Cvičení 13 - výsledky

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Příklad 1. *Nechť* $k, n \in \mathbb{N} \cup \{0\}$.

$$a)\ \int \cos^{2k+1} x \sin^n x \,\mathrm{d}x = \int (1-\sin^2 x)^{2k} \sin^n x \cos x \,\mathrm{d}x = \left| \begin{array}{c} y = \sin x \\ \mathrm{d}y = \cos x \,\mathrm{d}x \end{array} \right| =$$

$$= \int \sum_{l=0}^{2k} (-1)^l \left(\begin{array}{c} 2k \\ l \end{array} \right) y^{n+2l} \,\mathrm{d}y = \sum_{l=0}^{2k} (-1)^l \left(\begin{array}{c} 2k \\ l \end{array} \right) \frac{\sin^{n+2l+1}(x)}{n+l+1} + c, \quad x \in \mathbb{R}.$$

$$b)\ \int \sin^{2k+1} x \cos^n x \,\mathrm{d}x \int (1-cos^2 x)^{2k} \cos^n x \sin x \,\mathrm{d}x = \left| \begin{array}{c} y = \cos x \\ \mathrm{d}y = -\sin x \,\mathrm{d}x \end{array} \right| =$$

$$= \int \sum_{l=0}^{2k} (-1)^{l+1} \left(\begin{array}{c} 2k \\ l \end{array} \right) y^{n+2l} \,\mathrm{d}y = \sum_{l=0}^{2k} (-1)^{l+1} \left(\begin{array}{c} 2k \\ l \end{array} \right) \frac{\cos^{n+2l+1}(x)}{n+l+1} + c, \quad x \in \mathbb{R},$$

$$c)\ \int \cos^{-2}(4x+3) \,\mathrm{d}x = \left| \begin{array}{c} y = 4x+3 \\ \mathrm{d}y = 4 \,\mathrm{d}x \end{array} \right| = \frac{1}{4} \int \frac{1}{\cos^2 y} \,\mathrm{d}y = \mathrm{tg}(4x+3) + c, \quad x \in \mathbb{R} \setminus \{\tfrac{\pi}{2} + k\pi; k \in \mathbb{Z}\}.$$

Příklad 2. *Zakrývacím pravidlem získáme koeficienty u nejvyšších mocnin* $\frac{1}{x-a}$. *Ostatní musíme dopočítat.*

$$a)\ \frac{x-7}{x^2+x-2} = \frac{3}{x+2} + \frac{-2}{x-1},$$

$$b)\ \frac{3x^5-14x^4+20x^3-3x^2-17x+12}{x^3-4x^2+4x} = 3x^2 - 2x + \frac{3}{x} + \frac{2}{x-2} + \frac{-1}{(x-2)^2},$$

$$c)\ \frac{8x^2-21x+19}{(x+1)(x-3)^2} = \frac{3}{x+1} + \frac{5}{x-3} + \frac{7}{(x-3)^2},$$

$$d)\ \frac{5x^3+28x^2+37x+62}{(x^2-4)(x^2+x+3)} = \frac{8}{x-2} + \frac{-3}{x+2} + \frac{1}{x^2+x+3}.$$

Příklad 3. *Nechť $a \in \mathbb{R}$ a $n \in \mathbb{N}, n \geq 2$. Vypočtěte:*

a) $\int \frac{1}{x+a}\,\mathrm{d}x = \ln|x-a| + c, \quad x \in \mathbb{R} \setminus \{a\},$

b) $\int \frac{1}{(x+a)^n}\,\mathrm{d}x = \frac{1}{(1-n)(x-a)^{n-1}}, \quad x \in \mathbb{R} \setminus \{a\},$

c) $\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \int \frac{1}{\frac{1}{a^2}\left((\frac{x}{a})^2+1\right)}\,\mathrm{d}x = \frac{1}{a} \cdot \arctan\frac{x}{a} + c, \quad x \in \mathbb{R},\ a > 0$

