Cvičení 8 - výsledky

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Příklad 1.

$$a)\ \lim_{x\to 0}\frac{\sin x}{x}\overset{l'H}{=}\lim_{x\to 0}\frac{\cos x}{1}=\cos 0=1,$$
$$b)\ \lim_{x\to 1}\frac{\ln x}{x-1}\overset{l'H}{=}\lim_{x\to 1}\frac{1}{x\cdot 1}=1,$$
$$c)\ \lim_{x\to 0}\frac{e^x-1}{x}\overset{l'H}{=}\lim_{x\to 0}\frac{e^x}{1}=e^0=1,$$
$$d)\ \lim_{x\to 0}\frac{1-\cos x}{x^2}\overset{l'H}{=}\lim_{x\to 0}\frac{\sin x}{2x}\overset{l'H}{=}\lim_{x\to 0}\frac{\cos x}{2}=\frac{\cos 0}{2}=\frac{1}{2}.$$

Příklad 2.

$$a)\ \lim_{x\to\frac{\pi}{2}}\frac{1-\sin 5x}{1-\sin x}\overset{l'H}{=}\lim_{x\to\frac{\pi}{2}}\frac{-5\cos 5x}{-\cos x}\overset{l'H}{=}\lim_{x\to\frac{\pi}{2}}\frac{25\sin 5x}{\sin x}=25,$$
$$b)\ \lim_{x\to 0}\frac{\sin x^3}{x^3}\overset{l'H}{=}\lim_{x\to 0}\frac{3x^2\cos x^3}{3x^2}=\lim_{x\to 0}\cos x^3=1,$$
$$c)\ \lim_{x\to 0}\frac{1}{x}\cdot\ln(e^x+x)\overset{l'H}{=}\lim_{x\to 0}\frac{1}{e^x+x}\cdot(e^x+1)=2,$$
$$d)\ \lim_{x\to 0}\frac{\operatorname{tg}x-x}{x-\sin x}\overset{l'H}{=}\lim_{x\to 0}\frac{\frac{1}{\cos^2 x}-1}{1-\cos x}=\lim_{x\to 0}\frac{1+\cos x}{\cos^2 x}=2,$$
$$e)\ \lim_{x\to 0+}\frac{\ln(\sin ax)}{\ln(\sin bx)}\overset{l'H}{=}\lim_{x\to 0+}\frac{\cos ax}{\cos bx}\frac{a\sin bx}{b\sin ax}\cdot\frac{abx}{abx}=\frac{a}{b}\cdot\frac{b}{a}=1,$$
$$f)\ \lim_{x\to\infty}\frac{\log(x+1)}{2x+2}\overset{l'H}{=}\lim_{x\to\infty}\frac{1}{2\cdot(x+1)}=0,$$
$$g)\ \lim_{x\to\infty}\frac{e^x}{x^3}\overset{l'H}{=}\lim_{x\to\infty}\frac{e^x}{3x^2}\overset{l'H}{=}\lim_{x\to\infty}\frac{e^x}{6x}=\overset{l'H}{=}\lim_{x\to\infty}\frac{e^x}{6}=\infty,$$
$$h)\ \lim_{x\to-\frac{\pi}{2}-}\frac{3\operatorname{cotg}2x}{\operatorname{tg}x}\overset{l'H}{=}\lim_{x\to-\frac{\pi}{2}-}-6\frac{\cos^2 x}{\sin^2 2x}\overset{l'H}{=}\lim_{x\to-\frac{\pi}{2}-}6\frac{2\cos x\sin x}{4\sin 2x\cos 2x}=\lim_{x\to-\frac{\pi}{2}-}\frac{6}{4\cos 2x}=-\frac{6}{4}=-\frac{3}{2},$$
$$i)\ \lim_{x\to\infty}\frac{\ln x}{\log_b x}=\lim\frac{\frac{1}{x}}{\frac{1}{x\ln b}}=\ln b\ \textit{pro}\ b>1,$$
