

EE 261 The Fourier Transform and its Applications
Fall 2007
Solutions to Problem Set Five

1. (30 points) *Evaluating integrals with the help of Fourier transforms* Evaluate the following integrals using Parseval's Theorem and one other method. (Yes, we expect you to evaluate the integral twice, and if you do it right you should get the same answer for both approaches (obviously)):

(a)

$$\int_{-\infty}^{\infty} \text{sinc}^4(t) dt$$

(b)

$$\int_{-\infty}^{\infty} \frac{2}{1 + (2\pi t)^2} \text{sinc}(2t) dt$$

(c)

$$\int_{-\infty}^{\infty} t^2 \text{sinc}^4(t) dt$$

Solution:

(a) We first note that

$$\int_{-\infty}^{\infty} \text{sinc}^4(t) dt = \int_{-\infty}^{\infty} |\text{sinc}^2(t)|^2 dt,$$

since $\text{sinc}(t)$ is a real-valued function.

Using Parseval's theorem we have,

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\mathcal{F}f(s)|^2 ds.$$

We set $f(t) = \text{sinc}^2(t)$ and using $F(s) = \Lambda(s)$, we obtain

$$\begin{aligned}
 \int_{-\infty}^{\infty} \text{sinc}^4(t) dt &= \int_{-\infty}^{\infty} |\Lambda(s)|^2 ds \\
 &= \int_{-1}^1 \Lambda^2(s) ds \\
 &= 2 \int_0^1 (-s+1)^2 ds \quad (\text{since } \Lambda \text{ is an even function}) \\
 &= 2 \int_0^1 s^2 - 2s + 1 ds \\
 &= 2 \left[\frac{s^3}{3} - s^2 + s \right]_0^1 \\
 &= \frac{2}{3}.
 \end{aligned}$$

For the other approach, we recognize that $\text{sinc}^4(t)$ transforms to $(\Lambda * \Lambda)(s)$. Hence, the integral $\int_{-\infty}^{\infty} \text{sinc}^4(t) dt$ would simply be $(\Lambda * \Lambda)(0)$. Writing this out as an equation yields:

$$\begin{aligned}
 \int_{-\infty}^{\infty} \text{sinc}^4(t) dt &= \int_{-\infty}^{\infty} \Lambda(-s)\Lambda(s) ds \\
 &= \int_{-\infty}^{\infty} \Lambda^2(s) ds \quad (\text{since } \Lambda \text{ is an even function}) \\
 &= \int_{-1}^1 \Lambda^2(s) ds
 \end{aligned}$$

This is the same integral as the Parseval approach and hence we get $\frac{2}{3}$ again.

- (b) Here we have a multiplication of two functions whose Fourier transforms are known. The given integral can therefore be evaluated using Parseval's theorem in its general form:

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)} dt = \int_{-\infty}^{\infty} F(s)\overline{G(s)} ds.$$

For the functions we have, this becomes:

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{2}{1+(2\pi t)^2} \text{sinc}(2t) dt &= \int_{-\infty}^{\infty} \frac{2}{1+(2\pi t)^2} \overline{\text{sinc}(2t)} dt \\
 &= \int_{-\infty}^{\infty} e^{-|s|} \overline{\frac{1}{2}\Pi\left(\frac{s}{2}\right)} ds \\
 &= \int_{-1}^1 \frac{1}{2} e^{-|s|} ds \\
 &= 2 \int_0^1 \frac{1}{2} e^{-s} ds \\
 &= \int_0^1 e^{-s} ds \\
 &= 1 - \frac{1}{e}.
 \end{aligned}$$

For the other approach, we recognize that $\frac{2}{1+(2\pi t)^2} \text{sinc}(2t)$ transforms to $e^{-|s|} * \frac{1}{2}\Pi_2(s)$. Hence, the integral $\int_{-\infty}^{\infty} \frac{2}{1+(2\pi t)^2} \text{sinc}(2t) dt$ would be the convolution of $e^{-|s|}$ and $\frac{1}{2}\Pi_2(s)$ evaluated at zero.