d) $\int \frac{1}{x^2+x+1}\,\mathrm{d}x = \int \frac{1}{(x+\frac{1}{2})^2-\frac{1}{4}+1}\,\mathrm{d}x = \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,\mathrm{d}x = \left| \begin{array}{c} y = \frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3} \\ \mathrm{d}y = \frac{2\sqrt{3}}{3}\,\mathrm{d}x \end{array} \right| = \frac{2\sqrt{3}}{3}\arctan(\frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}) + c,$
$x \in \mathbb{R},$

e) $\int \frac{x}{a^2 \pm x^2}\,\mathrm{d}x = \left| \begin{array}{c} y = \pm x^2 + a^2 \\ \mathrm{d}y = \pm 2x\,\mathrm{d}x \end{array} \right| = \pm\frac{1}{2}\ln|a^2 \pm x^2| + c, a > 0, \quad x \in \mathbb{R},$ *respektive* $x \in \mathbb{R} \setminus \{a\},$

f) $\int \frac{x+\frac{1}{2}}{(x^2+x+1)^2}\,\mathrm{d}x = \left| \begin{array}{c} y = x^2 + x + 1, \\ \mathrm{d}y = 2x + 1\,\mathrm{d}x \end{array} \right| = \int \frac{1}{2y^2}\,\mathrm{d}y = -\frac{1}{2x^2+2x+2} + c, \quad x \in \mathbb{R}.$

Příklad 4. *Vypočtěte:*

a) $\int \frac{-2x-6}{x^4-1}\,\mathrm{d}x = \int \frac{-2x-6}{(x-1)(x+1)(x^2+1)}\,\mathrm{d}x = \int \frac{1}{x+1} + \frac{-2}{x-1} + \frac{x}{x^2+1} + \frac{3}{x^2+1}\,\mathrm{d}x =$
$= \ln|x+1| - 2\ln|x-1| + \frac{1}{2}\ln(x^2+1) + 3\arctan x + c, \quad x \in \mathbb{R}\setminus\{-1,1\},$
Koeficienty $\frac{A}{x-1}, \frac{B}{x+1}$ *spočteme zakrývací metodou, zbylé dopočítáme roznásobením.*

b) $\int \frac{4x-5}{(x-2)(x-3)}\,\mathrm{d}x = \int \frac{-3}{x-2} + \frac{7}{x-3}\,\mathrm{d}x = -3\ln|x-2| + 7\ln|x-3| + c, \quad x \in \mathbb{R}\setminus\{2;3\},$

c) $\int \frac{x^2}{x^6+7x^3-8}\,\mathrm{d}x = \int \frac{x^2}{(x^3+8)(x^3-1)}\,\mathrm{d}x = \left| \begin{array}{l} y = x^3, \\ \mathrm{d}y = 3x^2\,\mathrm{d}x \end{array} \right| = \frac{1}{3}\int \frac{1}{(y-1)(y+2)}\,\mathrm{d}y =$
$= \frac{1}{3}\int \frac{\frac{1}{3}}{y-1} + \frac{-\frac{1}{3}}{y+2}\,\mathrm{d}y = \frac{1}{9}(\ln|x^3-1| - \ln|x^3+8|) + c, \quad x \in \mathbb{R}\setminus\{-2;1\},$
Kdybychom rozkládali původní integrál před substitucí, hledali bychom rozklad ve tvaru:
$\frac{x^2}{(x+2)(x^2-2x+4)(x-1)(x^2+x+1)} = \frac{-\frac{1}{27}}{x+2} + \frac{Ax+B}{x^2-2x+4} + \frac{\frac{1}{27}}{x-1} + \frac{Cx+D}{x^2+x+1},$

d) $\int \frac{x^7}{x^2+x-2}\,\mathrm{d}x = \int x^5 - x^4 + 3x^3 - 5x^2 + 11x - 21 + \frac{43x-42}{(x+2)(x-1)}\,\mathrm{d}x =$
$= \frac{x^6}{6} - \frac{x^5}{5} + \frac{3x^4}{4} - \frac{5x^3}{3} + \frac{11x^2}{2} - 21x + \int \frac{\frac{1}{3}}{x-1} + \frac{\frac{128}{3}}{x+2}\,\mathrm{d}x =$
$= \frac{x^6}{6} - \frac{x^5}{5} + \frac{3x^4}{4} - \frac{5x^3}{3} + \frac{11x^2}{2} - 21x + \frac{1}{3}\ln|x-1| + \frac{128}{3}\ln|x+2| + c, \quad x \in \mathbb{R}\setminus\{-2;1\},$