$$j)\ \lim_{x\to 0}\frac{x^3-6x+6\sin x}{2x^5}\overset{l'H}{=}\lim_{x\to 0}\frac{3x^2-6+6\cos x}{10x^4}\overset{l'H}{=}\lim_{x\to 0}\frac{6x-6\sin x}{40x^3}\overset{l'H}{=}\lim_{x\to 0}\frac{6-6\cos x}{120x^2}\overset{l'H}{=}\lim_{x\to 0}\frac{6\sin x}{240x}=$$
$$\overset{l'H}{=}\lim_{x\to 0}\frac{6\cos x}{240}=\frac{1}{40},$$
$$k)\ \lim_{x\to\frac{\pi}{2}}(\operatorname{tg}x-\frac{1}{\cos x})=\lim_{x\to\frac{\pi}{2}}\frac{\sin x-1}{\cos x}\overset{l'H}{=}\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{-\sin x}=\frac{0}{-1}=0,$$
$$l)\ \lim_{x\to 0+}\sin x\cdot\ln x=\lim_{x\to 0+}\frac{\ln x}{\frac{1}{\sin x}}\overset{l'H}{=}\lim_{x\to 0+}\frac{\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}}=\lim_{x\to 0+}\frac{\sin x}{x}\cdot\frac{-\sin x}{\cos x}=1\cdot 0=0,$$
$$m)\ \lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{\ln x}\right)=\lim_{x\to 1}\frac{1-x}{\ln x}\overset{l'H}{=}\lim_{x\to 1}\frac{-1}{\frac{1}{x}}=\lim_{x\to 1}-x=-1,$$
$$n)\ \lim_{x\to 0}(e^x-1)\cdot\operatorname{cotg}x=\lim_{x\to 0}\frac{e^x-1}{\operatorname{tg}x}\overset{l'H}{=}\lim_{x\to 0}\frac{e^x}{\frac{1}{\cos^2 x}}=\lim_{x\to 0}e^x\cdot\cos^2 x=1,$$
$$o)\ \lim_{x\to 1}\frac{4\cdot 3^x-3\cdot 4^x}{6^x-2^x-4\cdot 3^{x-1}}=\lim_{x\to 1}\frac{4\cdot e^{x\ln 3}-3\cdot e^{x\ln 4}}{e^{x\ln 6}-e^{x\ln 2}-4\cdot e^{(x-1)\ln 3}}\overset{l'H}{=}\lim_{x\to 1}\frac{4\cdot\ln 3\cdot e^{x\ln 3}-3\cdot\ln 4\cdot e^{x\ln 4}}{\ln 6\cdot e^{x\ln 6}-\ln 2\cdot e^{x\ln 2}-4\cdot\ln 3\cdot e^{(x-1)\ln 3}}=$$
$$=\frac{4\cdot\ln 3\cdot 3-3\cdot\ln 4\cdot 4}{\ln 6\cdot 6-2\cdot\ln 2-4\cdot\ln 3\cdot 1}=\frac{12\ln\frac{4}{3}}{6\ln 6-2\ln 18}=\frac{6\ln\frac{3}{4}}{\ln 12},$$
$$p)\ \lim_{x\to 1}x^{\frac{2}{x^2-3x+2}}=\lim_{x\to 1}\exp\left(\frac{2}{x^2-3x+2}\cdot\ln x\right)=\exp(\lim_{x\to 1}\frac{2\ln x}{x^2-3x+2}\overset{l'H}{=}\exp(\lim_{x\to 1}\frac{\frac{2}{x}}{(2x-3)})=e^{-2}.$$

Příklad 3. *Taylorův polynom funkce f stupně n v bodě x_0:*

$$\mathbf{T_n^{x_0}(x)=f(x_0)+\frac{f'(x_0)}{1!}\cdot(x-x_0)+\frac{f''(x_0)}{2!}\cdot(x-x_0)^2+\dots+\frac{f^{(n)}(x_0)}{n!}\cdot(x-x_0)^n.}$$

