

EE 261 The Fourier Transform and its Applications

Fall 2007

Solutions to Problem Set Four

1. (15 points) *Cross Correlation* The *cross-correlation* (sometimes just called correlation) of two real-valued signals $f(t)$ and $g(t)$ is defined by

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(y)g(x+y) dy.$$

- (a) $f \star g$ is often described as ‘a measure of how well g , shifted by x , matches f ’, thus how much the values of g are correlated with those of f after a shift. Explain this, e.g. what would it mean for $f \star g$ to be large? small? positive? negative? You could illustrate this with some plots.
- (b) Cross-correlation is similar to convolution, with some important differences. Show that $f \star g = f^- * g = (f * g^-)^-$. Is it true that $f \star g = g \star f$?
- (c) *Cross-correlation and delays* Show that

$$f \star (\tau_b g) = \tau_b(f \star g).$$

What about $(\tau_b f) \star g$?

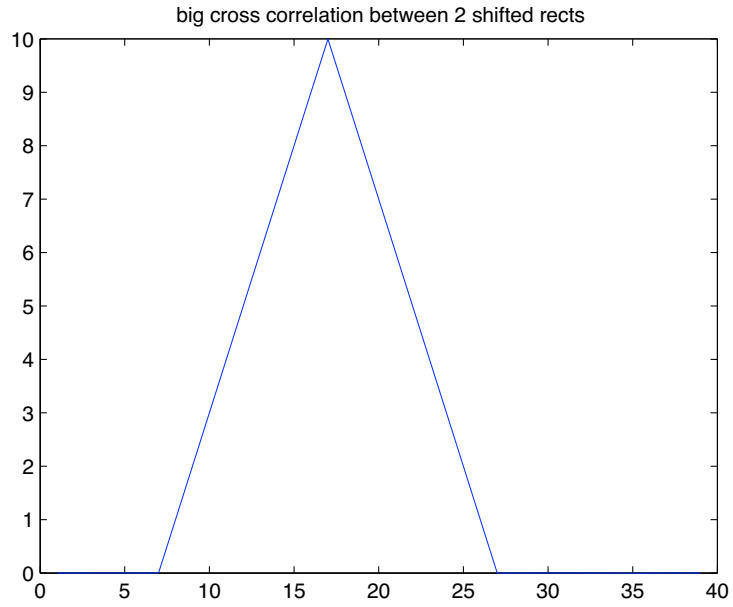
Solutions:

- (a) The cross-correlation is

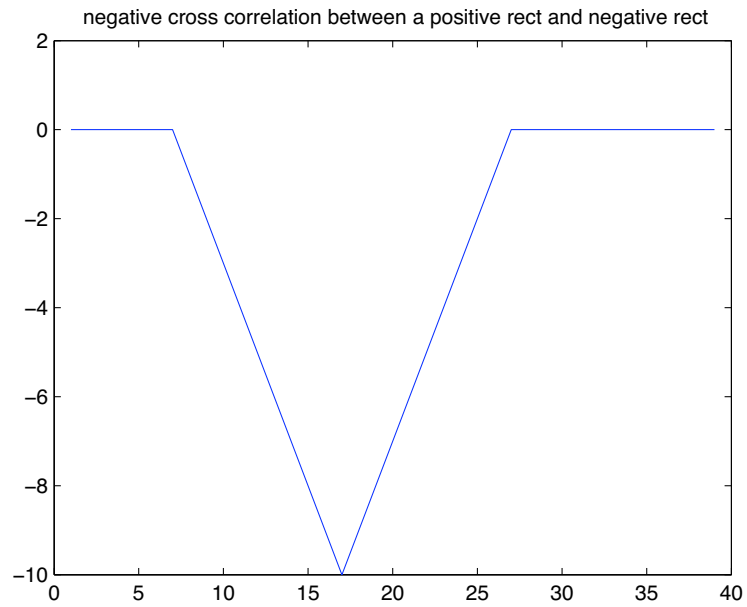
$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x)g(x+y) dy.$$