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{2}{1+(2\pi t)^2} \text{sinc}(2t) dt &= \int_{-\infty}^{\infty} e^{-|s|} \frac{1}{2}\Pi_2(-s) ds \\ &= \int_{-1}^1 \frac{1}{2} e^{-|s|} ds \\ &= 2 \int_0^1 \frac{1}{2} e^{-s} ds \\ &= \int_0^1 e^{-s} ds \\ &= 1 - \frac{1}{e}. \end{aligned}$$

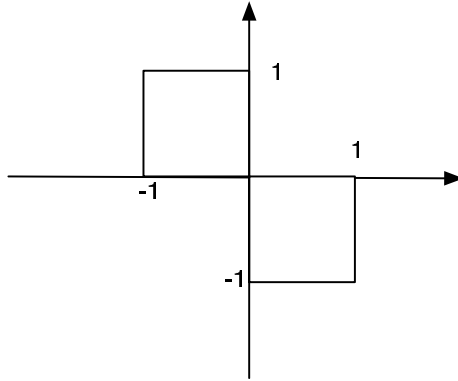
(c) The integral can be written as:

$$\int_{-\infty}^{\infty} t^2 \text{sinc}^4(t) dt = \int_{-\infty}^{\infty} (t \text{sinc}^2(t)) (t \text{sinc}^2(t)) dt$$

Using the derivative property, which states:

$$\mathcal{F}\{t^n g(t)\} = \left(\frac{i}{2\pi}\right)^n G^{(n)}(s).$$

we can deduce that $t \text{sinc}^2(t)$ transforms to $\frac{i}{2\pi}\Lambda'(s)$. Here we take the derivative of $\Lambda(s)$ in a piecewise manner. The graph of $\Lambda'(s)$ is shown below.



Hence, from Parseval's we have:

$$\begin{aligned} \int_{-\infty}^{\infty} t^2 \text{sinc}^4(t) dt &= \int_{-\infty}^{\infty} \frac{i}{2\pi}\Lambda'(s) \overline{\frac{i}{2\pi}\Lambda'(s)} ds \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \Lambda'(s)\Lambda'(s) ds \\ &= \frac{1}{4\pi^2}(2) \\ &= \frac{1}{2\pi^2} \end{aligned}$$

An alternative method would be to again use the derivative property in conjunction with convolution. Writing the left hand side as an integral,

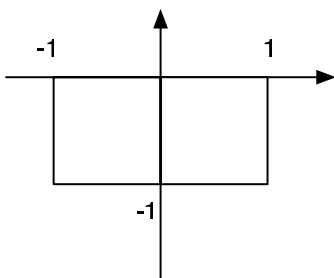
$$\int_{-\infty}^{\infty} t^n g(t) e^{-i2\pi st} dt = \left(\frac{i}{2\pi}\right)^n G^{(n)}(s).$$

Evaluating both sides at $s = 0$ then gives us

$$\int_{-\infty}^{\infty} t^n g(t) dt = \left(\frac{i}{2\pi}\right)^n G^{(n)}(0).$$

Comparing the left hand side with the integral of interest, $\int_{-\infty}^{\infty} t^2 \text{sinc}^4(t) dt$, we can see that $n = 2$ and $g(t) = \text{sinc}^4(t)$. What is left is to find the second derivative of $G(s)$ evaluated at zero.

From the convolution theorem of Fourier transforms, we know that since $g(t) = \text{sinc}^2(t)\text{sinc}^2(t)$, then $G(s) = (\Lambda * \Lambda)(s)$. We now use the useful property that differentiation distributes over convolution. Hence, $G^{(2)}(s) = (\Lambda' * \Lambda')(s)$. The convolution of this graph with itself, evaluated at zero, would just be the area under the following curve:



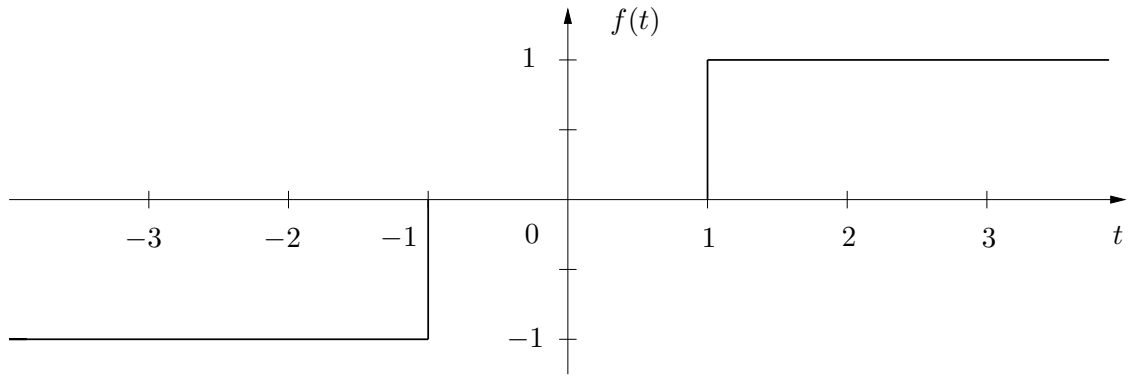
which is -2 , i.e., $G^{(2)}(0) = -2$. Putting the results together we have

