TheFourierTransformAndItsApplications-Lecture26

Instructor (Brad Osgood): Relax, but no, no, no, the TV is on. It's time to hit the road. Time to rock and roll. We're going to now turn to our last topic of the quarter, and that is space, the final frontier, a quick trip to some of the fascination and mysteries of the higher-dimensional Fourier transform. Actually, one of the things that I want to convince you of is that there are no mysteries to the higher-dimensional Fourier transform, or at least not so many because our goal is to try to make the higher-dimensional case look as much as possible like the one-dimensional case. So all that you learned, that hard one knowledge – sort of the same thing we did when we were looking at the Discrete Fourier transform, to try to carry over your intuition from one setting to another setting. So all the stuff you learned, all the formulas you learned, we'll find counterparts in the higher – all the stuff you learned in the one-dimensional case, we'll find counterparts in the higher-dimensional case. This is not just – so here's the topic. Higher-dimensional Fourier transforms. By that, I really mean Fourier transforms as functions of several variables. That's what I mean when I talk about higher-dimensional Fourier transforms, i.e. Fourier transforms of functions of more than one variable. Now this is not an idle generalization by any means. It's not a generalization for generalization's sake. These days, you're as likely to find applications of Fourier analysis, and single processing and so on to functions of more than one variable as you are functions of one variable. At least two or three variables. I think a leading example of this, for instance, is in the Fourier analysis of imaging. There are many applications of a higher-dimensional theory, higher-dimensional Fourier transforms. For example, EG images or the Fourier analysis of images. Spectral analysis. Let me put it that way. Spectral analysis and what is almost the same thing, signal processing for images.

For what is an image, after all? What is a mathematical description of an image? Well, at least not a two-dimensional image. At least mathematically, it's given by a function of two variables, say X1 and X2. Function F of X1, X2, where X1 and X2 are varying over some part of the X1, X2 plane. At each point, what the function prescribes is the intensity. I'm thinking about black and white images here. So you think F of X1 and X2 as a range of numbers from zero to one, from black to white. So you think of F of X1, X2 as the intensity from black to white, say, at the point X1, X2. And then as X1 and X2 vary over region the plane, the intensity varies, and that what makes an image. That's what makes a black and white image. Color's more complicated, but at least for black and white images, that's what you get. Now, you're used to the more Discrete version of this in the digital version where the [inaudible] quantals at 256 levels and so on. But you can think of this first in the continuous case where X1 and X2 were varying continuously over a region. The intensity is varying continuously from black to white. That's what describes an image.

Now, the question is – this you might think of as the spacial description of an image. It's what you see with your senses. It's like the temporal of the time description of sound, what you hear with your ears. So this is the spacial description of an image. It's what you perceive. The question the Fourier analysis raises is, is there another description? Is there a spectral description? Can you describe an image in terms of its component parts like you can describe a sound in terms of its component parts? Is there a spectral description? Can you analyze an image into simpler components, like you can analyze a sound into simpler components. If you can, of course that gives you a certain amount of control over it. If you can manipulate the component parts, you can manipulate the image. In the same way as if you can manipulate the harmonics that go into a sound, you can manipulate the sounds, to pick one example of it. There are other sorts of examples like this, but just keep this one in mind because it's a very natural example. It's a very important example. The answer, of course, is yes, and it's provided, exactly, in this case, by the two-dimensional Fourier transform and in the higher dimensions, similar sorts of problems are provided by the higher-dimensional Fourier transform. So yes, the spectral analysis, spectral description, is provided by the two-dimensional Fourier transform. The component parts are the two-dimensional complex exponentials. It's a very close analogy to what happens in the one-dimensional case, and that's really what I want to push. The components are the 2D complex exponentials. The same things hold in higher dimensions. So again, as we understand these things, and as we go through the formulas, the thing that I want to emphasize is that the two-dimensional case and the higher-dimensional case can be made to look very much like the one-dimensional case. Furthermore, there's no essential difference between the two-dimensional case and the higher-dimensional case.

