

# EE 261 The Fourier Transform and its Applications

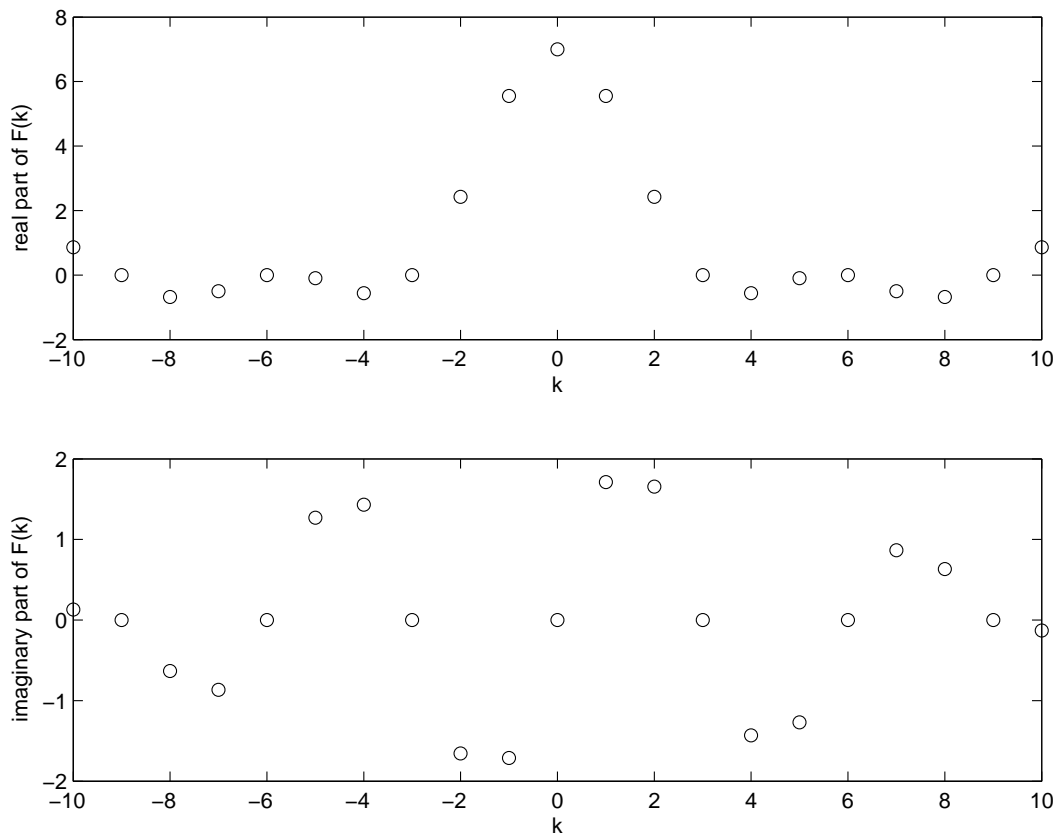
## Fall 2007

### Problem Set Eight Solutions

1. (20 points) *A True Story:* Professor Osgood and a graduate student were working on a discrete form of the sampling theorem. This included looking at the DFT of the discrete rect function

$$\underline{f}[n] = \begin{cases} 1, & |n| \leq \frac{N}{4} \\ 0, & -\frac{N}{2} + 1 \leq n < -\frac{N}{4}, \quad \frac{N}{4} < n \leq \frac{N}{2} \end{cases}$$

The grad student, ever eager, said ‘Let me work this out.’ A short time later the student came back saying ‘I took a particular value of  $N$  and I plotted the DFT using MATLAB (their FFT routine). Here are plots of the real part and the imaginary part.’



- (a) Produce these figures.

Professor Osgood said, ‘That can’t be correct.’

- (b) Is Professor Osgood right to object? If so, what is the basis of his objection, and produce the correct plot. If not, explain why the student is correct.