$$\begin{aligned} \int_{-\infty}^{\infty} t^2 \text{sinc}^4(t) dt &= \left(\frac{i}{2\pi}\right)^2 G^{(2)}(0) \\ &= \frac{-1}{4\pi^2} (-2) \\ &= \frac{1}{2\pi^2}. \end{aligned}$$

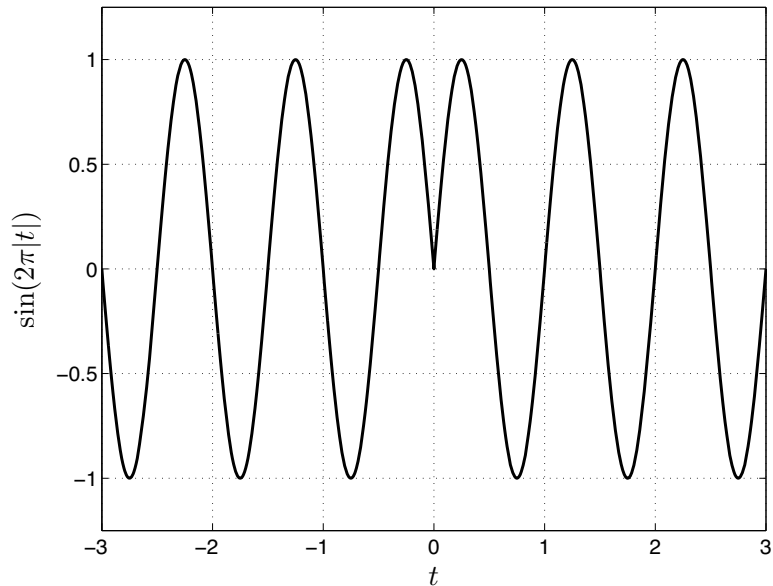
This is the same value obtained by applying Parseval's theorem.

2. (20 points) *Generalized Fourier transforms*

- (a) Find the Fourier transform of the signal shown below.



- (b) Find the Fourier transform of the function $f(t) = \sin(2\pi|t|)$ plotted below. (Simplify your expression as *much* as possible.)



Solution:

- (a) The signal is made out of the unit step function $H(t)$, so for sure we'll need to use

$$\mathcal{F}H(s) = \frac{1}{2}\left(\delta + \frac{1}{\pi i s}\right).$$

The part of the signal on the right is just a shift of $H(t)$ to the right, and that's $H(t-1)$. To get the part on the left we first reverse to H^- , then shift to the left, making $H^-(t+1)$ (which you could also write as $H(-(t+1)) = H(-t-1)$, but I won't) and then flip it upside down, to $-H^-(t+1)$. The whole signal is then

$$f(t) = H(t-1) - H^-(t+1)$$

The Fourier transform of this is

$$\begin{aligned}
 \mathcal{F}f(s) &= e^{-2\pi is} \mathcal{F}H(s) - e^{2\pi is} (\mathcal{F}H^-)(s) \\
 &= e^{-2\pi is} \mathcal{F}H(s) - e^{2\pi is} (\mathcal{F}H)^-(s) \\
 &= e^{-2\pi is} \mathcal{F}H(s) - e^{2\pi is} \mathcal{F}H(-s) \\
 &= e^{-2\pi is} \left(\frac{1}{2} \delta(s) + \frac{1}{2\pi is} \right) - e^{2\pi is} \left(\frac{1}{2} \delta(-s) + \frac{1}{2\pi i(-s)} \right) \\
 &= \frac{1}{2} (e^{-2\pi is} - e^{2\pi is}) \delta(s) + \frac{1}{2\pi is} (e^{-2\pi is} + e^{2\pi is}) \\
 &= -i(\sin 2\pi s) \delta(s) + \frac{1}{\pi is} \cos 2\pi s \\
 &= \frac{1}{\pi is} \cos 2\pi s.
 \end{aligned}$$

As a check, notice that the Fourier transform is purely imaginary and odd, which it should be.

