Cvičení 6 - výsledky

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Příklad 1.

a) $\lim_{x\to 0}\frac{\sin^2 2x}{1-cos2x}=\lim_{x\to 0}\frac{4\sin^2 x\cdot\cos^2 x}{1-(\cos^2 x-\sin^2 x)}=\lim_{x\to 0}\frac{4\sin^2 x\cdot\cos^2 x}{1-(1-\sin^2 x-\sin^2 x)}=\lim_{x\to 0}\frac{4\sin^2 x\cdot\cos^2 x}{2\sin^2 x}=$
$=\lim_{x\to 0}2\cos^2 x=2,$

b) $\lim_{x\to 0}\frac{\operatorname{tg} x}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}=\lim_{x\to 0}\frac{\sin x}{\cos x}\cdot\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{(1-\sin x)-(1+\sin x)}=\lim_{x\to 0}\frac{\sin x}{\cos x}\cdot\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{-2\sin x}=$
$=\lim_{x\to 0}\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{-2\cos x}=-\frac{2}{2}=-1,$

c) $\lim_{x\to 0}\frac{\sin x+\sin 8x}{x}=\lim_{x\to 0}\frac{\sin x}{x}+8\cdot\frac{\sin 8x}{8x}=1+8=9,$

d) $\lim_{x\to 0}\frac{\cos^2 x-1+\sin 2x}{x}=\lim_{x\to 0}\frac{-\sin^2 x+2\sin x\cdot\cos x}{x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot(2\cos x-\sin x)=1\cdot(2\cdot 1-0)=2,$

e) $\lim_{x\to\pi}\frac{1-\sqrt{\cos x+2}}{\sin^2 2x}=\lim_{x\to\pi}\frac{1-(\cos x+2)}{(1+\sqrt{\cos x+2})\cdot 4\cdot\sin^2 x\cdot\cos^2 x}=\lim_{x\to\pi}\frac{-1-\cos x}{(1+\sqrt{\cos x+2})\cdot 4\cdot(1-\cos x)\cdot(1+\cos x)\cdot\cos^2 x}=$
$=\lim_{x\to\pi}\frac{-1}{4(1+\sqrt{\cos x+2})\cdot(1-\cos x)\cdot\cos^2 x}=\frac{-1}{4\cdot(1+1)\cdot(1-(-1))\cdot(-1)^2}=-\frac{1}{16},$

f) $\lim_{x\to 0}\frac{5\ln(x+1)}{\sin 3x}=\lim_{x\to 0}5\cdot\frac{\ln(x+1)}{x}\cdot\frac{3x}{3\sin 3x}=\frac{5}{3},$

g) $\lim_{x\to 0}\operatorname{sign} x\cdot(x^2+2x+1)$ *neexistuje, protože* $\lim_{x\to 0}x^2+2x+1=1$ *a* $\lim_{x\to 0}\operatorname{sign} x$ *neexistuje,*

h) $\lim_{x\to 1}\ln x\cdot\operatorname{sign}(x-1)=0,$ *protože* $\operatorname{sign}(x-1)$ *je omezená a* $\lim_{x\to 1}\ln x=0,$

i) $\lim_{x\to 0}(1-3x)^{\frac{x^2+x-6}{x^2+x}}=\lim_{x\to 0}\exp\left(\frac{x^2+x-6}{x^2+x}\cdot\ln(1-3x)\right)=\exp\left(\lim_{x\to 0}\frac{x^2+x-6}{x^2+x}\cdot\frac{\ln(1-3x)}{-3x}\cdot(-3x)\right)=$
$=\exp(-3\cdot\lim_{x\to 0}\frac{x^2+x-6}{x+1})=e^{(-3)\cdot(-6)}=e^{18},$

j) $\lim_{x\to 0}(1+7x)^{\frac{1}{\sin 2x}}=\lim_{x\to 0}\exp\left(\frac{1}{\sin 2x}\cdot\ln(1+7x)\right)=\exp\left(\lim_{x\to 0}\frac{2x}{2\sin 2x}\cdot\frac{7\ln(1+7x)}{7x}\right)=e^{\frac{7}{2}},$

k) $\lim_{x\to\infty}e^x\cdot\cos x$ *neexistuje, protože pro* $e^{\frac{\pi}{2}+k\pi}\cdot\cos(\frac{\pi}{2}+k\pi)=0$ *a* $|e^{k\pi}\cdot\cos k\pi|=|(-1)^k\cdot e^{k\pi}|>k,$

l) $\lim_{x\to-\infty}e^x\cdot\cos x=0,$ *protože* $|\cos x|\le 1$ *a* $\lim_{x\to-\infty}e^x=0.$

