Cvičení 12 - výsledky

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Příklad 1. $F'(x) = \left(3 + \ln\sqrt{\left|\frac{x+1}{1-x}\right|}\right)' = \sqrt{\left|\frac{1-x}{x+1}\right|} \cdot \frac{1}{2} \cdot \sqrt{\left|\frac{1-x}{x+1}\right|} \cdot \operatorname{sign}\left(\frac{x+1}{1-x}\right) \cdot \frac{2}{(1-x)^2} = \frac{1-x}{x+1} \cdot \frac{1}{(1-x)^2} = \frac{1}{1-x^2}.$

Příklad 2. *Můžeme postupovat dvěma způsoby. Buď obě funkce zderivujeme a zkontrolujeme, jestli obě derivace jsou stejné, nebo dokážeme, že $F_1(x) - F_2(x) =$konstanta.*

a) $F_1'(x) = (\ln\sqrt{x-2} - 5)' = \frac{1}{\sqrt{x-2}} \cdot \frac{1}{2\sqrt{x-2}} = \frac{1}{2x-4}$,
$F_2'(x) = (\ln\sqrt{2x-4})' = \frac{1}{\sqrt{2x-4}} \cdot \frac{1}{\sqrt{2x-4}} = \frac{1}{2x-4}$,

b) $F_1(x) = \cos 2x$,
$F_2(x) = 6\cos^2 x + 4\sin^2 x = 5\cos^2 x + 5\sin^2 x + \cos^2 x - \sin^2 x = \cos 2x + 5$,

c) $F_1(x) = \sin 2x$,
$F_2(x) = (\cos x + \sin x)^2 = \cos^2 x + 2\cos x \sin x + \sin^2 x = \sin 2x + 1$.

Příklad 3.

a) $\int 6x^4 - 7x + 2 \,\mathrm{d}x = \frac{6}{5}x^5 - \frac{7}{2}x^2 + 2x + c, \quad x \in \mathbb{R}$,

b) $\int \frac{x^2}{\sqrt{x}} \,\mathrm{d}x = \int x^{\frac{3}{2}} \,\mathrm{d}x = \frac{2}{5}x^{\frac{5}{2}} + c, \quad x \in (0; \infty)$,

c) $\int \frac{x^4 - 1 + \sqrt{x}}{x^3} \,\mathrm{d}x = \int x - x^{-3} + x^{-\frac{5}{2}} \,\mathrm{d}x = \frac{x^2}{2} + \frac{1}{2x^2} - \frac{2}{3\sqrt{x^3}} + c, \quad x \in (0; \infty)$,

d) $\int 4 + \frac{1}{x} + \sqrt[3]{x} \,\mathrm{d}x = 4x + \ln|x| + \frac{3\sqrt[3]{x^4}}{4} + c, \quad x \in \mathbb{R} \setminus \{0\}$,

e) $\int \frac{1}{5x+25} \,\mathrm{d}x = \frac{1}{5}\int \frac{1}{x+5} \,\mathrm{d}x = \frac{1}{5}\ln|x+5| + c, \quad x \in \mathbb{R} \setminus \{-5\}$,

f) $\int \frac{x^2-4x+4}{4x^3-2x^4} \,\mathrm{d}x = \int \frac{(x-2)^2}{-2x^3(x-2)} \,\mathrm{d}x = \int \frac{x-2}{-2x^3} \,\mathrm{d}x = \int -\frac{1}{2x^2} + \frac{1}{x^3} \,\mathrm{d}x = \frac{1}{2x} - \frac{1}{2x^2} + c = \frac{x-1}{2x^2} + c$,
$x \in \mathbb{R} \setminus \{0; 2\}$,

g) $\int \sin^5 x \,\mathrm{d}x = \int (1 - \cos x)^2 \sin x \,\mathrm{d}x = |y = \cos x,\ \mathrm{d}y = -\sin x \,\mathrm{d}x| = -\int 1 - 2y + y^2 \,\mathrm{d}y =$
$= -\cos x + \cos^2 x - \frac{\cos^3 x}{3} + c, \quad x \in \mathbb{R}$,

h) $\int \frac{\operatorname{tg} x}{\sin 2x} \,\mathrm{d}x = \int \frac{\frac{\sin x}{\cos x}}{2\sin x \cos x} \,\mathrm{d}x = \int \frac{1}{2\cos^2 x} \,\mathrm{d}x = \frac{1}{2}\operatorname{tg} x + c, \quad x \in \mathbb{R} \setminus \{k\frac{\pi}{2}; k \in \mathbb{Z}\}$,

