Cvičení 14 - výsledky

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Příklad 1. *Vypočítejte pomocí vhodné substituce:*

$a)\int \frac{1}{x}\left(7\ln^6 x - 8\ln^5 x + 9\ln x - 1\right)\,\mathrm{d}x = \left|\begin{array}{c} y=\ln x \\ \mathrm{d}y = \frac{1}{x}\,\mathrm{d}x \end{array}\right| = \int 7y^6 - 8y^5 + 9y - 1 =$
$= \ln^7 x - \frac{4}{3}\ln^6 x + \frac{9}{2}\ln^2 x - \ln x + c, x \in (0;\infty),$

$b)\int \frac{e^{2x}-3e^x+1}{(e^x-2)(e^x+6)}\,\mathrm{d}x = \left|\begin{array}{c} y=e^x \\ \mathrm{d}y = e^x\,\mathrm{d}x \end{array}\right| = \int \frac{y^2-3y+1}{y(y-2)(y+6)} = \int -\frac{1}{12y} - \frac{1}{16(y-2)} + \frac{55}{48(y+6)}\,\mathrm{d}y =$
$= -\frac{1}{12}x - \frac{1}{16}\ln|e^x-2| + \frac{55}{48}\ln|e^x+6| + c,\ \ x \in \mathbb{R}\setminus\{\ln 2\},$

$c)\int \frac{1}{x(4+4\ln^2 2x)}\,\mathrm{d}x = \left|\begin{array}{c} y=\ln 2x \\ \mathrm{d}y = \frac{1}{x}\,\mathrm{d}x \end{array}\right| = \int \frac{1}{4(1+y^2)}\,\mathrm{d}y = \frac{1}{4}\arctan\ln 2x + c,\ \ x \in (0;\infty)$

$d)\int \frac{3\ln^2 x}{x+x\ln^3 x}\,\mathrm{d}x = \left|\begin{array}{c} y=\ln x \\ \mathrm{d}y = \frac{1}{x}\,\mathrm{d}x \end{array}\right| = \int \frac{3y^2}{1+y^3}\,\mathrm{d}y = \left|\begin{array}{c} z=y^3+1 \\ \mathrm{d}z = 3y^2\,\mathrm{d}y \end{array}\right| = \int z^{-1}\,\mathrm{d}z = \ln(1+\ln^3 x) + c,$
$x \in (e^{-1};\infty)$

$e)\int \frac{e^{8x}-1}{e^{4x}(e^{8x}+2e^{4x}+1)}\,\mathrm{d}x = \left|\begin{array}{c} y=e^{4x} \\ \mathrm{d}y = 4e^{4x} \end{array}\right| = \frac{1}{4}\int \frac{y^2-1}{y^2(y+1)^2}\,\mathrm{d}y = \frac{1}{4}\int \frac{y-1}{y^2(y+1)}\,\mathrm{d}y = \frac{1}{4}\int \frac{2}{y} - \frac{1}{y^2} - \frac{2}{y+1}\,\mathrm{d}y =$
$= \frac{1}{4}\left(2\ln(e^{4x}) + \frac{1}{e^{4x}} - 2\ln|e^{4x}+1|\right) + c = 2x + \frac{1}{4}e^{-4x} - \frac{1}{2}\ln|e^{4x}+1| + c,\ \ x \in \mathbb{R}$

$f)\int \frac{e^{-4x}+e^{-2x}}{e^{-6x}-3e^{-4x}+3e^{-2x}-1}\,\mathrm{d}x = \left|\begin{array}{c} y=e^{-2x} \\ \mathrm{d}y = -2e^{-2x}\,\mathrm{d}x \end{array}\right| = \int \frac{y+1}{(y-1)^3}\,\mathrm{d}y = \int \frac{1}{(y-1)^2} + \frac{2}{(y-1)^3}\,\mathrm{d}y =$
$= -\frac{1}{e^{-2x}-1} - \frac{1}{(e^{-2x}-1)^2} + c,\ \ x \in \mathbb{R}\setminus\{0\}.$