*Solution*

The student executed the following MATLAB command.

```
f = [0 0 0 0 0 0 0 1 1 1 1 1 1 1 0 0 0 0 0 0 0];
```

```
F = fftshift(fft(fftshift(f)));
```

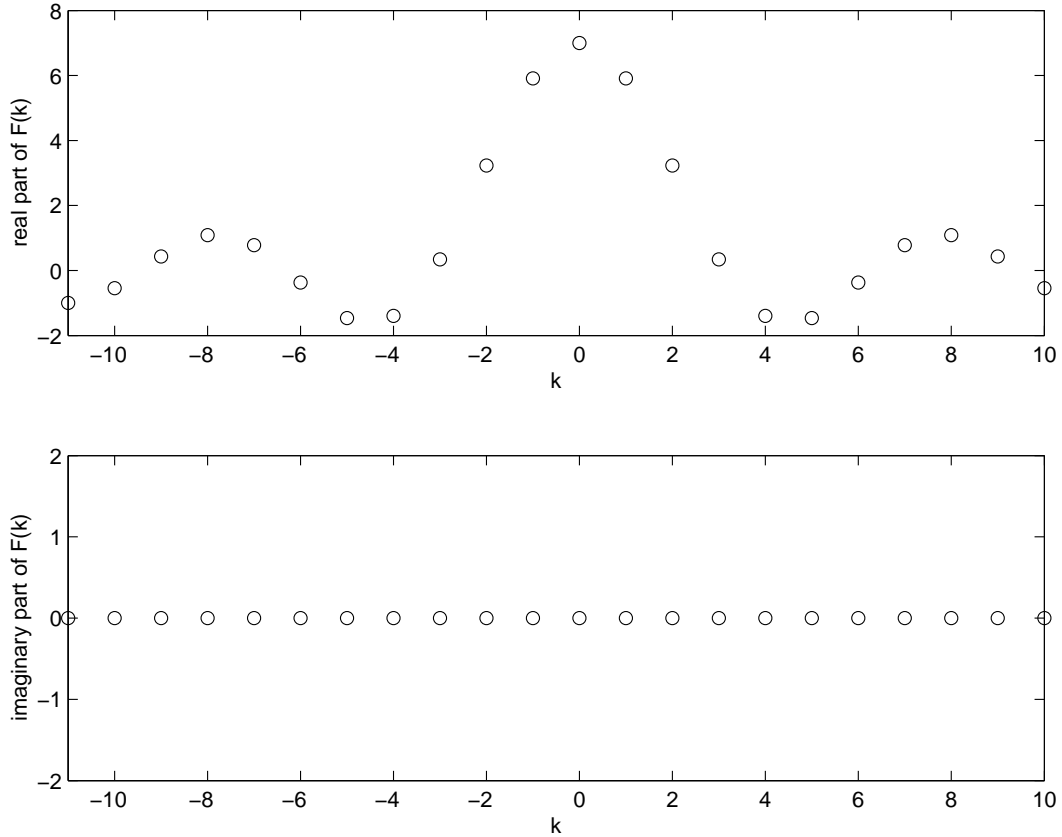
DFT in MATLAB is done assuming DC at the first component of the array. Therefore the command `fftshift` is used to alter the center of the input array. There is a fine detail here that is worth noting. When the array size is  $N$ , `fftshift` shifts the  $\text{ceil}(N/2)+1$  component to be the first component of the array.

Therefore, to obtain the desired result, the `rect` function should be centered around the  $\text{ceil}(N/2)+1$  component. The following command will give  $F$  without any imaginary component.

```
f = [0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 0 0 0 0 0 0];
```

```
F = fftshift(fft(fftshift(f)));
```

Here’s the correct plot:



2. (20 points) *Linearity and time-invariance.* State whether the following systems are linear or non-linear, time-invariant or time-variant, and why. Assume that  $v(t)$  is the input and  $w(t)$  is the output for all systems. *No credit will be given for answers without explanations or proofs!*

- (a)  $w(t) = v(t) \cos(\omega t)$
- (b)  $w(t) = \sin(v(t))$
- (c)  $w(t) = \int_{-\infty}^{\infty} v(\tau) e^{-2\pi i t \tau} d\tau$
- (d)  $w(t) = \frac{d}{dt} v(t)$
- (e)  $w(t) = \cos(\omega t + v(t))$

*Solution:*

In each case, we need to use the definition of linearity and time-invariance to prove or disprove the claim. A system  $T$  is linear if  $T\{\alpha_1 v_1(t) + \alpha_2 v_2(t)\} = \alpha_1 T\{v_1(t)\} + \alpha_2 T\{v_2(t)\}$ . Given that  $w(t) = T\{v(t)\}$ , a system  $T$  is time-invariant if  $w(t - \tau) = T\{v(t - \tau)\}$ .

