\mathrm{d}y = \left| \begin{array}{l} \sinh t = y \\ \cosh t\,\mathrm{d}t = \mathrm{d}y \\ \mathrm{d}t = \frac{1}{\sqrt{y^2+1}}\,\mathrm{d}y \end{array} \right| = \int 1\,\mathrm{d}t = \operatorname{arcsinh}(\frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}) + c, x \in \mathbb{R},$

c) $\int \frac{x}{2\sqrt{1-x^2}}\,\mathrm{d}x = \left| \begin{array}{l} y = 1 - x^2 \\ \mathrm{d}y = -2x\,\mathrm{d}x \end{array} \right| = \int \frac{-2x}{-4\sqrt{1-x^2}}\,\mathrm{d}x = -\frac{1}{4}\int y^{-\frac{1}{2}}\,\mathrm{d}y = 2\sqrt{1-x^2} + c, x \in (-1;1),$

d) $\int \frac{1}{\sqrt{x^2-4x}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(x-2)^2-4}}\,\mathrm{d}x = \int \frac{1}{2\sqrt{(\frac{x}{2}-1)^2-1}}\,\mathrm{d}x = \left| \begin{array}{l} y = \frac{x}{2} - 1 \\ \mathrm{d}y = \frac{1}{2}\,\mathrm{d}x \end{array} \right| = \int \frac{1}{\sqrt{y^2-1}}\,\mathrm{d}y =$
$= \left| \begin{array}{l} \operatorname{sign} t \cosh t = y \\ \mathrm{d}t = \frac{1}{\sqrt{y^2-1}}\,\mathrm{d}y \end{array} \right| = \int 1\,\mathrm{d}t = \begin{cases} \operatorname{arccosh}(\frac{x}{2} - 1) + c, x \in (4;\infty) \\ \operatorname{arccosh}(1 - \frac{x}{2}) + c, x \in (-\infty;0) \end{cases},$

e) $\int \frac{1-x+x^2}{\sqrt{1+x-x^2}}\,\mathrm{d}x = \int \frac{1-x+x^2}{\sqrt{-(x-\frac{1}{2})^2+\frac{5}{4}}}\,\mathrm{d}x = \int \frac{1-x+x^2}{\frac{\sqrt{5}}{2}\sqrt{1-(\frac{2\sqrt{5}}{5}x-\frac{\sqrt{5}}{5})^2}}\,\mathrm{d}x = \left| \begin{array}{l} y = \frac{2\sqrt{5}}{5}x - \frac{\sqrt{5}}{5} \\ \mathrm{d}y = \frac{2\sqrt{5}}{5}\,\mathrm{d}x \end{array} \right| = \frac{1}{4}\int \frac{y^2+1}{\sqrt{1-y^2}}\,\mathrm{d}y =$
$= \left| \begin{array}{l} \sin t = y \\ \cos t\,\mathrm{d}t = \mathrm{d}y \\ \mathrm{d}t = \frac{1}{\sqrt{1-y^2}}\,\mathrm{d}y \end{array} \right| = \frac{1}{4}\int \sin^2 t + 1\,\mathrm{d}t = \frac{1}{8}\int 3 - \cos 2t\,\mathrm{d}t = \frac{3}{8}\arcsin(\frac{2\sqrt{5}}{5}x - \frac{\sqrt{5}}{5}) - \frac{1}{8}(\frac{2\sqrt{5}}{5}x - \frac{\sqrt{5}}{5}) + c,$
$x \in (\frac{1-\sqrt{5}}{2}; \frac{1+\sqrt{5}}{2}),$

f) $\int \frac{1}{1+\sqrt{x}}\,\mathrm{d}x = \left| \begin{array}{l} y = 1 + \sqrt{x} \\ \mathrm{d}y = \frac{1}{2\sqrt{x}}\,\mathrm{d}x \end{array} \right| = \int \frac{1}{1+\sqrt{x}} \cdot \frac{2\sqrt{x}}{2\sqrt{x}}\,\mathrm{d}x = 2\int \frac{y-1}{y}\,\mathrm{d}y = 2\int 1 - \frac{1}{y}\,\mathrm{d}y =$
$= 2(1+\sqrt{x}) - 2\ln(1+\sqrt{x}) + c = 2\sqrt{x} - 2\ln(1+\sqrt{x}) + c, \quad x \in \langle 0;\infty).$