To get a sense of this, think about when it’s positive (and large) or negative (and large) or zero (or near zero). If, for a given x , the values $f(y)$ and $g(x+y)$ are tracking each other – both positive or both negative – then the integral will be positive and so the value $(f \star g)(x)$ will be positive. The closer the match between $f(x)$ and $g(x+y)$ (as y varies) the larger the integral and the larger the cross-correlation. In the other direction, if, for example, $f(y)$ and $g(x+y)$ maintain opposite signs as y varies (so are negatively correlated) then the integral will be negative and $f \star g(x) < 0$. The more the negatively they are correlated the more negative $(f \star g)(x)$. Finally, it might be that the values of $f(x)$ and $g(x+y)$ jump around as y varies; sometimes positive and sometimes negative, and it may then be that the values cancel out in taking the integral, making $(f \star g)(x)$ near zero. One might say – one does say – that f and g are uncorrelated if $(f \star g)(x) = 0$ for all x .

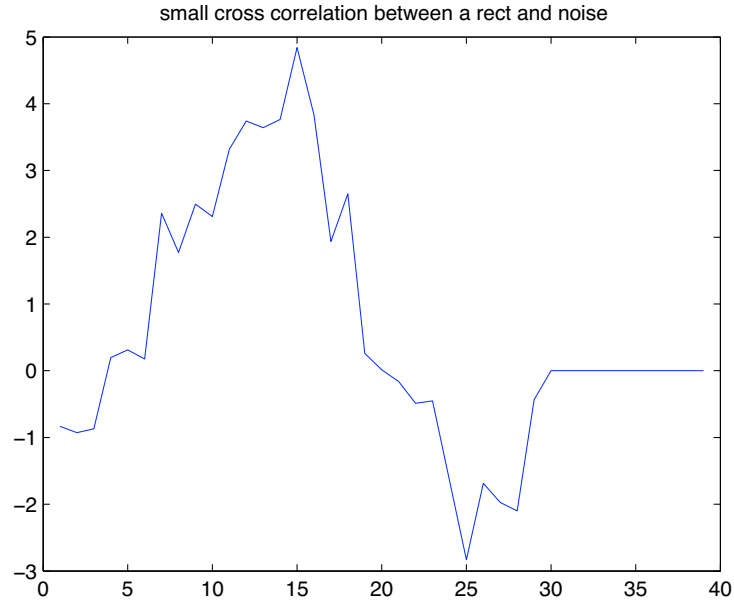
Here are some plots. First take two Π ’s, one shifted, Their cross-correlation looks like



Here's a plot of the cross-correlation between a Π and a shifted, *negative* Π .



Finally, here's a plot of the cross-correlation between a rect and a random signal (noise).



(b) For the cross-correlation we have

$$\begin{aligned}
 (f \star g)(x) &= \int_{-\infty}^{\infty} f(y)g(x+y) dy \\
 &= \int_{-\infty}^{\infty} f(u-x)g(u) du \quad (\text{substituting } u = x+y) \\
 &= \int_{-\infty}^{\infty} f(-(x-u))g(u) du \\
 &= \int_{-\infty}^{\infty} f^{-}(x-u)g(u) du \\
 &= (f^{-} \star g)(x).
 \end{aligned}$$

Next, starting from the convolution $(f \star g^{-})^{-}$ we have

$$\begin{aligned}
 (f \star g^{-})^{-}(x) &= (f \star g^{-})(-x) \\
 &= \int_{-\infty}^{\infty} f(y)g^{-}(-x-y) dy \\
 &= \int_{-\infty}^{\infty} f(y)g(-(-x-y)) dy \\
 &= \int_{-\infty}^{\infty} f(y)g(x+y) dy \\
 &= (f \star g)(x).
 \end{aligned}$$

Finally, no, cross-correlation does not, in general, commute. Using the previous identities

$$f \star g = f^{-} \star g$$

while then

$$g \star f = g^{-} \star f$$

and while we have

$$f^- * g = g * f^-$$

we do not necessarily then have

$$g * f^- = g^- * f.$$

We would have $f \star g = g \star f$ if all the functions in question were even, for example.