- (b) We can directly observe from the plot that given function is a multiplication of $\sin(2\pi t)$ and the sign (signum) function $\text{sgn}(t)$ which has the Fourier transform $\mathcal{F}(\text{sgn})(s) = 1/(\pi is)$.

The alternative definition of $\sin(2\pi|t|)$ can also be derived mathematically as

$$\begin{aligned}
 \sin(2\pi|t|) &= \begin{cases} \sin(2\pi t) & \text{for } t > 0 \\ \sin(2\pi(-t)) & \text{for } t < 0 \end{cases} \\
 &= \begin{cases} \sin(2\pi t) & \text{for } t > 0 \\ -\sin(2\pi t) & \text{for } t < 0 \end{cases} \\
 &= \sin(2\pi t) \text{sgn}(t).
 \end{aligned}$$

Taking the Fourier transform we get

$$\begin{aligned}
 \mathcal{F}(\sin(2\pi|t|))(s) &= \mathcal{F}(\sin(2\pi t) \text{sgn}(t))(s) \\
 &= \mathcal{F}(\sin(2\pi t))(s) * \mathcal{F}(\text{sgn}(t))(s) \\
 &= \frac{1}{2i} (\delta(s-1) - \delta(s+1)) * \frac{1}{\pi is} \\
 &= \frac{-1}{2\pi(s-1)} + \frac{1}{2\pi(s+1)} \\
 &= \frac{-s-1+s-1}{2\pi(s-1)(s+1)}.
 \end{aligned}$$

After simplification, the final answer is

$$\mathcal{F}(\sin(2\pi|t|))(s) = \frac{1}{\pi(1-s^2)}.$$

3. (20 points) *System identification*

- (a) Consider a filter with *real and even* transfer function $H(s)$. If the input to the filter is $\cos(2\pi\nu t)$, show that the output is $H(\nu)\cos(2\pi\nu t)$.
- (b) We shall use the result in part (a) to identify an unknown filter. Download **identme.m** and **transferfcn.m** from the class web page

<http://see.stanford.edu/see/materials/ISOFAEE261/assignments.aspx>

The file **identme.m** is the filter which takes an input signal and produces an output. The file **transferfcn.m** stores the transfer function and is called by **identme.m**. Let the input be a sinusoid with some frequency ν . Estimate $H(\nu)$ from the amplitude of the output. Sketch $H(s)$ by varying ν . What kind of filter is this? What is/are the cutoff frequency(ies)?

[Hint: (1) Try frequencies in the range 0 - 20 Hz. (2) Ignore the fact that the output signal is not a perfect sinusoid. This is because the signal is not infinite duration.]

- (c) Plot $H(s)$ from **transferfcn.m**. Compare with part (b).

Solution:

- (a) (5 points) Since the Fourier transform of $\cos(2\pi\nu t)$ is $\frac{1}{2}(\delta(s - \nu) + \delta(s + \nu))$, the output of the filter is

$$\begin{aligned} & \mathcal{F}^{-1} \left\{ \frac{1}{2}(\delta(s - \nu) + \delta(s + \nu))H(s) \right\} \\ &= \mathcal{F}^{-1} \left\{ \frac{1}{2}(\delta(s - \nu)H(\nu) + \delta(s + \nu)H(-\nu)) \right\} \\ &= \mathcal{F}^{-1} \left\{ \frac{1}{2}(\delta(s - \nu) + \delta(s + \nu))H(\nu) \right\} \quad (\text{because } H \text{ is even}) \\ &= \cos(2\pi\nu t)H(\nu). \end{aligned}$$