Příklad 2. *Vypočítejte pomocí vhodné substituce:*

$$a)\int \frac{\sin^2 x+\cos x}{(\sin^2 x+\frac{3}{2}\cos x)^2}\sin x\,\mathrm{d}x = \left|\begin{array}{l} y=\cos x \\ \mathrm{d}y = -\sin x\,\mathrm{d}x \end{array}\right| = -\int \frac{1-y^2+y}{(1-y^2+\frac{3}{2}y)^2}\,\mathrm{d}y = \int \frac{y^2-y-1}{(y-2)^2(y+\frac{1}{2})^2}\,\mathrm{d}y =$$
$$= \int \frac{44}{125(y-2)} + \frac{4}{25(y-2)^2} - \frac{44}{125(y+\frac{1}{2})} - \frac{1}{25(y+\frac{1}{2})^2}\,\mathrm{d}y =$$
$$= \frac{44}{125}\ln|\cos x - 2| - \frac{44}{125}\ln|\cos x + \tfrac{1}{2}| - \frac{4}{25(y-2)} + \frac{1}{25(y+\frac{1}{2})} + c,\ \ x \in \mathbb{R}\setminus\{l\tfrac{2\pi}{3}+2k\pi; l=1,2, k\in\mathbb{Z}\},$$

$$b)\int \mathrm{tg}^5 x\,\mathrm{d}x = \int \frac{\sin^4 x}{\cos^5 x}\sin x\,\mathrm{d}x = \left|\begin{array}{l} y=\cos x \\ \mathrm{d}y = -\sin x\,\mathrm{d}x \end{array}\right| = -\int \frac{(1-y^2)^2}{y^5}\,\mathrm{d}y = -\int y^{-5} - 2y^{-3} + y^{-1}\,\mathrm{d}y =$$
$$= \frac{1}{4\cos^4 x} - \frac{1}{\cos^2 x} + \ln|\cos x| + c, x \in \mathbb{R}\setminus\{\tfrac{\pi}{2}+k\pi; k\in\mathbb{Z}\},$$

$$c)\int \frac{1}{\cos^3 x}\,\mathrm{d}x = \int \frac{\cos x}{\cos^4 x}\,\mathrm{d}x = \left|\begin{array}{l} y=\sin x \\ \cos x\,\mathrm{d}y = \ \mathrm{d}x \end{array}\right| = \int \frac{1}{(1-y^2)^2}\,\mathrm{d}y = \frac{1}{4}\int \frac{1}{1-y} + \frac{1}{(1-y)^2} + \frac{1}{1+y} + \frac{1}{(1+y)^2}\,\mathrm{d}y =$$
$$= \frac{1}{4}\left(-\ln|1-\sin x| + \frac{1}{1-\sin x} + \ln|1+\sin x| - \frac{1}{1+\sin x}\right) + c,\ \ x\in\mathbb{R}\setminus\{\tfrac{\pi}{2}+k\pi; k\in\mathbb{Z}\}$$

$$d)\int \frac{2\sin x - \cos x}{3\sin^2 x + 4\cos^2 x}\,\mathrm{d}x = 2I - II,$$
$$I = \int \frac{\sin x}{3\sin^2 x+4\cos^2 x}\,\mathrm{d}x = \left|\begin{array}{l} y=\cos x \\ \mathrm{d}y = -\sin x\,\mathrm{d}x \end{array}\right| = \int \frac{1}{3+y^2}\,\mathrm{d}y = \frac{\sqrt{3}}{3}\arctan(\tfrac{\sqrt{3}}{3}\cos x) + c,\ \ x\in\mathbb{R}$$
$$II = \int \frac{\cos x}{3\sin^2 x+4\cos^2 x}\,\mathrm{d}x = \left|\begin{array}{l} y=\sin x \\ \mathrm{d}y = \cos x\,\mathrm{d}x \end{array}\right| = \int \frac{1}{4-y^2}\,\mathrm{d}y = \frac{1}{4}\int -\frac{1}{2+y} + \frac{1}{y-2}\,\mathrm{d}y =$$
$$= \frac{1}{4}\left(-\ln|2+\sin x| + \ln|\sin x - 2|\right) + c,\ \ x\in\mathbb{R},$$