Příklad 2.

$$a)\ \int \frac{\cos x+3\sin x\cos x}{\sin x-\cos^2 x+3\cos^2 x\sin^2 x}\,\mathrm{d}x = \int \frac{1+3\sin x}{\sin x-(1-\sin^2 x)+3(1-\sin^2 x)\sin^2 x}\cdot\cos x = \left|\begin{array}{l} y=\sin x\\ \mathrm{d}y=\cos x\,\mathrm{d}x\end{array}\right| =$$
$$= \int \frac{1+3y}{4y^2-3y^4+y-1}\,\mathrm{d}y,$$

$$b)\ \int \frac{\cos x+3\sin^2 x\cos^2 x}{\sin x-\sin^3 x+3\cos^2 x\sin x}\,\mathrm{d}x = \int \frac{\cos x+3(1-\cos^2 x)\cos^2 x}{(1-\cos^2 x)-(1-\cos^2 x)^2+3\cos^2 x(1-\cos^2 x)}\sin x\,\mathrm{d}x = \left|\begin{array}{l} y=\cos x\\ \mathrm{d}y=-\sin x\,\mathrm{d}x\end{array}\right| =$$
$$= -\int \frac{y+3y^2-3y^4}{-4y^4+4y^2}\,\mathrm{d}y = \frac{1}{4}\int \frac{1+3y-3y^3}{y^3-y}\,\mathrm{d}y = \frac{1}{4}\int \frac{1}{y(y-1)(y+1)} - 3\,\mathrm{d}y,$$

$$c)\ \int \frac{6\cos 2x-4e^{-4x}+18x^5}{3\sin 2x+e^{-4x}+3x^6+5}\,\mathrm{d}x = \left|\begin{array}{l} y=3\sin 2x+e^{-4x}+3x^6+5\\ \mathrm{d}y=6\cos 2x-4e^{-4x}+18x^5\,\mathrm{d}x\end{array}\right| = \int \frac{1}{y}\,\mathrm{d}y =$$
$$= \ln(3\sin 2x+e^{-4x}+3x^6+5)+c,\ x\in\mathbb{R},$$

$$d)\ \int \frac{\cosh x+3e^{3x}}{\sqrt[4]{\sinh x+e^{3x}}}\,\mathrm{d}x = \left|\begin{array}{l} y=\sinh x+e^{3x}\\ \mathrm{d}y=\cosh x+3e^{3x}\,\mathrm{d}x\end{array}\right| = \int y^{-\frac{1}{4}}\,\mathrm{d}y = \frac{4(\sinh x+e^{3x})^{\frac{3}{4}}}{3},\ x\in(\tfrac{1}{2}\ln\tfrac{1}{2};\infty),$$

$$e)\ \int \frac{1}{2+\sqrt{(x+3)(x-2)}}\,\mathrm{d}x = \int \frac{1}{2+(x-2)\sqrt{\frac{x+3}{x-2}}}\,\mathrm{d}x = \left|\begin{array}{l} y=\sqrt{\frac{x+3}{x-2}}\\ \mathrm{d}y=\frac{1}{2}\sqrt{\frac{x-2}{x+3}}\frac{-5}{(x-2)^2}\,\mathrm{d}x\end{array}\right| =$$
$$= \int \frac{1}{2+(x-2)\sqrt{\frac{x+3}{x-2}}}\cdot\sqrt{\frac{x+3}{x-2}}\cdot\sqrt{\frac{x-2}{x+3}}\cdot\frac{-10(x-2)^2}{-10(x-2)^2}\,\mathrm{d}x = -\frac{2}{5}\int \frac{y}{2+\frac{5y}{y^2-1}}\cdot\frac{25}{(y^2-1)^2}\,\mathrm{d}y = 10\int \frac{y}{(2y^2+5y-2)(1-y^2)}\,\mathrm{d}y,$$