i) $\int \ln x \,\mathrm{d}x = \int 1 \cdot \ln x \,\mathrm{d}x \overset{per\ partes}{=} \left| \begin{array}{ll} u = \ln x, & u' = \frac{1}{x} \\ v' = 1, & v = x \end{array} \right| = x\ln x - \int x \cdot \frac{1}{x} = x\ln x - x + c, \quad x \in (0; \infty)$,

j) $\int (x+2)^2 \cos x \,\mathrm{d}x \overset{per\ partes}{=} \left| \begin{array}{ll} u = (x+2)^2, & u' = 2x + 4 \\ v' = \cos x, & v = \sin x \end{array} \right| = (x+2)^2 \sin x - \int (2x+4)\sin x \,\mathrm{d}x \overset{p.\ p.}{=}$
$= \left| \begin{array}{ll} u = 2x + 4, & u' = 2 \\ v' = \sin x, & v = -\cos x \end{array} \right| = (x+2)^2 \sin x + 2(x+2)\cos x - \int 2\cos x \,\mathrm{d}x =$
$= (x+2)^2 \sin x + 2(x+2)\cos x - 2\sin x + c, \quad x \in \mathbb{R}$,

$$k)\ \int \sin x \cos x \,\mathrm{d}x = \left| \begin{array}{l} y = \sin x, \\ \mathrm{d}y = \cos x \end{array} \right| = \int y \,\mathrm{d}y = \tfrac{\sin^2 x}{2} + c, \quad x \in \mathbb{R},$$

$$l)\ \int \sin ax \,\mathrm{d}x = \begin{cases} -\frac{\cos ax}{a} + c & a \neq 0 \\ 0 + c & a = 0 \end{cases}, \ a \in \mathbb{R}, \quad x \in \mathbb{R},$$

$$m)\ \int x \sin \tfrac{x}{2} \,\mathrm{d}x \overset{p.\,p.}{=} \left| \begin{array}{ll} u = x, & u' = 1 \\ v' = \sin \frac{x}{2}, & v = -2\cos \frac{x}{2} \end{array} \right| = -2x \cos \tfrac{x}{2} + 2 \int \cos \tfrac{x}{2} \,\mathrm{d}x = -2x \cos \tfrac{x}{2} + 4 \sin \tfrac{x}{2} + c,$$
$$x \in \mathbb{R},$$

$$n)\ \int (x^3 - x + 5)e^x \,\mathrm{d}x \overset{p.\,p.}{=} \left| \begin{array}{ll} u = x^3 - x + 5, & u' = 3x^2 - 1 \\ v' = e^x, & v = e^x \end{array} \right| = (x^3 - x + 5)e^x - \int (3x^2 - 1)e^x \,\mathrm{d}x \overset{p.\,p.}{=}$$
$$= \left| \begin{array}{ll} u = 3x^2 - 1, & u' = 6x \\ v' = e^x, & v = e^x \end{array} \right| = e^x(x^3 - 3x^2 - x + 6) + 6 \int x e^x \,\mathrm{d}x \overset{p.\,p.}{=} \left| \begin{array}{ll} u = x, & u' = 1 \\ v' = e^x, & v = e^x \end{array} \right| =$$
$$= e^x(x^3 - 3x^2 + 5x + 6) - 6 \int e^x \,\mathrm{d}x = e^x(x^3 - 3x^2 + 5x) + c, \quad x \in \mathbb{R},$$

$$o)\ \int \sin^3 x \cos x + \cos(-8x) \,\mathrm{d}x = \int \sin^3 x \cos x \,\mathrm{d}x - \tfrac{1}{8} \sin(-8x) = \left| \begin{array}{l} y = \sin x \\ \mathrm{d}y = \cos x \,\mathrm{d}x \end{array} \right| =$$
$$= \int y^3 \,\mathrm{d}y + \tfrac{1}{8} \sin(8x) = \tfrac{1}{4} \sin^4 x + \tfrac{1}{8} \sin(8x) + c, \quad x \in \mathbb{R},$$