(a) Linearity:  $(\alpha_1 v_1(t) + \alpha_2 v_2(t)) \cos(\omega t) = \alpha_1 v_1(t) \cos(\omega t) + \alpha_2 v_2(t) \cos(\omega t)$ , so the system is clearly linear.

Time-Invariance:  $v(t - \tau) \cos(\omega t) \neq w(t - \tau)$ , so the system is time-variant.

(b) Linearity:  $\sin(\alpha_1 v_1(t) + \alpha_2 v_2(t)) \neq \alpha_1 \sin(v_1(t)) + \alpha_2 \sin(v_2(t))$ , so the system is non-linear.

Time-Invariance:  $\sin(v(t - \tau)) = w(t - \tau)$ , so the system is time-invariant.

(c) Linearity:  $\int_{-\infty}^{\infty} (\alpha_1 v_1(\tau) + \alpha_2 v_2(\tau)) e^{-2\pi i t \tau} d\tau = \alpha_1 \int_{-\infty}^{\infty} v_1(\tau) e^{-2\pi i t \tau} d\tau + \alpha_2 \int_{-\infty}^{\infty} v_2(\tau) e^{-2\pi i t \tau} d\tau$ , so the system is linear.

Time-Invariance:  $\int_{-\infty}^{\infty} v(\tau - \alpha) e^{-2\pi i t \tau} d\tau \neq w(t - \alpha)$ , so the system is time-variant.

(d) Linearity:  $\frac{d}{dt}(\alpha_1 v_1(t) + \alpha_2 v_2(t)) = \alpha_1 \frac{d}{dt} v_1(t) + \alpha_2 \frac{d}{dt} v_2(t)$ , so the system is linear.

Time-Invariance:  $\frac{d}{dt} v(t - \tau) = w(t - \tau)$ , so the system is time-invariant.

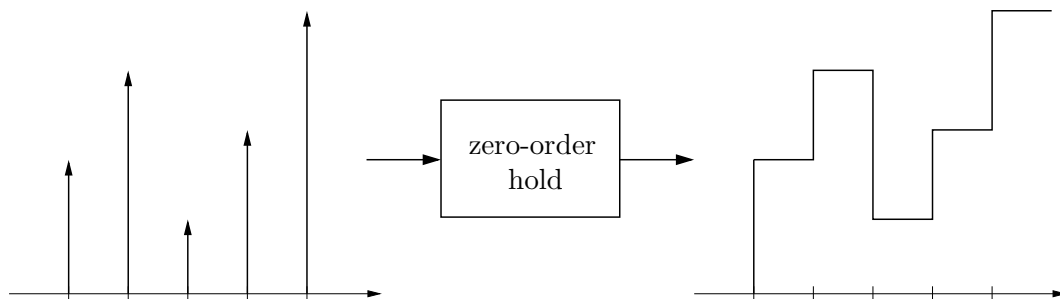
(e) Linearity:  $\cos(\omega t + \alpha_1 v_1(t) + \alpha_2 v_2(t)) \neq \alpha_1 \cos(\omega t + v_1(t)) + \alpha_2 \cos(\omega t + v_2(t))$ , so the system is non-linear.

Time-Invariance:  $\cos(\omega t + v(t - \tau)) \neq w(t - \tau)$ , so the system is time-variant.

3. (15 points) *Zero-order hold*. The music on your CD has been sampled at the rate 44.1 kHz. This sampling rate comes from the Sampling Theorem together with experimental observations that your ear cannot respond to sounds with frequencies above about 20 kHz. (The precise value 44.1 kHz comes from the technical specs of the earlier audio tape machines that were used when CDs were first getting started.)

A problem with reconstructing the original music from samples is that interpolation based on the sinc function is not physically realizable – for one thing, the sinc function is not time-limited. Cheap CD players use what is known as ‘zero-order hold’. This means that the value of a given sample is *held* until the next sample is read, at which point that sample value is held, and so on.

Suppose the input is represented by a train of  $\delta$ -functions, spaced  $T = 1/44.1$  msec apart with strengths determined by the sampled values of the music, and the output looks like a staircase function. The system for carrying out zero-order hold then looks like the diagram, below. (The scales on the axes are the same for both the input and the output.)