Příklad 2. *Použijeme věty o součtu a podílu limit:* $\lim_{n\to\infty}\frac{a_n-1}{2b_n+2}=\frac{4-1}{2\cdot 7+2}=\frac{3}{16}.$

Příklad 3.

a) $\lim_{n\to\infty}\sqrt{n}(\sqrt{n+3}-\sqrt{n+2})=\lim_{n\to\infty}\sqrt{n}\frac{(n+3)-(n+2)}{\sqrt{n+3}+\sqrt{n+2}}=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+3}+\sqrt{n+2}}=\frac{1}{2},$

b) $\lim_{n\to\infty}\sin\frac{n\pi}{2}$ *neexistuje, protože* $\sin\frac{2k\pi}{2}=0$ *a* $\sin\frac{(2k+1)\pi}{2}=(-1)^k, k\in\mathbb{Z},$

c) $\lim_{n\to\infty}\frac{(-1)^n+2}{n^2+1}=0,$ *protože* $0=\lim_{n\to\infty}\frac{1}{n^2+1}\le\lim_{n\to\infty}\frac{(-1)^n+2}{n^2+1}\le\lim_{n\to\infty}\frac{3}{n^2+1}=0,$

d) $\lim_{n\to\infty}\frac{\sin n}{n^5}=0,$ *protože* $0\le|\lim_{n\to\infty}\frac{\sin n}{n^5}|\le\lim_{n\to\infty}\frac{1}{n^5}=0,$

e) $\lim_{n\to\infty}\frac{4n-1}{n+2}\cdot\sqrt{\frac{n}{9n+1}}=\frac{4}{3},$

f) $\lim_{n\to\infty}\frac{1+2+\cdots+n}{n^2}=\lim_{n\to\infty}\frac{\frac{n(n+1)}{2}}{n^2}=\lim_{n\to\infty}\frac{n^2+n}{2n^2}=\frac{1}{2},$

g) $\lim_{n\to\infty}\frac{2(n+3)!+5n!}{n\cdot(n+2)!+3(n-1)!}=\lim_{n\to\infty}\frac{n!}{n!}\frac{2(n+3)(n+2)(n+1)+5}{n(n+2)(n+1)+\frac{3}{n}}=2,$

h) $\lim_{n\to\infty}\frac{(-2)^{n+1}}{1+(-2)^n}=\lim_{n\to\infty}\frac{(-2)^n}{(-2)^n}\cdot\frac{-2}{(-\frac{1}{2})^n+1}=\frac{-2}{0+1}=-2,$

i) $\lim_{n\to\infty}\frac{1+2+\cdots+2^n}{1+5+\cdots+5^n}=\lim_{n\to\infty}\frac{\frac{2^n-1}{2-1}}{\frac{5^n-1}{5-1}}=\lim_{n\to\infty}\frac{4\cdot(2^n-1)}{5^n-1}=0,$

j) $\lim_{n\to\infty}\left(\frac{3n+1}{4n+5}\right)^n=0,$ *protože* $0\le\lim_{n\to\infty}\left(\frac{3n+1}{4n+5}\right)^n\le\lim_{n\to\infty}(\frac{3}{4})^n=0,$

k) $\lim_{n\to\infty}\cos n^2\cdot\ln^2\frac{3}{2}$ *neexistuje,*

l) $\lim_{n\to\infty}\sin^2\frac{n\pi}{2}\cdot\frac{\pi}{n}=0,$ *protože* $|\sin^2\frac{n\pi}{2}\cdot\pi|\le\pi$ *a* $\lim_{n\to\infty}\frac{1}{n}=0,$

m) $\lim_{n\to\infty}\left(\frac{n^3+1}{n^3}\right)^{n^3+2}=\lim_{n\to\infty}\left(1+\frac{1}{n^3}\right)^2\cdot\left(1+\frac{1}{n^3}\right)^{n^3}=1\cdot e,$ *protože* $\left(1+\frac{1}{n^3}\right)^{n^3}$ *je vybraná posloupnost z* $\left(1+\frac{1}{n}\right)^n.$

Příklad 4. *Pro* $n = 1$ *máme* $\lim_{x\to a} \frac{x-a}{x-a} = 1$.

Pro $n > 1$ *platí* $x^n - a^n = (x-a)\cdot(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$. *Tedy*

$$\lim_{x\to a} \frac{x^n - a^n}{x-a} = \lim_{x\to a}(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) = n \cdot a^{n-1}.$$