There are differences in going from one to two dimensions. Some of the formulas are a little bit different. There are new phenomenon that come in, in going from one dimensions to two dimensions. You have two degrees of freedom instead of one degree of freedom. Naturally, some things are going to be richer in the two-dimensional case. That's true. But there's more similarity than there is difference, I'd say, between the one-dimensional case and the two-dimensional case and certainly there is very little difference between the two-dimensional case and the higher-dimensional case. So we want to make the 2D case and higher look like the 1D case. There are differences, and as a matter of fact, by making them look as similar as we can, it'll make the differences also stand out when there are differences. But certainly, there's more similarities than there are differences, and there are very few differences between the two-dimensional case and higher-dimensional cases. So if you get that down, then you're in pretty good shape. I consider this good news. That is, the things we've learned so far will have counterparts and analogues in the higher-dimensional setting. Now, the key to doing this, the key to making it look very similar, is really how you write the formulas, and the way you write the formulas is with vector notation. So the key to doing this, that is, to making the cases look that same, is to use – whenever possible, is to think in terms of vectors. Write the formulas, etc et al, with vectors. There are different ways of motivating this. I'm going to really pretty much jump into it. So let me write down the one-dimensional case, the one-dimensional formula for the Fourier transform and see what has to be replaced and what has to change and again, how I can make the higher-dimensional case look like the one-dimensional case. Here's the one-dimensional Fourier transform as we've written down many times. Fourier transform function is the integral from minus infinity to infinity of either the minus two pi I ST, F of T, DT. Wonderful.

Now, what goes into that? So I have one-dimensional continuous variables as a T, or, so to speak, 1D continuous variables. The function if I take the Fourier transform as a function of that one variable, T, and the Fourier transform is a function also, so that one variable S. F of T is a function of T. The Fourier transform, F of S, is a function of S. Of course, the other thing that happens here, the other big element that enters into the definition of the Fourier transform is the complex exponential. There, it's the product of the variables, S and T, that come in. That's what it takes to define the Fourier transform. Remember, of course, there was the integral and everything else. It's not that many symbols, but it's a pretty complicated expression. In either the minus two pi I ST, it's the product of S and T enters. Now, what about the higher-dimensional case? Let's just do the two-dimensional case. What do I replace the variables by? What do I replace the functions by, and so on? So instead of single functions of a continuous variable, I have a vector function X, a vector variable X, is a function of two variables. That'll be the so-called spacial variable. So you can think of it as just a pair of numbers, X1 and X2, but again, I'm going to write that as a vector in most cases. Then the frequency, I'm going to write with a Greek letter, xi. That's a pretty standard term, and if you haven't learned to write a Greek letter, xi, this is your big chance.

That's a pair of numbers, xi 1, xi 2. That'll be thought of as the frequency variable. The functions that I'm going to be taking the transform of is not a function just of one variable alone, but it's a function of a vector variable, X, or it's a functions of X1 and X2. Function to be transformed is F of X. If I write it just like that in vector notation, or F of X1, X2. The Fourier transform is likewise, going to be a function of the frequency variable, which is the pair, xi 1 and xi 2. The Fourier transform will be something like the Fourier transform of F, I use the same notation of the vector variable, the frequency variable, xi, or if I write it out as a pair, xi 1, xi 2. Now the big [inaudible] how will I actually make the definition? What happens to the complex exponential? How do I replace multiplication of the one-dimensional variables S and T in the one-dimensional case by a higher-dimensional analogue? So what happens to the complex exponential? You replace the product, ST, in the one-dimensional case, by the dot product or the inner product of the two variables that are into the higher-dimensional case. That is the spacial variable and the frequency variable. You replace this by X dot xi, the inner product. So that's X1 times xi 1 plus X2 times xi 2.