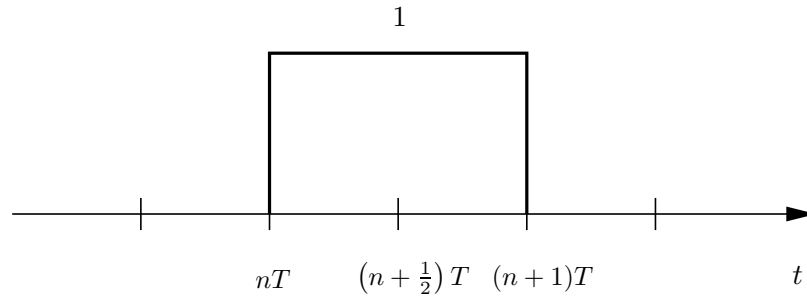
- (a) Is this a linear system? Is it time invariant for shifts of integer multiples of the sampling period?
- (b) Find the impulse response for this system.
- (c) Find the transfer function.

*Solution:*

- (a) Say the music is represented by a signal  $u(t)$ . We sample  $u(t)$  with time spacing  $T$  and so the input can be written as

$$v(t) = \sum_{n=-\infty}^{\infty} u(nT) \delta(t - nT).$$

The output,  $w(t)$ , takes the value  $u(nT)$  and holds it for  $T$  seconds, then takes the value  $u((n+1)T)$  and holds it for  $T$  seconds, and so on. We can express this as follows. Start with the rect function of width  $T$ , that is  $\Pi_T(t)$ , and center it on the interval from  $nT$  to  $(n+1)T$ , that is  $\Pi_T(t - (n + \frac{1}{2})T)$ . This shifted rect looks like



Then the output is

$$w(t) = Lv(t) = \sum_{n=-\infty}^{\infty} u(nT) \Pi_T \left( t - \left( n + \frac{1}{2} \right) T \right).$$

From this expression we can easily see that the system is linear, for

$$\begin{aligned} L(v_1(t) + v_2(t)) &= \sum_{n=-\infty}^{\infty} (u_1(nT) + u_2(nT)) \Pi_T \left( t - \left( n + \frac{1}{2} \right) T \right) \\ &= \sum_{n=-\infty}^{\infty} u_1(nT) \Pi_T \left( t - \left( n + \frac{1}{2} \right) T \right) + \sum_{n=-\infty}^{\infty} u_2(nT) \Pi_T \left( t - \left( n + \frac{1}{2} \right) T \right) \\ &= Lv_1(t) + Lv_2(t) \end{aligned}$$

Similarly,

$$L(\alpha v(t)) = \alpha Lv(t).$$

Next, suppose we shift the time by an integer multiple of  $T$ , say  $mT$ . Then the input is

$$\begin{aligned} v(t - mT) &= \sum_{n=-\infty}^{\infty} u(nT) \delta(t - mT - nT) \\ &= \sum_{n=-\infty}^{\infty} u(nT) \delta(t - (n + m)T) \quad (\text{now let } k = m + n \text{ to write the sum as}) \\ &= \sum_{k=-\infty}^{\infty} u((k - m)T) \delta(t - kT) \end{aligned}$$

If  $w(t) = Lv(t)$  then the output resulting from the shift is

$$\begin{aligned}
 L(v(t - mT)) &= \sum_{k=-\infty}^{\infty} u((k - m)T) \Pi_T \left( t - \left( k + \frac{1}{2} \right) T \right) \\
 &\quad \text{(now, undoing what we did before, let } n = k - m \text{ to write the sum as)} \\
 &= \sum_{n=-\infty}^{\infty} u(nT) \Pi_T \left( t - \left( n + m + \frac{1}{2} \right) T \right) \\
 &= \sum_{n=-\infty}^{\infty} u(nT) \Pi_T \left( t - \left( n + \frac{1}{2} \right) T - mT \right) \\
 &= w(t - mT)
 \end{aligned}$$

Thus the system is time invariant for shifts by integer multiples of  $T$ .