e) $\int \frac{x^4}{x^4+5x^2+4}\,\mathrm{d}x = \int 1 - \frac{5x^2+4}{(x^2+4)(x^2+1)} = x - \int \frac{-\frac{1}{3}}{x^2+1} + \frac{\frac{16}{3}}{x^2+4}\,\mathrm{d}x = x + \frac{1}{3}\arctan x - \frac{8}{3}\arctan\frac{x}{2} + c,$

f) $\int \frac{1}{x^5+x^4-2x^3-2x^2+x+1}\,\mathrm{d}x = \int \frac{1}{(x-1)^2(x+1)^3}\,\mathrm{d}x = \int \frac{-\frac{3}{16}}{x-1} + \frac{\frac{1}{4}}{(x-1)^2} + \frac{\frac{3}{16}}{x+1} + \frac{\frac{1}{4}}{(x+1)^2} + \frac{\frac{1}{8}}{(x+1)^3}\,\mathrm{d}x =$
$= -\frac{3}{16}\ln|x-1| - \frac{1}{4}\frac{1}{x-1} + \frac{3}{16}\ln|x+1| - \frac{1}{4}\frac{1}{x+1} - \frac{1}{16}\frac{1}{(x+1)^2} + c, \quad x \in \mathbb{R}\setminus\{-1;1\},$

g) $\int \frac{x}{(x-1)^2(x+1)}\,\mathrm{d}x = \int \frac{-\frac{1}{4}}{x+1} + \frac{\frac{1}{4}}{x-1} + \frac{\frac{1}{2}}{(x-1)^2}\,\mathrm{d}x = \frac{1}{4}(-\ln|x+1| + \ln|x-1| - 2\frac{1}{x-1}) + c,$
$x \in \mathbb{R}\setminus\{-1;1\},$

h) $\int \frac{x^2}{(x^2+2x+2)^2}\,\mathrm{d}x = \int \frac{1}{x^2+2x+2} + \frac{-2x-2}{(x^2+2x+2)^2}\,\mathrm{d}x = \arctan(x+1) - \ln|x^2+2x+1| + c, \quad x \in \mathbb{R}.$
$\int \frac{1}{x^2+2x+2}\,\mathrm{d}x = \int \frac{1}{(x+1)^2+1}\,\mathrm{d}x = \arctan(x+1) + c, \quad x \in \mathbb{R},$
$\int \frac{2x+2}{(x^2+2x+2)^2}\,\mathrm{d}x = \left| \begin{array}{l} y = x^2+2x+1 \\ \mathrm{d}y = 2x+2\,\mathrm{d}x \end{array} \right| = \int \frac{1}{y}\,\mathrm{d}y = \ln|x^2+2x+1| + c, \quad x \in \mathbb{R}.$

i) $\int \frac{2}{x(x+1)(x^2+x+1)}\,\mathrm{d}x = \int \frac{2}{x} + \frac{-2}{x+1} + \frac{-2}{x^2+x+1}\,\mathrm{d}x = 2\ln|x| - 2\ln|x+1| - \frac{4\sqrt{3}}{3}\arctan(\frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}) + c,$
$x \in \mathbb{R}\setminus\{0,-1\},$

j) $\int \frac{x^2-2x+3}{(x-1)^2(x^2+1)}\,\mathrm{d}x = \int \frac{-1}{x-1} + \frac{1}{(x-1)^2} + \frac{x+1}{x^2+1}\,\mathrm{d}x = -\ln|x-1| - \frac{1}{x-1} + \frac{1}{2}\ln(x^2+1) + \arctan x + c,$
$x \in \mathbb{R}\setminus\{1\}.$