So the complex exponential, then, is E to the minus two pi I, X dot xi – or xi dot X. It doesn't matter, it's the same thing – which is E to the minus two pi I, X1 times xi 1 plus X2 times xi 2. That's what replaces the complex exponential. If you write it like this, in vector notation with the inner product, it looks pretty much, as much as you can make it, like the one-dimensional case. Putting all this together, what is the definition of the Fourier transform? Now, realize here, by the way, this is still – I'm computing a scale here. I haven't done any – this is the ordinary exponential function because X dot xi is a number, is a scaler. So I'm taking E to the minus two pi I times a scaler. That's nothing new. The fact that that scaler is arising as the inner product of two vectors, well, okay, that's something new. But nevertheless, I haven't introduced a new function here. I've only replaced the multiplication of the one-dimensional variables, S and T, by the inner product of the two-dimensional variables, X and xi. So with all this, what is the definition of the Fourier transform?

Let me write it in vector form first, and then I'll write it out in terms of components. The Fourier transform of F at the vector variable, xi, is the interval over R2, interval over the entire plane, of E to the minus two pi I, X dot xi, F of X, DX. All right? Looks as much like the one-dimensional case as I can make it look. I've replaced scaler variables, one-dimensional variables, by vector variables, X and xi. For that matter, this same definition holds in – well, let me write it out in components before I say anything about higher-dimensions. In components, this says that the Fourier transform F at xi 1, xi 2, the pair of frequency variables, that integral over R2 becomes a double integral. That's the integral from minus infinity to infinity. The integral from minus infinity to infinity of E to the minus two pi I, X1 times xi 1 plus X2 times xi 2, times F of X1, X2 times DX1, DX2. Same formula, but now everything is written out in terms of components. Now, which would you rather write? This, which took the entire blackboard, or this, which took the entire blackboard, but a little bit less of the entire blackboard? This, you can recognize as the same form as the one-dimensional Fourier transform. This sort of looks the same, but it's a little bit more complicated. You need both. This we're going to need to write things out in terms of components when we actually compute for specific Fourier transforms, but if we want to understand the Fourier transform in higher-dimensions, and we want to work with it conceptually, then it's by far better to write things in vector form.

The formulas, the theorems that govern the higher-dimensional case into the extent that we can make them analogous of the one-dimensional case, and it really is, to a great extent, they are much better understood if we use a vector notation in that sort of form for the Fourier transform. So let me just say that now the higher-dimensional case is exactly an extension of this, from two dimensions to N dimensions. N dimensions, the spacial variable is a function of N variables. The frequency variable also is an N-tuple, xi 1 up through xi N. Again, multiplication's replaced by the inner product. X dot xi is X1 times xi 1 plus – up to XN times xi N. So this time, the function you're taking the transform of is the function of N variables. The Fourier transform is also a function of N variables. But the definition looks the same, and I will not write it out in terms of components. Let me just write out the vector form of it. So the Fourier transform of F at xi, the vector variable, is the integral over RN of same thing, E to the minus two pi I, X dot xi times F of X, DX. Same thing. Same formula, except this time, I'm integrating over all of N-dimensional space instead of just the plane. I will demure from trying to write this out in components, X1 through XN, but you can.

You see, the economy of writing things in the vector notation. It just makes things much simpler to write. Again, I contend, and I hope to show you, that it also makes it much simpler to understand things conceptually, understand how the formulas look conceptually if you stick with the vector case. I should mention that of course one also has the inverse Fourier transform and by analogy, it looks very similar to the right of the Fourier – to the forward Fourier transform, except I changed the plus to a minus rather than a minus to a plus in the complex exponential. So the inverse Fourier transform, let's say it's the inverse Fourier transform of G. This function of the spacial variable is the interval over RN. I'll do it in the N-dimensional case for the thrill of it. E to the plus two pi I, X dot xi, say G of xi D xi, which looks the same as the one-dimensional inverse Fourier transform. Now, I'm not going to – when time comes, I'll talk about Fourier inversion and all the rest of that stuff, but it's going to work out the same way as in the one-dimensional case. There are some differences, again, to be sure, but the basic phenomenon that hold in the one-dimensional case are also going to hold in the two-dimensional case and the higher-dimensional case. It works out just beautifully. I'll say one other thing here. What about the dimensions of the variables involved – dimensions, not units so much. So again, there is a – and we'll see different instances of this. There's a reciprocal relationship between the spacial domain and the frequency domain. In the one-dimensional case, we refer to the time domain and the frequency domain. In the higher-dimensional case, we think more in terms of X as a spacial variable rather than a time variable, naturally, because that's where the problems come from. There is a reciprocal relationship between the spacial domain and the frequency domain. So this time, you're talking about the two domains as a spacial domain and a frequency domain.