- (b) In order to determine the impulse response, we input the delta function,  $\delta(t)$ , and have to determine the output. The input is concentrated at  $t = 0$  with strength 1 and is thereafter 0. So, starting at  $t = 0$ , the output holds the value 1 for  $T$  seconds and is then 0 for  $t \geq T$ . The output is also 0 for  $t < 0$ . In other words, the impulse response is

$$h(t) = \begin{cases} 1, & 0 \leq t < T \\ 0, & \text{otherwise.} \end{cases}$$

- (c) We can now calculate the transfer function from the obtained impulse response. This is

$$\begin{aligned}
 H(s) &= \mathcal{F}h(s) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i s t} dt \\
 &= \int_0^T e^{-2\pi i s t} dt \\
 &= \left. \frac{e^{-2\pi i s t}}{-2\pi i s} \right|_0^T \\
 &= \frac{1 - e^{-2\pi i s T}}{2\pi i s} \\
 &= T e^{-\pi i s T} \frac{\sin \pi s T}{\pi s T} \\
 &= T e^{-\pi i s T} \text{sinc}(sT).
 \end{aligned}$$

4. (20 points) Let  $w(t) = Lv(t)$  be an LTI system with impulse response  $h(t)$ ; thus  $w(t) = (h * v)(t)$ .

- (a) If  $h(t)$  is bandlimited show that  $w(t) = Lv(t)$  is bandlimited regardless of whether  $v(t)$  is bandlimited.
- (b) Suppose  $v(t)$ ,  $h(t)$ , and  $w(t) = Lv(t)$  are all bandlimited with spectrum contained in  $-1/2 < s < 1/2$ . According to the sampling theorem we can write

$$v(t) = \sum_{k=-\infty}^{\infty} v(k) \text{sinc}(t - k) \quad h(t) = \sum_{k=-\infty}^{\infty} h(k) \text{sinc}(t - k) \quad w(t) = \sum_{k=-\infty}^{\infty} w(k) \text{sinc}(t - k)$$

Express the sample values  $w(k)$  in terms of the sample values of  $v(t)$  and  $h(t)$ . (You will need to consider the convolution of shifted sincs.)

*Solution:*

- (a) Take the Fourier transform of  $w = h * v$  to obtain

$$\mathcal{F}w = (\mathcal{F}h)(\mathcal{F}v).$$

From here we see that the spectrum of  $w$  can extend no farther than the spectrum of  $h$ , and hence if  $h$  is bandlimited so too is  $w$  whether  $v$  is or not.

- (b) We work in the time domain and consider the convolution

$$\begin{aligned} \sum_{m=-\infty}^{\infty} w(m) \operatorname{sinc}(t - m) &= w(t) = (h * v)(t) \\ &= \left( \sum_{k=-\infty}^{\infty} h(k) \operatorname{sinc}(t - k) \right) * \left( \sum_{n=-\infty}^{\infty} v(n) \operatorname{sinc}(t - n) \right) \\ &= \sum_{k,n=-\infty}^{\infty} h(k)v(n) \operatorname{sinc}(t - k) * \operatorname{sinc}(t - n) \end{aligned}$$

We had to use different indices on the sums on the right to combine them in this way. You'll see in a moment why we used still another index for the sum on the left.

Now let's work with the convolution of the shifted sincs. Taking the Fourier transform we have

$$\begin{aligned} \mathcal{F}(\operatorname{sinc}(t - k) * \operatorname{sinc}(t - n)) &= e^{-2\pi i k s} \Pi(s) e^{-2\pi i n s} \Pi(s) \\ &= e^{-2\pi i (k+n) s} \Pi^2(s) \\ &= e^{-2\pi i (k+n) s} \Pi(s), \end{aligned}$$

since  $\Pi^2 = \Pi$ . And now taking the inverse Fourier transform we get

$$\operatorname{sinc}(t - k) * \operatorname{sinc}(t - n) = \operatorname{sinc}(t - (k + n)).$$

Thus

$$\sum_{m=-\infty}^{\infty} w(m) \operatorname{sinc}(t - m) = \sum_{k,n=-\infty}^{\infty} h(k)v(n) \operatorname{sinc}(t - (k + n))$$

Take a particular  $m_0$  and let  $t = m_0$ . Then the left hand side is just  $w(m_0)$  so we have

$$w(m_0) = \sum_{k,n=-\infty}^{\infty} h(k)v(n) \operatorname{sinc}(m_0 - (k + n)).$$