$$e)\int \frac{1}{\sin^6 x+\cos^6 x}\,\mathrm{d}x = \int \frac{1}{(\sin^2 x+\cos^2 x)\cdot(\sin^4 x - \cos^2 x\sin^2 x + \cos^4 x)} = \left|\begin{array}{l} y=\mathrm{tg}\, x \\ \mathrm{d}y = \frac{1}{\cos^2 x}\,\mathrm{d}x \end{array}\right| =$$
$$= \int \frac{1}{(\frac{y^2}{1+y^2})^2 - (\frac{y}{1+y^2})^2 + (\frac{1}{1+y^2})^2}\cdot\frac{1}{1+y^2}\,\mathrm{d}y = \int \frac{1+y^2}{y^4-y^2+1}\,\mathrm{d}y = \frac{1}{2}\int \frac{1}{y^2-\sqrt{3}y+1} + \frac{1}{y^2+\sqrt{3}y+1}\,\mathrm{d}y =$$
$$= 8\left(\arctan(\tfrac{2y+\sqrt{3}}{4}) + \arctan(\tfrac{2y-\sqrt{3}}{4})\right) + c,\ \ x\in(-\tfrac{\pi}{2}+k\pi; \tfrac{\pi}{2}+k\pi), k\in\mathbb{Z},$$
$y = \mathrm{tg}\, x$:
$$\cos^2 x = \frac{\cos^2 x}{\cos^2 x+\sin^2 x} = \frac{1}{1+\mathrm{tg}^2 x} = \frac{1}{1+y^2}$$
$$\sin^2 x = \frac{\sin^2 x}{\cos^2 x+\sin^2 x} = \frac{\mathrm{tg}^2 x}{1+\mathrm{tg}^2 x} = \frac{y^2}{1+y^2}$$
$$\cos x\sin x = \frac{\sin x}{\cos x}\cos^2 x = \frac{y}{1+y^2}$$

$$f)\int \frac{\cos^2 x}{\sin x\cos x+\sin^2 x}\,\mathrm{d}x = \left|\begin{array}{l} y=\mathrm{tg}\, x \\ \mathrm{d}y = \frac{1}{\cos^2 x}\,\mathrm{d}x \end{array}\right| = \int \frac{\frac{1}{1+y^2}}{\frac{y}{1+y^2}+\frac{y^2}{1+y^2}}\cdot\frac{1}{1+y^2}\,\mathrm{d}y = \int \frac{1}{y(1+y)(1+y^2)}\,\mathrm{d}y =$$
$$= \int \frac{1}{y} + \frac{-1}{2(1+y)} + \frac{-y-1}{2(1+y^2)}\,\mathrm{d}y = \ln|\mathrm{tg}\, x| - \frac{1}{2}\ln|1+\mathrm{tg}\, x| - \frac{1}{4}\ln|1+\mathrm{tg}^2 x| - \frac{1}{2}x + c,$$
$$x\in(-\tfrac{\pi}{2}+k\pi; \tfrac{\pi}{2}+k\pi), k\in\mathbb{Z},$$

$$g)\int \cos x\sqrt{1+\sin^2 x}\,\mathrm{d}x = \left|\begin{array}{l} y=\sin x \\ \mathrm{d}y = \cos x\,\mathrm{d}x \end{array}\right| = \int\sqrt{1+y^2}\,\mathrm{d}y = \left|\begin{array}{l} y=\sinh t \\ \mathrm{d}y = \cosh t\,\mathrm{d}t \end{array}\right| = \int \cosh^2 t\,\mathrm{d}t =$$
$$= \frac{1}{4}\int e^{2t} + e^{-2t} + 2\,\mathrm{d}t = \frac{1}{4}(\sinh(2t) + 2t) = \frac{1}{2}\sinh t\cosh t + \frac{1}{2}t + c =$$
$$= \frac{1}{2}\sin x\sqrt{1+\sin^2 x} + \frac{1}{2}\operatorname{arcsinh}(\sin x) + c, x\in\mathbb{R},$$

$h)\int \frac{\sin x\cos x+\sin^2 x}{\cos^2 x}\,\mathrm{d}x = \begin{vmatrix} y=\operatorname{tg} x \\ \mathrm{d}y=\frac{1}{\cos^2 x}\,\mathrm{d}x \end{vmatrix} = \int \frac{y}{1+y^2}+\frac{y^2}{1+y^2}\,\mathrm{d}y = \frac{1}{2}\ln|\operatorname{tg} x| + \operatorname{tg} x - x + c,$
$x\in(-\frac{\pi}{2}+k\pi;k\pi)\cup(k\pi;\frac{\pi}{2}+k\pi), k\in\mathbb{Z},$