What is the first instance of a reciprocal relationship between the spacial domain and the frequency domain? Now, we'll see various instances of this. Again, most of the analogates to the one-dimensional case, but even in the two-dimensional case, what do I mean by this? Well, if I think of the vector variable X as X1 through XN as the spacial variable, then I would imagine the XIs each have dimension length. So if this is a spacial variable, then XI has dimension length, right? So what dimension should the frequency variable have? If you think physically? Mathematically, we never care about dimensions. The people who work in physics, especially in applications, they always like to attach units to their variables as a way of sort of checking the formulas make sense. So to form X dot xi, and have it make sense physically, that's X1 times xi 1 plus X2 times xi 2 and so on, to XN times xi N. To make sense of this physically, you would want the xis to have dimension one over length. So it's length times one over length gives you a number, gives you a scale so you can add them all up. You would want xi I to have dimension one over length. That's the reciprocal relationship between space and frequency. If the Xs have dimension length describing the spacial variable, then in the frequency domain, the variables have dimension one over length. Its analogous to time and frequency, time having the dimension time, or units of seconds. And frequency having units of one over second, or hertz. Okay? Same thing.

So this is sort of the first natural example of the reciprocal relationship between the two domains, again, analogous to what happens in the one-dimensional case. Cool? I think so. I like this stuff. Now, let me talk about – this is the formula. Again, I presented this to you as directly analogous, as close as I can make it, to the one-dimensional case. Of course, there are ways of – you can get to the higher-dimensional Fourier transform in a similar way as you can get to the one-dimensional Fourier transform by considering higher-dimensional Fourier series and sort of limiting cases of higher-dimensional Fourier series and so on. I'm not going to do that. There's a lot of water under the bridge between where we started and where we are now, so I'm not going to revisit all those ideas. I'm not going to talk about – there is a section in the notes on a nice application, I think, of higher-dimensional Fourier series, but it's not something I want to talk about in public. So we're going to really go pretty much directly for the main ideas and the main applications of this thing without unfortunately somehow taking some of the many fascinating little side routes that you could go on. Again, there are ways of motivating the definition of a higher-dimensional Fourier transform. That is, there are ways of motivating why you replace multiplication of the one-dimensional variables, S and T, with inner product. Why that's a natural thing to do rather than just my presenting it as my [inaudible], that this is how we're going to do it.

Now, let me write down the formula again in vector form. I want to say something more about – again, as I said, if you consider that what we want to do is try to develop the spectral picture of higher-dimensional signals, then the aspects of that are defined [inaudible] Fourier transform, and understanding what the components are. The components, in this case, are these higher-dimensional complex exponentials, so I would like to spend a little bit of time making sure we understand and helping to get a feeling for what the higher-dimensional complex exponentials are. How you can understand then geometrically, how you can – if you can't quite visualize them, at least how you can put them in your head in some reasonable way other than just the formula. So again, let me write down the formula for the Fourier transform. Let me just do the two-dimensional case. So in the two-dimensional cases, the integral over the plane, E to the minus two pi I, X dot xi, F of X, VX. What I want to talk a little bit about is the nature of the 2D complex exponential. E to the plus or minus two pi I, X – see, I put plus or minus in there because it's a minus sign that comes in for the Fourier transform, it's a plus sign that comes in for the inversed Fourier transform. It doesn't matter. They have the same nature, whether or not you consider plus or minus two pi I times X dot xi. Okay. Now you can't draw – they're complex functions, so you can't really draw a picture of them. You can't draw a graph of these things, but there are ways of understanding it geometrically. So again, let's think of –