Next,  $\operatorname{sinc}(m_0 - (k + n))$  will be zero except when  $k + n = m_0$ , in which case it's 1. So the double sum on the right hand side reduces to a sum over indices  $k$  and  $n$  for which  $k + n = m_0$ . Or, writing  $n = m_0 - k$ , we can express the result as

$$w(m_0) = \sum_{k=-\infty}^{\infty} h(k)v(m_0 - k)$$

Since this holds for all  $m_0$  let's just write this as

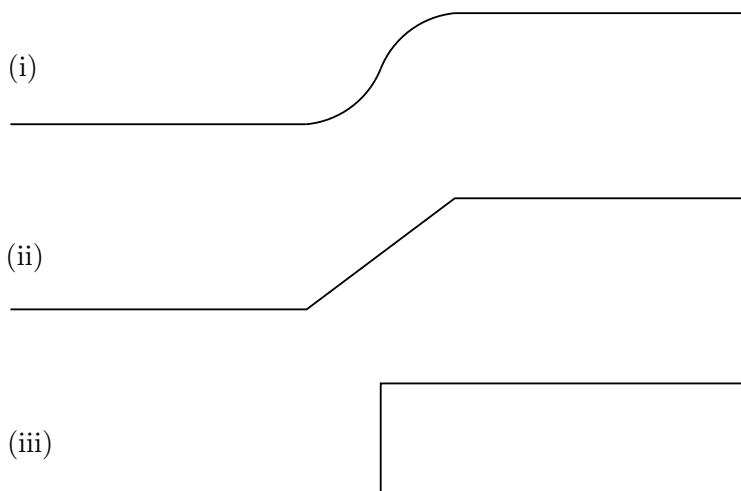
$$w(m) = \sum_{k=-\infty}^{\infty} h(k)v(m-k).$$

Yep. The values of the output  $w(t)$  at the sample points are obtained by taking the convolutions of the (infinite) sequences of the sample points of the impulse response and the input.

5. (20 points) Let  $a > 0$  and define the (continuous) running average of a signal  $f(t)$  to be

$$Lf(t) = \frac{1}{2a} \int_{t-a}^{t+a} f(x) dx$$

- Show that  $L$  is an LTI system and find its impulse response and transfer function.
- What happens to the system as  $a \rightarrow 0$ ? Justify your answer based on properties of the impulse response.
- Find the impulse response and the transfer function for the cascaded system  $M = L(Lf)$ . Simplify your answer as much as possible, *i.e.*, don't simply express the impulse response as a convolution.
- Three signals are plotted (approximately) below. One is an input signal  $f(t)$ . One is  $Lf(t)$  and one is  $Mf(t)$ . Which is which, and why?



*Solution:*

- (a) First let's show that  $L$  is linear:

$$\begin{aligned} L(\alpha_1 f_1(t) + \alpha_2 f_2(t)) &= \frac{1}{2a} \int_{t-a}^{t+a} (\alpha_1 f_1(x) + \alpha_2 f_2(x)) dx \\ &= \alpha_1 \frac{1}{2a} \int_{t-a}^{t+a} f_1(x) dx + \alpha_2 \frac{1}{2a} \int_{t-a}^{t+a} f_2(x) dx \\ &= \alpha_1 L(f_1(t)) + \alpha_2 L(f_2(t)) \end{aligned}$$



Hence,  $L$  is linear. Now let's show that  $L$  is time-invariant:

$$\begin{aligned} L(\tau_{t_0}f(t)) &= \frac{1}{2a} \int_{t-a}^{t+a} f(x-t_0) dx \\ &= \frac{1}{2a} \int_{(t-t_0)-a}^{(t-t_0)+a} f(y) dy \\ &= \tau_{t_0}L(f(t)) \end{aligned}$$

where  $\tau_{t_0}$  is the shift-operator, i.e.,  $\tau_{t_0}f(t) = f(t-t_0)$ . Hence, we have shown that  $L$  is indeed time-invariant. If we define  $h_a(t) = \frac{1}{2a}\Pi(\frac{t}{2a})$ , then clearly  $L(f(t)) = h_a(t) * f(t)$ . Hence, the impulse response of the system is given by  $h(t)$ , and the transfer function is simply  $H(s) = \text{sinc}(2as)$ .