$$p)\ \int x\sqrt{x^2 + 1} \,\mathrm{d}x = \left| \begin{array}{l} y = x^2 + 1 \\ \mathrm{d}y = 2x \,\mathrm{d}x \end{array} \right| = \int \tfrac{1}{2} \sqrt{y} \,\mathrm{d}y = \tfrac{1}{3} \sqrt{(x^2 + 1)^3} + c, \quad x \in \mathbb{R},$$

$$q)\ \int \cos^3 x \,\mathrm{d}x = \int (1 - \sin^2 x) \cos x \,\mathrm{d}x = \left| \begin{array}{l} y = \sin x \\ \mathrm{d}y = \cos x \,\mathrm{d}x \end{array} \right| = \int 1 - y^2 \,\mathrm{d}y = \sin x - \tfrac{1}{3} \sin^3 x + c =$$
$$= \sin x (1 - \tfrac{1}{3} \sin^2 x) + c\ , \quad x \in \mathbb{R},$$

$$s)\ \int \tfrac{\sqrt{x^2+1} - x\sqrt{1-x^2}}{\sqrt{1-x^4}} \,\mathrm{d}x = \int \sqrt{\tfrac{x^2+1}{(1-x^2)(x^2+1)}} - x\sqrt{\tfrac{1-x^2}{(1-x^2)(x^2+1)}} \,\mathrm{d}x = \int \tfrac{1}{\sqrt{1-x^2}} - \tfrac{x}{\sqrt{1+x^2}} \,\mathrm{d}x =$$
$$= \arcsin x - \int \tfrac{x}{\sqrt{1+x^2}} \,\mathrm{d}x = \left| \begin{array}{l} y = x^2 + 1 \\ \mathrm{d}y = 2x \,\mathrm{d}x \end{array} \right| = \arcsin x - \int \tfrac{1}{2\sqrt{y}} \,\mathrm{d}y = \arcsin x - \sqrt{x^2 + 1} + c,$$
$$x \in (-1; 1),$$

$$t)\ \int \operatorname{cotg} x \,\mathrm{d}x = \int \tfrac{\cos x}{\sin x} \,\mathrm{d}x = \left| \begin{array}{l} y = \sin x \\ \mathrm{d}y = \cos x \,\mathrm{d}x \end{array} \right| = \int y^{-1} \,\mathrm{d}y = \ln |\sin x| + c, \quad x \in \mathbb{R} \setminus \{k\pi; k \in \mathbb{Z}\},$$

$$u)\ \int \tfrac{6}{\sqrt{9 - 9x^2}} \,\mathrm{d}x = \tfrac{6}{3} \int \tfrac{1}{\sqrt{1-x^2}} \,\mathrm{d}x = 2 \arcsin x + c, \quad x \in (-1; 1),$$

$$v)\ \int \tfrac{1}{\sin^2(\frac{2x+3}{5})} \,\mathrm{d}x = \left| \begin{array}{l} y = \frac{2x+3}{5} \\ \mathrm{d}y = \frac{2}{5} \,\mathrm{d}x \end{array} \right| = \tfrac{5}{2} \int \tfrac{1}{\sin^2 y} \,\mathrm{d}y = -\tfrac{5}{2} \operatorname{cotg}(\tfrac{2x+3}{5}) + c, \quad x \in \mathbb{R} \setminus \{-\tfrac{3}{2} + \tfrac{5}{2}k\pi; k \in \mathbb{Z}\},$$

$$w)\ \int \tfrac{1}{1 + (3 - 5x)^2} \,\mathrm{d}x = \left| \begin{array}{l} y = 3 - 5x \\ \mathrm{d}y = -5 \,\mathrm{d}x \end{array} \right| = -\tfrac{1}{5} \int \tfrac{1}{1 + y^2} \,\mathrm{d}y = -\tfrac{1}{5} \arctan(3 - 5x) + c, \quad x \in \mathbb{R}$$