You can't draw a graph like you can draw a graph of sines and cosines, but you can understand it geometrically. In particular, you can understand frequency almost in terms of – you get a very intuitive sense of tactiles sense of frequency, what it means to have high frequency, what it means to have low frequency and so on. You can get an understanding, you can get a feeling for this sort of vector frequency. Now, let's go back for a second to the one-dimensional case. I'm just going to set the stage for this because again, I want to pursue by analogy here. So let's go back to the 1D case where I'm looking at either the – I'll look at the plus sign, okay? Just so I don't have to write the minus sign in there. So E to the two pi I ST where S is the frequency variable and T is the time variable. So imagine – fix the frequency. Fix S. Fix the frequency S, and look at this as a function of T. Look at E to the two pi I ST as a function of T. Then it is periodic of period one over S. As a function of T, it's periodic of period one over S. That's E to the two pi I, S times T plus one over S is E to the two pi I ST times E to the two pi I, S times one over S. E to the two pi I, E to the two pi I. It's just E to the two pi I ST. S is fixed here, two, three, five, twelve, whatever.

So that already tells you something about how rapidly – can't draw a picture here, but you can say the words, how rapidly is this complex exponential oscillating depending on S?

Well, if S is large, then one over S is small. The period is small, so it's oscillating fast. It's returning to the same value over and over again very quickly. If S is small, one over S is large. The period is large. So that gives you a sense of how fast the function's oscillating, how fast the function E to the two pi I ST is oscillating as a function of T, depending on the size of S. Or another way of looking at this is E to the two pi I ST will be equal to one at the point – well, T equals zero. T equals one over S or minus one over S, so plus or minus one over S. T equals plus or minus two over S, and so on and so on. So although it's a complex function and you can't draw the graph, you can say, well, it's returning to the value one at point space one over S apart. All right? So E to the two pi S is equal to one, E to the two pi I ST is equal to one at points spaced one over S apart. So again, if S is large, if you have a high frequency, then those spacings are very small. If S is small, you have a low frequency, then one over S is large. The points are spaced far apart. So it gives it a tactile description of how fast the function is oscillating, how often it assumes the value one. This is all in the one-dimensional case.

I don't know if I ever really spoke in these terms when we spoke about the complex exponentials, but should not be too much for you to get your head around. We worked with one-dimensional complex exponentials a lot. If I didn't say this before, maybe I should have. Now what about the two-dimensional case? The 2D case, where my complex exponentials will form E to the two pi I X dot xi, or if I write it out in components, that's E to the two pi I, X1 times xi 1 plus X2 times xi 2. Okay? So again, I'm going to fix a frequency and look at this as a function of X1 and X2. So I want to fix xi. Xi 1, xi 2, a given frequency. I'm not going to talk about high frequencies or low frequencies yet, but I will. But just imagine fixing a frequency, and look at [inaudible] complex exponential as a function of X1 and X2. That is, it's a function on the X1, X2 plane. Look at E to the two pi I – I'll write it out in components – xi 1 plus X2 times xi 2 as a function of X1 and X2. Now, by analogy to what I did in the one-dimensional case, when does a complex exponential equal to one? When does this function of a function of – xi 1 and xi 2 are fixed. When does this function assume the value one? Where is E to the two pi I, X1 times xi 1 plus X2 times xi 2 equal to one? Well, don't get too far ahead of things here. That will be true for points X1 and X2. So again, xi 1 and xi 2 is fixed. I'm asking this as a function of X1 and X2.