(b) If we let  $a \rightarrow 0$ , then the impulse response function becomes:

$$h(t) = \lim_{a \rightarrow 0} \frac{1}{2a} \Pi\left(\frac{t}{2a}\right) = \delta(t)$$

In the frequency domain, the transfer function becomes:

$$H(s) = \lim_{a \rightarrow 0} \text{sinc}(2as) = 1$$

which makes perfect sense.

(c) Note that  $M(t) = L(L(f(t))) = h_a(t) * (h_a(t) * f(t)) = (h_a(t) * h_a(t)) * f(t)$ . Hence, the impulse response of  $M$  is given by:

$$\begin{aligned} h_a(t) * h_a(t) &= \left( \frac{1}{2a} \Pi\left(\frac{t}{2a}\right) \right) * \left( \frac{1}{2a} \Pi\left(\frac{t}{2a}\right) \right) \\ &= \frac{1}{2a} \Delta\left(\frac{t}{2a}\right) \end{aligned}$$

and its transfer function is given by  $H(s)^2 = \text{sinc}^2(2as)$ .

(d) From part (b), we know that the operator  $L(f(t))$  attenuates high frequencies (smoothing operator). It's reasonable to think that the third plot corresponds to  $f(t)$ , the second plot to  $L(f(t))$  and the first plot to  $M(f(t)) = L(L(f(t)))$ . Let's show this more formally. Let  $f(t) = u(t)$  (the rect function). From part (b), we know that

$$\begin{aligned} L(f(t)) &= \frac{1}{2a} u(t) * \Pi\left(\frac{t}{2a}\right) \\ &= \frac{1}{2a} \int_0^\infty \Pi\left(\frac{t-\tau}{2a}\right) d\tau \\ &= \begin{cases} 0 & , t \leq -a \\ \frac{1}{2} + \frac{t}{2a} & , -a < t \leq a \\ 1 & , t > a \end{cases} \end{aligned}$$

which is piecewise linear on  $t$ . Similarly, it can be shown that  $M(f(t))$  is piecewise quadratic on  $t$ . Hence, our guess was correct.

6. (15 points) *Upsampling and Downsampling discrete signals:* We are given a discrete-time signal  $x_n$ , which was obtained by sampling a continuous-time  $x(t)$  at frequency much higher than the Nyquist rate. Let  $x_n = x(nT_s)$ . Sometimes for an application, we need the samples of the signal  $x(t)$  at a different sampling rate, say  $\alpha T_s$ , where  $\alpha > 0$ . If  $\alpha > 1$ , it is downsampling else it is upsampling. One way to obtain the sample points at a different sampling frequency, is to reconstruct the signal  $x(t)$  (which can be done as the original samples were obtained by sampling above the Nyquist rate) from the samples  $x_n$  and then resample it at the new frequency. However, due to non-idealities of the D/A and the A/D converters, this approach is not desirable. In this question, we want to obtain the new sample points directly by using  $x_n$ .
- Find the expression for  $w_n = x(n(MT_s))$  in terms of  $x_n$ , where  $M$  is an integer.
  - Find the expression for  $y_n = x(n(\frac{T_s}{M}))$  in terms of  $x_n$ , where  $M$  is an integer. If the resulting expression looks complicated, it will motivate the use of linear interpolation or nearest-neighbor interpolation as used in HW-5.
  - Find the expression for  $z_n = x(n(\alpha T_s))$  in terms of  $x_n$ , where  $\alpha > 0$  is a rational number  $\alpha = \frac{P}{Q}$ , where  $P$  and  $Q$  are integers. Give some insight into how would you accomplish a change in sampling frequency for any  $\alpha > 0$  (real number).

*Solution:*

(a)

$$w_n = x_{nM}$$

(b)

$$y_n = \sum_{k=-\infty}^{\infty} x_k \frac{\sin(\pi(\frac{n-kM}{M}))}{\pi(\frac{n-kM}{M})}$$

- (c) For any rational  $P$  and  $Q$ ,  $z_n$  can be obtained by a combination of upsampling and downsampling as shown in parts (a) and (b). Depending upon the values of  $P$  and  $Q$ , we can decide if we first downsample or upsample. For any real number  $\alpha$ , we can choose  $P$  and  $Q$  such that to approximate  $\alpha$  closely.