$i)\int \sin^4 x\cos^2 x\,\mathrm{d}x = \frac{1}{8}\int(1-\cos 2x)^2(1+\cos 2x)\,\mathrm{d}x = \frac{1}{8}\int 1-\cos 2x-\cos^2 2x+\cos^3 2x\,\mathrm{d}x =$
$= \frac{1}{8}\int \sin^2 2x + \cos 2x(-\sin^2 2x)\,\mathrm{d}x = \frac{1}{8}(\frac{1}{2}\int 1-\cos 4x - \int \sin^2 2x\cos 2x\,\mathrm{d}x) =$
$= \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + c,\ \ x\in\mathbb{R}$

$j)\int \frac{1}{2+\cos x}+\frac{3}{2-\sin x}\,\mathrm{d}x = \begin{vmatrix} y=\operatorname{tg}\frac{x}{2} \\ \mathrm{d}y=\frac{1}{2\cos^2\frac{x}{2}}\,\mathrm{d}x \end{vmatrix} = \int\left(\frac{1}{2+\frac{1-y^2}{1+y^2}}+\frac{3}{2-\frac{2y}{1+y^2}}\right)\frac{1}{2(\frac{1-y^2}{1+y^2})^2}\,\mathrm{d}y =$
$= \int\left(\frac{1+y^2}{3+y^2}+\frac{3(1+y^2)}{2(y^2-y+1)}\right)\frac{(1+y^2)^2}{2(1-y^2)^2}\,\mathrm{d}y = \frac{1}{2}\int\frac{1+y^2}{3+y^2}+\frac{3(1+y^2)}{2(y^2-y+1)}\,\mathrm{d}y + \int\frac{2y^2(1+y^2)}{(3+y^2)(1-y^2)^2}+\frac{6y^2(1+y^2)}{2(y^2-y+1)(1-y^2)^2}\,\mathrm{d}y,$
$y=\operatorname{tg}\frac{x}{2}:$
$\cos x = \cos^2\frac{x}{2}-\sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{1-y^2}{1+y^2},$
$\sin x = 2\cos\frac{x}{2}\sin\frac{x}{2} = \frac{2\cos\frac{x}{2}\sin\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{2y}{1+y^2},$

$k)\ \int\sin ax\cdot\cos bx\,\mathrm{d}x = \frac{1}{2}\int\sin((a+b)x)\cdot\sin((a-b)x)\,\mathrm{d}x = \begin{vmatrix} y=(a+b)x \\ \mathrm{d}y=a+b\,\mathrm{d}x \end{vmatrix} =$
$= \frac{1}{2(a+b)}\int\sin y\sin(\frac{a-b}{a+b}y)\,\mathrm{d}y =$
$= \frac{a+b}{4ab}\left(\frac{a-b}{a+b}\cos((a-b)x)\sin((a+b)x) - \cos((a+b)x)\sin((a-b)x)\right) + c, x\in\mathbb{R}, a,b\neq 0, a\neq -b.$

$I = \int\sin y\sin(Cy)\,\mathrm{d}y \overset{p.\ p.}{=} \begin{vmatrix} u=\sin Cy, & u'=C\cos Cy \\ v'=\sin y, & v=-\cos y \end{vmatrix} = -\cos y\sin Cy + C\int\cos y\cos Cy\,\mathrm{d}y \overset{p.\ p.}{=}$
$= \begin{vmatrix} u=\cos Cy, & u'=-C\sin Cy \\ v'=\cos y, & v=\sin y \end{vmatrix} = -\cos y\sin Cy + C(\cos Cy\sin y + C\int\sin Cy\sin y\,\mathrm{d}y)$
$I = -\cos y\sin Cy + C\cos Cy\sin y + C^2 I$
$I(1-C^2) = -\cos y\sin Cy + C\cos Cy\sin y + c$
$I = \frac{C\cos Cy\sin y - \cos y\sin Cy}{1-C^2} + c$