$$f)\ \int \frac{2}{1+\sqrt{\frac{x+1}{x+2}}}\,\mathrm{d}x = \left|\begin{array}{l} y=\sqrt{\frac{x+1}{x+2}}\\ \mathrm{d}y=\frac{1}{2}\sqrt{\frac{x+2}{x+1}}\cdot\frac{1}{(x+2)^2}\,\mathrm{d}x\end{array}\right| =$$
$$= \int \frac{2}{1+\sqrt{\frac{x+1}{x+2}}}\cdot\sqrt{\frac{x+1}{x+2}}\cdot\sqrt{\frac{x+2}{x+1}}\cdot\frac{2(x+2)^2}{2(x+2)^2}\,\mathrm{d}x = 4\int \frac{y}{(1+y)(1-y^2)^2}\,\mathrm{d}y.$$

Příklad 3. *Vypočítejte Riemannův integrál:*

a) $\int_a^b kx + q\,\mathrm{d}x = \left[\frac{k}{2}x^2 + qx\right]_a^b = \frac{k}{2}(b^2 - a^2) + q(b-a)$,

b) $\int_{-4}^4 |2x+3|\,\mathrm{d}x = 2\int_0^4 2x + 3\,\mathrm{d}x = 2\left[x^2+3x\right]_0^4 = (16+12) - 0 = 28$,

c) $\int_{-\sqrt{3}}^1 x^3 - 4x^2 + x - 10\,\mathrm{d}x = \left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{x^2}{2} - 10x\right]_{-\sqrt{3}}^1 =$
$= (\frac{1}{4} - \frac{4}{3} + \frac{1}{2} - 10) - (\frac{9}{4} + \frac{12\sqrt{3}}{3} + \frac{3}{2} + 10\sqrt{3}) = \frac{3-16+6-120-27-18}{12} - 14\sqrt{3} = -\frac{43}{3} - 14\sqrt{3}$,

d) $\int_{-1}^0 \frac{x}{\sqrt{5-4x^2}}\,\mathrm{d}x = \left| \begin{array}{l} y = 5-4x^2 \\ \mathrm{d}y = -8x\,\mathrm{d}x \end{array} \right| = -\frac{1}{8}\int_{-1}^1 \frac{-8x}{\sqrt{5-4x^2}}\,\mathrm{d}x = -\frac{1}{8}\int_1^5 \frac{1}{\sqrt{y}}\,\mathrm{d}y = -\frac{1}{8}\left[y^{\frac{1}{2}}\right]_1^5 =$
$= \frac{1}{8}(1-\sqrt{5})$,

e) $\int_0^{\frac{\pi}{2}} \frac{1}{\cos^2\frac{x}{2}}\,\mathrm{d}x = \left| \begin{array}{l} y = \frac{x}{2} \\ \mathrm{d}y = \frac{x}{2}\,\mathrm{d}x \end{array} \right| = 2\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 y}\,\mathrm{d}y = [\operatorname{tg} y]_0^{\frac{\pi}{4}} = \operatorname{tg}\frac{\pi}{4} - \operatorname{tg} 0 = 1 - 0 = 1$,

f) $\int_\pi^{4\pi} \sin 6x\,\mathrm{d}x = \left| \begin{array}{l} y = 6x \\ \mathrm{d}y = 6\,\mathrm{d}x \end{array} \right| = \frac{1}{6}\int_{6\pi}^{24\pi} \sin y\,\mathrm{d}y = [-\cos y]_{6\pi}^{24\pi} = -\cos 6\pi + \cos 24\pi = -1 + 1 = 0$,