This will be so at points X1, X2 where the inner product, X1 times xi 1 plus X2 times xi 2 equals an integer, N. So N can be zero plus or minus one, plus or minus two and so on. At those points, in the X1, X2 plane, this complex exponential will be equal to one. That's just the property of the ordinary exponential. See, we've [inaudible] E to the two pi I, N. Now, what are these points? X1, X2. It's not hard to see. Let me show you how this goes. It's very pretty. You get a very pretty picture here. Again, this is a picture – if you've studies imaging already, you have probably seen this picture. I don't know how it's presented in those courses, but let's present it here. I'm going to push it here as an analogue of the one-dimensional case. Let the case then equal zero. What about the points X1, xi 1 plus X2, xi 2 equals zero? What are those points? Again, xi 1, xi 2 are fixed, like five times X1 plus three times X2 equals zero. Three times X1 plus five times X2 equals zero, minus X1 plus four times X2 equals zero and so on. So again, if xi 1 and xi 2 are fixed, what does this set of points look like in the complex plane? Well, you have to remember a little bit of analytic geometry here. You need to remember analytic – and let me just say, analytic vector geometry. That is, the set of points in the X1, X2 plane where X1 plus xi 1 plus X2 times xi 2 is equal to zero is a line through the origin. So here's the X1, X2 plane. If I wrote it like this, three X1 plus five X2 is equal to zero or minus X1 plus four X2 equals zero, you'd probably recognize – and if I said, what is that figure, you would probably say it's a line through the origin.

That's fine. But you also have to remember the relationship of xi 1 and xi 2 to that line. What is the relationship? It's a line through the origin with the vector, xi, as the normal vector. X1 times xi 1 plus X2 times xi 2 equals zero is a line through the origin in the X1, X2 plane with xi as the normal vector. Not the unit normal vector. It doesn't add length to one. I'm not assuming that, but it's perpendicular to the line. It's normal to the line. So going back to what we're describing here, for example, E to the two pi I, X1 times xi 1 plus X2 times xi 2 is equal to one along that line because along this line, X1 times xi 1 plus X2 times xi 2 is equal to zero. So it's E to the zero is one. Now, what about all the other places where it's equal to one? That is, X1 times xi 1 plus X2 times xi 2 is equal to N for N equals zero, plus or minus one, plus or minus two and so on. What are those configurations in the X1, X2 plane? So now what is a description of all of the points where X1, xi one plus X2 times xi 2 is equal to N? N equals zero, plus or minus one, plus or minus two and so on. Well, each one of these things – let me do just one more example. X1 times xi 1 plus X2 times xi 2 is equal to one is another line with the same normal vector, xi, but not through the origin.

As a matter of fact, the picture would look something like this. Here's the X1, X2 plane. Here's the line, X1. Let me write it in vector notation. X dot xi equals zero, and here is the line, X dot xi equals one. It's parallel to this line because it has the same normal vector, but it doesn't pass through the origin. This is xi, and this is xi. How far apart are these lines? What is the spacing? How far apart are these lines? So the two lines, this one and this one. There's a point here. How do I find the distance between the two lines? Well, I'm going to find it using vectors. I take a point, X1, X2 on that line, and that point satisfies X1 times xi 1 plus X2 times xi 2 is equal to one. That's the defining property of the line. Call this angle theta. Then what I want is I want the – let me call this vector X. Then the distance between the two lines is the magnitude of X times the cosine of theta for any point, X1, X2, on the line. But what is the magnitude of X times the cosine of theta in terms of inner products? Remember, X dot xi is equal to the magnitude of X times the magnitude of xi times cosine of theta. That basic geometric property of dot products. That is to say the magnitude of X times cosine of theta is X dot xi divided by the magnitude of xi. But what is X dot xi? One. Anywhere along this line, X dot xi is equal to one. So it is equal to one over the magnitude of xi. How far apart are the lines spaced? The lines are spaced one over the magnitude of xi apart.

So again, I'm going to draw the picture. Here's the line, X dot xi equals zero. Here's the line X dot xi equals one. How far apart are they? The spacing, distance, is equal to the reciprocal of the length of xi. Now, same thing holds for all those other lines. Which other lines? I mean, when the inner product is equal to an integer. Zero plus or minus one, plus or minus two and so on and so on. That gives you a family of parallel, evenly-spaced lines in the plane. So X dot xi equals N, for N equals zero plus or minus one, plus or minus two and so on, gives you a family of parallel – they all have normal vector xi – lines in the plane. Parallel, evenly-spaced lines. What is the spacing? The spacing is one over xi. So the picture is something like this – I can only draw so many of these things. Here's the line through the origin. Here's the line corresponding, say, N equals one. Here's the line corresponding to N equals two. Here's the line corresponding to N equals minus one, minus two, whatever. Here's X dot xi equals two. Here's X dot xi equals one. Here's X dot xi equals zero and so on.