- (b) (10 points) By using the example provided above, we can plot the output signal and estimate its amplitude. See a MATLAB code below.

```
%ee261 system identification solution
%
vectnu = 0:20;           % frequency range of interest
resolution = 0.001;
t = 0:resolution:10;    % create a time vector
amplitude = [];         % a vector to store amplitude

for index = 1:length(vectnu)
    nu = vectnu(index);
    input = cos(2*pi*nu*t);
    output = identme(input,resolution);

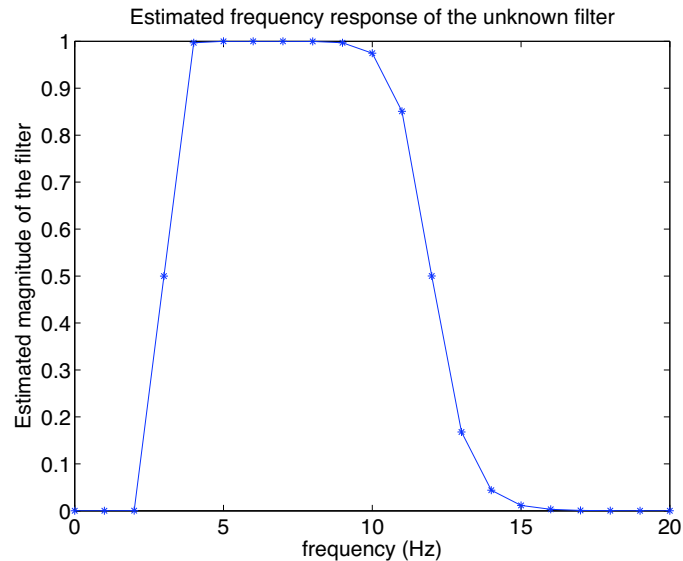
    % to avoid edge effect, pick the amplitude to be
    % maximum of the output signal in the middle time range
    amplitude = [amplitude max(output(3000:7000))];
end

set(gca, 'Linewidth',1.6);
set(gca, 'FontSize',16);
```

```

plot(vectnu,amplitude,'*-');
title('Estimated frequency response of the unknown filter');
xlabel('frequency (Hz)');
ylabel('Estimated magnitude of the filter');
print('-depsc','hw5prob4_1.eps');

```



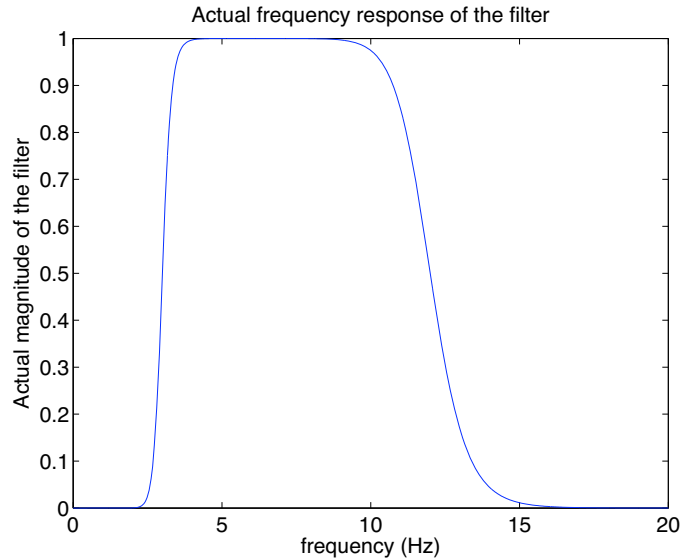
From the plot, this is a bandpass filter whose low cutoff frequency is $\sim 3 - 4$ Hz and high cutoff frequency is $\sim 10 - 14$ Hz. Note that in practice, we usually plot the frequency response in log-log scale.

- (c) (5 points) Simple MATLAB commands to plot $H(s)$ are given below. It is enough to plot $H(s)$ for $s \geq 0$.

```

s = 0:0.01:20;
H = transferfcn(s);
set(gca, 'Linewidth',1.6);
set(gca, 'FontSize',16);
plot(s,H);
title('Actual frequency response of the filter');
xlabel('frequency (Hz)');
ylabel('Actual magnitude of the filter');
print('-depsc','hw5prob4_2.eps');

```

This is similar to what we have in part (b). It is in fact a bandpass Butterworth filter with cutoff frequencies 3Hz and 12Hz.

4. (10 points) Consider the signal

$$f(t) = 1 + \Lambda(3t) * \text{III}_{1/3}(t).$$

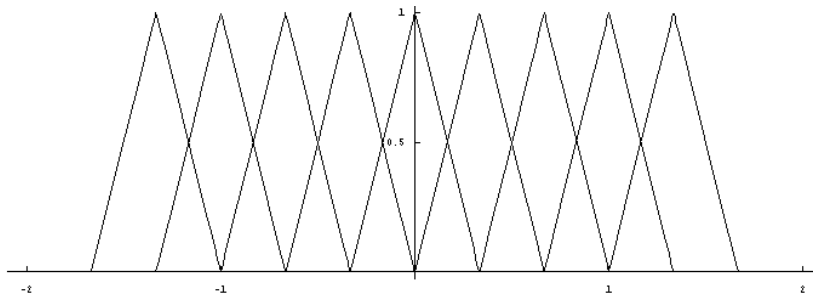
Sketch the graph of $f(t)$, find the Fourier transform $\mathcal{F}f(s)$, and sketch its graph. Comment on what you see.