g) $\int_{-\pi}^{\pi} |\sin x| + |\cos x|\,\mathrm{d}x = 4\int_0^{\frac{\pi}{2}} \sin x + \cos x\,\mathrm{d}x = [4(-\cos x + \sin x)]_0^{\frac{\pi}{2}} = 4(0+1) - 4(-1+0) = 8$,

h) $\int_2^4 \frac{x^2+x+1}{(x^2+1)^2}\,\mathrm{d}x = \int_2^4 \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}\,\mathrm{d}x = \left[\arctan x + \frac{1}{2}\ln(x^2+1)\right]_2^4 =$
$= \arctan 4 - \arctan 2 + \frac{1}{2}(\ln 17 - \ln 5)$,

i) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin x}\,\mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{1-\cos^2 x}\,\mathrm{d}x = \left| \begin{array}{l} y = \cos x \\ \mathrm{d}y = -\sin x\,\mathrm{d}x \end{array} \right| = -\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \frac{1}{(1-y)(1+y)}\,\mathrm{d}y = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{2}\left(\frac{1}{1-y} + \frac{1}{1+y}\right)\mathrm{d}y =$
$= \left[\frac{1}{2}(\ln|1-y| + \ln|1+y|)\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \frac{1}{2}\left(\ln(1-\frac{3}{4}) - \ln(1-\frac{1}{4})\right) = \frac{1}{2}\ln\frac{1}{3}$,

j) $\int_{-\frac{\sqrt{2}}{2}}^0 \frac{\arccos x}{\sqrt{1-x^2}}\,\mathrm{d}x = \left| \begin{array}{l} \cos y = x \\ -\sin y\,\mathrm{d}y = \mathrm{d}x \\ \mathrm{d}y = -\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \end{array} \right| = -\int_{\frac{3\pi}{4}}^{\frac{\pi}{2}} y\,\mathrm{d}y = \left[\frac{y^2}{2}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} = \frac{\pi^2}{2}(\frac{9}{16} - \frac{1}{4}) = \frac{5}{32}\pi^2$,

k) $\int_4^5 \frac{8}{x^4-4x^2}\,\mathrm{d}x = \int_4^5 \frac{-1}{x+2} + \frac{1}{x-2} + \frac{-2}{x^2}\,\mathrm{d}x = \left[-\ln|x+2| + \ln|x-2| + \frac{2}{x}\right]_4^5 =$
$= (\frac{2}{5} + \ln 3 - \ln 7) - (\frac{1}{2} + \ln 2 - \ln 6) = -\frac{1}{10} + \ln\frac{9}{7}$,

l) $\int_1^{5e} \frac{\ln x + 5}{x(\ln^2 x + 1)}\,\mathrm{d}x = \int_1^{5e} \frac{1}{2}\left(\frac{2(\ln x + 1)}{x(\ln^2 x + 1)} + \frac{8}{x(\ln^2 x + 1)}\right)\mathrm{d}x = \frac{1}{2}(\ln 26 + 8\arctan 5)$,

$\int_1^{5e} \frac{2(\ln x + 1)}{x(\ln^2 x + 1)}\,\mathrm{d}y = \left| \begin{array}{l} y = \ln^2 x + 1 \\ \mathrm{d}y = \frac{2(\ln x + 1)}{x}\,\mathrm{d}x \end{array} \right| = \int_1^{26} \frac{1}{y}\,\mathrm{d}y = \ln 26$,

