BINARY EQUALITY WORDS FOR PERIODIC MORPHISMS

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Abstract. Let g and h be binary morphisms defined on $\{a,b\}^*$, and let g be periodic and h nonperiodic. It is well known that their equality language is generated by at most one nonempty word. Suppose $|h(b)| \geq |h(a)|$. We show that then the equality word is equal to a^iba^j, with $i,j \geq 0$.

Binary equality sets are the simplest nontrivial equality languages. Nevertheless, their full description is still not known. They were for the first time extensively studied in [2]. The case of two nonperiodic morphisms was then examined in [3] and [4], where it is shown that their equality set is generated by at most two words, and if the rank of $\mathrm{Eq}(g,h)$ is two, the generators are of the form $\{a^ib, ba^i\}^*$.

The problem is easier for periodic cases. If both morphisms are periodic, and all images have the same primitive root, then the equality language consists of all words in which the ratio of letters a and b guarantees the length agreement. If just one morphism is periodic, then it is easy to show that $\mathrm{Eq}(g,h)$ is generated by at most one nonempty word. The aim of this note is to give a precise description of such a word.

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1. Preliminaries

We shall use standard notions of combinatorics on words (see e.g. [1] or [5]).

Recall that a word is called *primitive*, if it is not power of a shorter word. Each nonempty word is a power of a unique primitive word, called its *primitive root*. By u^+ we denote the set of all powers u^k, $k \geq 1$, and by u^* the set u^k, $k \geq 0$, where w^0 denotes the empty word. We say that u is a factor of v^+ if it is a factor of v^k for some k.

Morphism defined on an alphabet of cardinality two, say $\{a,b\}$, is called *binary*. A binary morphism g is called *periodic* if $g(a)$ and $g(b)$ commute, otherwise it is called *nonperiodic*. If g is a periodic binary morphism, and $g(ab)$ is nonempty, we say that the uniquely given primitive word p such that $g(a), g(b) \in p^+$ is the primitive root of g.

The *equality set* of morphisms g and h defined on an alphabet A is the set

$$\mathrm{Eq}(g,h) = \{u \mid g(u) = h(u);\ u \in A^*\}.$$

From the folklore we point out following facts.

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Lemma 1. *Two words satisfy a nontrivial relation if and only if they are powers of the same primitive root. In particular, a binary morphism h is nonperiodic if and only if $h(u) = h(v)$ implies $u = v$.*

Lemma 2. *Let p be primitive. If a pup is a factor of p^+ for some u, then $u \in p^+$.*

Lemma 3. *Let p be a primitive word. Any factor of p^+ of length $|p|$ is also primitive.*

2. Characterization

Consider a periodic morphism g and a nonperiodic morphism h. We can surely suppose that $g(ab)$ is nonempty, otherwise the case is trivial. First, let us prove an auxiliary lemma.

Lemma 4. *Suppose that $\mathrm{Eq}(g, h)$ contains a nonempty word. Denote by p the primitive root of g. Then $\mathrm{Eq}(g, h) = w^*$ for some w, and $h(u) \in p^+$ if and only if $u \in w^+$.*

Proof. If a nonempty word u is in $\mathrm{Eq}(g, h)$, then obviously $h(u) \in p^+$. Moreover, if $u \in \mathrm{Eq}(g, h)$, then also the primitive root of u is in $\mathrm{Eq}(g, h)$, since for any word t and any integer $k \geq 1$ the equality $g(t^k) = h(t^k)$ holds if and only if $g(t) = h(t)$.

Let w be a primitive element of $\mathrm{Eq}(g, h)$, and let v be a nonempty word such that $h(v)$ is a power of p. Then

$$h(w^{|v|}) = h(v^{|w|}).$$

Since h is not periodic, we deduce $w^{|v|} = v^{|w|}$, by Lemma 1. The primitivity of w implies that $v \in w^+$, again by Lemma 1, which concludes the proof. $\square$

Now we can prove the main theorem.

Theorem 1. *Let $|h(a)| \leq |h(b)|$. If $\mathrm{Eq}(g, h)$ contains a nonempty word, then either $\mathrm{Eq}(g, h) = a^*$, or $\mathrm{Eq}(g, h) = (a^i b a^j)^*$, for some $i, j \geq 0$.*

Proof. Denote by p the primitive root of g. Suppose that $\mathrm{Eq}(g, h)$ contains a nonempty word, and let w be its nonempty generator, which exists by Lemma 4.

If $h(c)$ is a power of p for some letter $c \in \{a, b\}$, then Lemma 4 implies that $w = c$. Suppose therefore that neither $h(a)$ nor $h(b)$ is a power of p.

Let first $|h(a)| \geq |p|$ and $|h(b)| \geq |p|$. Suppose that a is the first letter of w. Then p is a prefix of $h(a)$. Since $h(a)$ is a factor of p^+ and $h(a)$ is not a power of p, the word p is not a suffix of $h(a)$, by Lemma 2. This implies that the last letter of w is b, and p is a suffix of $h(b)$. Clearly, the word ab is a factor of w. Since p is a prefix and a suffix of $h(ab)$, which is a factor of p^+, the word $h(ab)$ is a power of p, by Lemma 2, and we conclude that $w = ab$. Similarly, we obtain $w = ba$, if b is the first letter of w.

If $|h(a)| < |p|$ and $|h(b)| < |p|$, then $g(v)$ is strictly longer than $h(v)$ for any nonepty word v, a contradiction.

It remains to consider the case when $|h(a)| < |p|$, and $|h(b)| \geq |p|$. Denote by q the prefix of $h(b)$ of length $|p|$. Note that q is primitive, by Lemma 3. The word w contains at least one b, by a length argument. Therefore $ba^m b$ is a factor of ww for some m. This implies that $h(ba^m b)$ is a factor of p^+ and thus also of q^+. Whence $h(ba^m)$ is a power of q, by Lemma 2. Suppose that $ba^k b$ is a factor of ww for some $k \neq m$. Then also $h(ba^k)$ is a power of q, which implies that $h(a)$ commutes with

q. But this is a contradiction, since q is primitive, and $|h(a)| < |p| = |q|$. We have proved that $ww = a^i(ba^m)^\ell a^j$, where $m = i + j$. Minimality of w yields $w = a^i b a^j$, and the proof is complete. $\square$

For each $i, j \geq 0$ there is a periodic morphism g and a nonperiodic morphism h, such that $\mathrm{Eq}(g, h) = (a^i b a^j)^*$. Consider for example morphisms

$$g(a) = a^i b a^j, \qquad h(a) = a,$$
$$g(b) = a^i b a^j, \qquad h(b) = (ba^{i+j})^{i+j} b.$$

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