Solution:

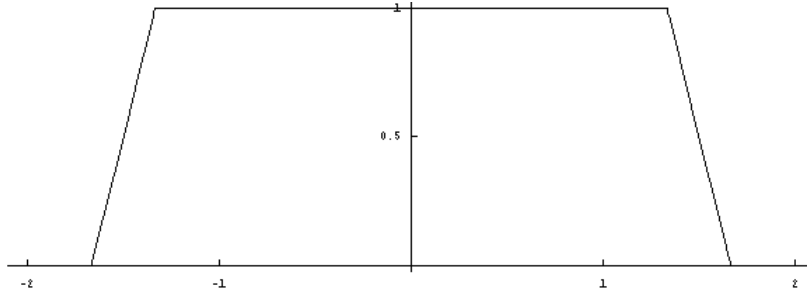
Taking the convolution $\Lambda(3t) * \text{III}_{1/3}(t)$ produces the sum

$$\Lambda(3t) * \text{III}_{1/3}(t) = \sum_{k=-\infty}^{\infty} \Lambda\left(3t - \frac{k}{3}\right) = \sum_{k=-\infty}^{\infty} \Lambda(3t - k)$$

A plot of the shifted triangles from $k = -4$ to 4 looks like this:



If you draw your graph carefully enough (or actually do the algebra) you'll see that the overlaps are such that when you *add* the shifted triangles you get a plot that looks like:



And you would have concluded that $\Lambda(3t) * \text{III}_{1/3}(t) = 1$, and that $f(t) = 2$, and, so, for that matter that $\mathcal{F}f(s) = 2\delta$

However, if you didn't draw the graph carefully enough you had a second chance to come to the same conclusion by taking the Fourier transform:

$$\begin{aligned} \mathcal{F}f &= \delta + \left(\frac{1}{3}\text{sinc}^2\left(\frac{1}{3}s\right)\right)(3\text{III}_3(s)) \\ &= \delta + \sum_{k=-\infty}^{\infty} \text{sinc}^2\left(\frac{k}{3}\right)\delta(s - 3k) \end{aligned}$$

Now the thing you have to realize is that $\text{sinc}^2(k/3)$ is 1 at $k = 0$ and is zero at the points where $k/3$ is an integer, *i.e.* where k is a multiple of 3. The zeros of sinc^2 line up with the impulses $\delta(s - 3k)$, $k \neq 0$ and hence kill them off. What remains is

$$\mathcal{F}f = \delta + \delta = 2\delta.$$

I won't plot this. But having seen it, if you didn't get the graph of $f(t)$ right you should now see that it's just $2\mathcal{F}^{-1}\delta = 2$.

5. (10 points) *Sampling Images*

Thomas and Lykomidis are chatting after class.

Thomas: What's up Lykomidis? You look too skeptical today. Is everything ok?

Lykomidis: Well ... I was watching this movie yesterday and I am still thinking about it...

Thomas: It must have been a great movie!

Lykomidis: Actually, it was rather boring. However, I observed something which I cannot really explain. In one of the scenes, there was a fan in the background. It was a hot summer day, so at some point, the guy in the movie turned the fan on. My problem is that the fan did not seem to rotate even after that point. That guy commented on how much nicer it was, after the fan had been turned on. I mean.. it was not the greatest movie ever, but I am pretty sure they would not have used a fan that was not functioning..

Thomas: You know, it is funny that you mention that. I had observed something similar a few years ago.

Brad, who was standing right behind them all this time, joins the conversation.

Brad: Well, maybe this can help you: What happens when you sample a periodic signal with a sampling period equal to the period of the signal? For what other sampling periods do you observe the same phenomenon?

Can you figure out what is happening? Why does a rotating fan appear stopped in the movie? You are further told that the particular fan has a three-blade propeller. How does that impact

your answer?

Solution:

A film is a collection of sampled images. Let T_{img} be the period of the image sampling in the movie. Also let T_{fan} be the period for a full fan rotation. Now, observe that if $T_{img} = kT_{fan}$ where $k = 1, 2, 3, \dots$ then the fan will appear stopped. However, since that image has a three-blade propeller, this is also true if $T_{img} = \frac{k}{3}T_{fan}$ where $k = 1, 2, 3, \dots$