$\int_1^{5e} \frac{8}{x(\ln^2 x + 1)}\,\mathrm{d}x = \left| \begin{array}{l} y = \ln x \\ \mathrm{d}y = \frac{1}{x}\,\mathrm{d}x \end{array} \right| = 8\int_0^5 \frac{1}{y^2+1}\,\mathrm{d}y = [8\arctan y]_0^5 = 8\arctan 5$,

m) $\int_1^\infty \frac{1}{x^3}\,\mathrm{d}x = \left[-\frac{1}{2x^2}\right]_1^\infty = \frac{1}{2} - 0 = \frac{1}{2}$, *n)* $\int_{-\infty}^{0} e^x \,\mathrm{d}x = [e^x]_{-\infty}^{0} = 1 - 0 = 1,$

o) $\int_2^5 \frac{1}{x-3} \,\mathrm{d}x$ *diverguje vlivem funkce,*

p) $\int_0^{\infty} x^{\frac{8}{7}} \,\mathrm{d}x$ *diverguje vlivem meze.*

Příklad 4.

$$l(f) = \int_0^4 \sqrt{1 + (f'(x))^2} \,\mathrm{d}x.$$

$$\int_0^4 \sqrt{1 + (\frac{3}{2}\sqrt{x})^2} \,\mathrm{d}x = \int_0^4 \sqrt{1 + \frac{9}{4}x} \,\mathrm{d}x = \left| \begin{array}{c} y = 1 + \frac{9}{4}x \\ \mathrm{d}y = \frac{9}{4}\,\mathrm{d}x \end{array} \right| = \frac{4}{9}\int_1^{10} \sqrt{y} \,\mathrm{d}y = \left[\frac{8}{27} y^{\frac{3}{2}}\right]_1^{10} = \frac{8}{27}(10^{\frac{3}{2}} - 1).$$

Délka grafu funkce $f(x) = x^{\frac{3}{2}}$*, kde* $D(f) = \langle 0; 4\rangle$*, je* $\frac{8}{27}(10\sqrt{10} - 1)$.

Příklad 5. *obsah útvaru omezeného parabolou* $4y = x^2$ *a přímkou danou rovnicí* $4x + 2y + 6 = 0$*, tj.* $y = -2x - 3$*, je dán rozdílem jejich určitých integrálů.*

Průsečíky $2y = -4x - 6$ *a* $4x = x^2$ *jsou kořeny rovnice* $0 = x^2 + 8x + 12 = (x+2)(x+6)$. *Přičemž* $\frac{x^2}{4} \leq -2x - 3$ *na* $\langle -6; -2\rangle$.

$$O = \int_{-6}^{-2} (-2x - 3) - \frac{x^2}{4} \,\mathrm{d}x = \left[-x^2 - 3x - \frac{x^3}{12}\right]_{-6}^{-2} = (-4 + 6 + \frac{8}{12}) - (-36 + 18 + \frac{216}{12}) = 20 - \frac{208}{12} =$$

$$= 3 - \frac{4}{12} = \frac{8}{3}.$$

Obsah daného útvaru je $\frac{8}{3}$.

Příklad 6. *Objem tělesa vzniklého rotací kolem osy x plochy omezené grafy funkcí* $f(x) = 9$ *a* $g(x) = (x-2)^2$ *je dán rozdílem objemů těles vzniklých rotací daných funkcí a omezených průsečíky daných funkcí.*

Průsečíky f a g jsou -1 *a* 5.

$$V = V_1 - V_2 = \pi \int_{-1}^{5} 9^2 - \left((x-2)^2\right)^2 \,\mathrm{d}x = \pi \int_{-1}^{5} 9^2 - (x-2)^4 \,\mathrm{d}x =$$

$$= \left[\pi\left(81x - \frac{(x-2)^5}{5}\right)\right]_{-1}^{5} = \pi(81 \cdot 6 - \frac{(5-2)^5 - (-1-2)^5}{5}) =$$

$$= \pi(468 - \frac{3^5 + 3^5}{5}) = \pi(486 - \frac{486}{5}) = \pi(389 - \frac{1}{5}).$$

Objem tělesa vzniklého rotací kolem osy x plochy omezené grafy funkcí f a g je $\pi(389 - \frac{1}{5})$.