$x)$ $\int e^{4x+3}(x+x^2)+6^x - x\cosh x^2\,\mathrm{d}x = \int e^{4x+3}(x+x^2)\,\mathrm{d}x + \int 6^x\,\mathrm{d}x - \int x\cosh x^2\,\mathrm{d}x =$
$= \frac{1}{4}e^{4x+3}\left(x^2+\frac{1}{2}x-\frac{1}{8}\right)+\frac{6^x}{\ln 6}-\frac{1}{2}\sinh x^2 + c, \quad x\in\mathbb{R}$

$$\int e^{4x+3}(x+x^2)\,\mathrm{d}x \overset{p.\ p.}{=} \left| \begin{array}{ll} u = x+x^2, & u' = 1+2x \\ v' = e^{4x+3}, & v = \frac{1}{4}e^{4x+3} \end{array} \right| = \frac{1}{4}e^{4x+3}(x+x^2) - \frac{1}{4}\int e^{4x+3}(1+2x)\,\mathrm{d}x \overset{p.\ p.}{=}$$
$$= \left| \begin{array}{ll} u = 1+2x, & u' = 2 \\ v' = e^{4x+3}, & v = \frac{1}{4}e^{4x+3} \end{array} \right| = \frac{1}{4}e^{4x+3}(x+x^2) - \frac{1}{16}e^{4x+3}(1+2x) + \frac{1}{8}\int e^{4x+3}\,\mathrm{d}x =$$
$$= \frac{1}{4}e^{4x+3}\left(x^2+x-\frac{1}{4}(2x+1)+\frac{1}{8}\right)+c, \quad x\in\mathbb{R},$$
$$\int 6^x\,\mathrm{d}x = \frac{6^x}{\ln 6}+c, \quad x\in\mathbb{R},$$
$$\int x\cosh x^2\,\mathrm{d}x = \left| \begin{array}{l} y = x^2 \\ \mathrm{d}y = 2x\,\mathrm{d}x \end{array} \right| = \frac{1}{2}\int \cosh y\,\mathrm{d}y = \frac{1}{2}\sinh x^2 + c, \quad x\in\mathbb{R},$$

$y)$ $$\int \frac{\sin\ln x}{x}\,\mathrm{d}x = \left| \begin{array}{l} y = \ln x \\ \mathrm{d}y = \frac{\mathrm{d}x}{x} \end{array} \right| = \int \sin y\,\mathrm{d}y = -\cos\ln x + c, \quad x\in(0;\infty),$$

$z)$ $$\int \sin\frac{x}{2}\ln\operatorname{tg}\frac{x}{4}\,\mathrm{d}x \overset{p.\ p.}{=} \left| \begin{array}{ll} u = \ln\operatorname{tg}\frac{x}{4} & u' = \frac{1}{4\sin\frac{x}{4}\cos\frac{x}{4}} \\ v' = \sin\frac{x}{2} & v = -2\cos\frac{x}{2} \end{array} \right| = -2\ln\operatorname{tg}\frac{x}{4}\cdot\cos\frac{x}{2} + \int \frac{\cos\frac{x}{2}}{2\sin\frac{x}{4}\cos\frac{x}{4}}\,\mathrm{d}x =$$
$$= -2\ln\operatorname{tg}\frac{x}{4}\cdot\cos\frac{x}{2} + \int \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\,\mathrm{d}x = \left| \begin{array}{l} y = \sin\frac{x}{2} \\ \mathrm{d}y = \frac{1}{2}\cos\frac{x}{2}\,\mathrm{d}x \end{array} \right| = -2\ln\operatorname{tg}\frac{x}{4}\cdot\cos\frac{x}{2} + 2\int\frac{1}{y}\,\mathrm{d}y =$$
$$= -2\ln\operatorname{tg}\frac{x}{4}\cdot\cos\frac{x}{2} + 2\ln\left|\sin\frac{x}{2}\right| + c, \quad x\in\left(k\pi;\frac{\pi}{2}+k\pi\right), k\in\mathbb{Z}.$$

$\check{z})$ $$I = \int e^x\sin x\,\mathrm{d}x \overset{p.\ p.}{=} \left| \begin{array}{ll} u = e^x, & u' = e^x \\ v' = \sin x, & v = -\cos x \end{array} \right| = -e^x\cos x + \int e^x\cos x\,\mathrm{d}x \overset{p.\ p.}{=}$$
$$\left| \begin{array}{ll} u = e^x, & u' = e^x \\ v' = \cos x, & v = \sin x \end{array} \right| = -e^x\cos x + e^x\sin x - \int e^x\sin x\,\mathrm{d}x = e^x(\sin x - \cos x) - I$$
$$I = e^x(\sin x - \cos x) - I$$
$$I = \frac{1}{2}e^x(\sin x - \cos x), \quad x\in\mathbb{R}.$$