Taylorův polynom splňuje $f^{(k)}(x_0) = (T_n^{x_0}(x_0))^{(k)}$ *pro* $k = 0, 1, \ldots, n$.

a) $f(x) = e^{2x}\sin 5x + 6$ *v bodě* $x_0 = \frac{\pi}{2}$,

$$f^{(4)}(x) = (e^{2x}\sin 5x + 6)^{(4)} = (2e^{2x}\sin 5x + 5e^{2x}\cos 5x)^{(3)} = (4e^{2x}\sin 5x + 20e^{2x}\cos 5x - 25e^{2x}\sin 5x)'' =$$
$$= (-21e^{2x}\sin 5x + 20e^{2x}\cos 5x)'' = (-42e^{2x}\sin 5x - 105e^{2x}\cos 5x + 40e^{2x}\cos 5x - 100e^{2x}\sin 5x)' =$$
$$= (-65e^{2x}\cos 5x - 142e^{2x}\sin 5x)' = 41e^{2x}\sin 5x - 840e^{2x}\cos 5x,$$

$$T_4^{\frac{\pi}{2}}(x) = (e^{\pi} + 6) + 2e^{\pi}(x - \tfrac{\pi}{2}) - \tfrac{21}{2}e^{\pi}(x - \tfrac{\pi}{2})^2 - \tfrac{142}{6}e^{\pi}(x - \tfrac{\pi}{2})^3 + \tfrac{41}{24}e^{\pi}(x - \tfrac{\pi}{2})^4,$$

b) $f(x) = \frac{1}{4}\ln\frac{x^2-1}{x^2+1}$ *v bodě* $x_0 = 2$,

$$f^{(4)}(x) = (\tfrac{1}{4}\ln\tfrac{x^2-1}{x^2+1})^{(4)} = (\tfrac{x}{x^4-1})^{(3)} = (\tfrac{-3x^4-1}{(x^4-1)^2})'' = (\tfrac{20x^3+12x^4}{(x^4-1)^3})' = \tfrac{-60x^2-264x^6-60x^{10}}{(x^4-1)^4},$$

$$T_4^2(x) = \tfrac{1}{4}\ln\tfrac{1}{5} + \tfrac{2}{15}(x-2) - \tfrac{49}{30}(x-2)^2 + \tfrac{352}{20250}(x-2)^3 - \tfrac{78576}{1215000}(x-2)^4,$$

c) $f(x) = \sqrt[3]{x}$ *v bodě* $x_0 = 3$,

$$f^{(4)}(x) = (x^{\frac{1}{3}})^{(4)} = (\tfrac{1}{3}x^{-\frac{2}{3}})^{(3)} = (-\tfrac{2}{9}x^{-\frac{5}{3}})'' = (\tfrac{10}{27}x^{-\frac{8}{3}})' = -\tfrac{80}{81}x^{-\frac{11}{3}},$$

$$T_4^3(x) = \sqrt[3]{3} + \tfrac{1}{3\sqrt[3]{3^2}}(x-3) - \tfrac{2}{18\sqrt[3]{3^5}}(x-3)^2 + \tfrac{10}{162\sqrt[3]{3^8}}(x-3)^3 - \tfrac{80}{1944\sqrt[3]{3^{11}}}(x-3)^4,$$

d) $f(x) = \sin x$ *v bodě* $x_0 = \frac{\pi}{6}$,

$$f^{(4)}(x) = (\sin x)^{(4)} = (\cos x)^{(3)} = (-\sin x)'' = (-\cos x)' = \sin x.$$

$$T_4^{\frac{\pi}{6}}(x) = \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}(x - \tfrac{\pi}{6}) - \tfrac{1}{4}(x - \tfrac{\pi}{6})^2 - \tfrac{\sqrt{3}}{12}(x - \tfrac{\pi}{6})^3 + \tfrac{1}{48}(x - \tfrac{\pi}{6})^4.$$