They're all parallel. They all have the same normal vector, and the spacing is the same. The spacing between any two lines, any two adjacent lines, is one over xi apart. Now, I'll remind you – let's go back to the complex exponential. So that says E to the two pi I, X dot xi, is equal to one along each one of these lines. So it's equal to one here. Then it oscillates, and it's equal to one, equal to one, equal to one and so on. It's a complex function. You can't draw the picture, but you can say that it's somehow – it's oscillating up and down. It's not really the right thing to say because it's complex. You can't talk about up and down with complex, but it's oscillating, and it's equal to one. Then it oscillates. Then it's one, one, one and so on. As a matter of fact, to be a little more precise here – I'll let you sort this out. It's in the notes – you can say that E to the two pi I X dot xi is periodic in the direction xi with period one over the magnitude of xi. I will leave it to you. I think this is discussed in the notes, but I will leave it to you to read that or make that precise, but that's the closest analogy you can have in the two-dimensional case to the one-dimensional case. If I just imagine myself in the X1, X2 plane, going off in a certain direction, and if I go off in that direction, this function's going to be periodic, and it's going to periodic of period one over xi. It oscillates. It goes up to one, down. Again, I can't resist saying up and down, but the idea is that is oscillates and it returns to the value on along each one of these lines. If you look back, this is not a bad analogy. It's a pretty close analogy to the way we look at the complex exponential, for the way you can visualize or imagine the complex exponential in the one-dimensional case.

In particular, it gives you a sense of what it means to have a high frequency and low frequency harmonic. So if you think of these complex exponentials as a two-dimensional harmonic, you get a sense of what it means to have a high-frequency harmonic or a low-frequency harmonic. A high-frequency harmonic would be if the magnitude of xi is large. High frequency means the magnitude of xi is large. That means one over xi is small. That means these lines are spaced close together. That means there's rapid oscillation. So close spacing of the lines, rapid oscillation of the complex exponential. And low frequency would meant the magnitude of xi is small. That means that one over the magnitude xi is large. That means the line spacing is large, and I have slow oscillation. So far spacing of the lines, and I have a slow oscillation of the complex exponential. Now, again, you have to be a little more careful. It's richer in two dimensions than it is in one dimension because harmonics don't just have a magnitude associated with it, don't just have a – you can't just say how fast or how slow it's oscillating. You also have the specify the direction. The frequency is a vector quantity in two dimensions and in higher dimensions, not a scaler quantity. So you have to talk about it's oscillating slowly in a given direction. It'll be different depending on the frequency, right? If you change the frequency, you're changing – you might change both its magnitude and its direction. So you can change those two independently. It might be oscillating rapidly in one direction and slowing in another direction. So it's a richer picture.

We'll talk about this, and there are pictures of it, also, that are given in the notes. But this is the geometric interpretation. Again, you can't – I hesitate to say this is how you visualize complex exponentials – vector, two-dimensional complex exponentials because unless you're really visualizing them because they're complex functions. But again, you are seeing from this how they're oscillating, in what ways they're oscillating, in what way they generalize the one-dimensional complex exponential. In this case, you have lines where the functions are equal to one. In some areas, in optics, for example, usually refer to these lines as lines of constant phase. They complex exponential's real on those lines, so we sometimes refer to that as constant phase. It varies. I think the terminology varies. I don't necessarily want to attach any one particular interpretation or any one particular terminology to it. I think the picture itself holds, regardless of the interpretation and regardless of the setting, and that's what you should think of. Okay? All right. So we have now done – we have the basic formula for the Fourier transform, the components to go into it, that is the particular elements that are added into it, and some understanding, I hope, of the components into which a higher-dimensional signal is broken. That is, these complex exponentials and what they represent. Next time, we will get [inaudible] higher-dimensional Fourier transform, and you will see how the theorems in the one-dimensional case carry over to the higher-dimensional case, where they're similar and where they're different. All that is waiting for us. See you then.

[End of Audio]

Duration: 54 minutes