(c) We compute $f \star (\tau_b g)$:

$$\begin{aligned}(f \star (\tau_b g))(x) &= \int_{-\infty}^{\infty} f(y) \tau_b(x+y) dy \\ &= \int_{-\infty}^{\infty} f(y) g(x+y-b) dy \\ &= \int_{-\infty}^{\infty} f(y) g((x-b)+y) dy \\ &= (f \star g)(x-b) \\ &= \tau_b(f \star g)(x)\end{aligned}$$

For $(\tau_b f) \star g$ we have

$$\begin{aligned}((\tau_b f) \star g)(x) &= \int_{-\infty}^{\infty} (\tau_b f)(y) g(x+y) dy \\ &= \int_{-\infty}^{\infty} f(y-b) g(x+y) dy \\ &= \int_{-\infty}^{\infty} f(u) g(x+u+b) du \quad (\text{substituting } u = y-b) \\ &= \int_{-\infty}^{\infty} f(u) g(u+(x+b)) du \\ &= (f \star g)(x+b) \\ &= \tau_{-b}(f \star g)(x).\end{aligned}$$

That is,

$$(\tau_b f) \star g = \tau_{-b}(f \star g).$$

2. (20 points) The *autocorrelation* of a signal $f(t)$ with itself is defined to be

$$(f \star f)(x) = \int_{-\infty}^{\infty} f(y) f(x+y) dy.$$

This is a measure of how much the values of f are correlated, so how much they match up under shifts. For example, ideal *white noise* has uncorrelated values, thus one definition of white noise is a signal with $f \star f = 0$.

(a) Find $\Pi \star \Pi$, without much work. [Use the relationship between correlation and convolution.]

(b) Show that

$$(f \star f)(x) \leq (f \star f)(0)$$

To show this use the *Cauchy-Schwarz* inequality. See the Course Reader, Chapter 1, for a discussion of this in the context of general inner products. In the present context the Cauchy-Schwarz inequality states that

$$\int_{-\infty}^{\infty} f(t)g(t) dt \leq \left\{ \int_{-\infty}^{\infty} f(t)^2 dt \right\}^{1/2} \left\{ \int_{-\infty}^{\infty} g(t)^2 dt \right\}^{1/2},$$

with equality holding if and only if $f(t)$ and $g(t)$ are proportional.

(c) Show that

$$\mathcal{F}(f \star f) = |\mathcal{F}f|^2.$$

This is sometimes known as the *Wiener-Khintchine Theorem*, though they actually proved a more general, limiting result that applies when the transform may not exist. It's many uses include the study of noise in electronic systems.

(d) *Correlation and radar detection* Here's an application of the maximum value property, $(f \star f)(t) \leq (f \star f)(0)$, above. A radar signal $f(t)$ is sent out, takes a time T to reach an object, and is reflected back and received by the station. In total, the signal is delayed by a time $2T$, attenuated by an amount, say α , and subject to noise, say given by a function $n(t)$. Thus the received signal, $f_r(t)$, is modeled by

$$f_r(t) = \alpha f(t - 2T) + n(t) = \alpha(\tau_{2T}f)(t) + n(t).$$

You know $f(t)$ and $f_r(t)$, but because of the noise you are uncertain about T , which is what you want to know to determine the distance of the object from the radar station. You also don't know much about the noise, but one assumption that is made is that the cross-correlation of f with n is constant,

$$(f \star n)(t) = C, \quad \text{a constant.}$$

You can compute $f \star f_r$ (so do that) and you can tell (in practice, given the data) where it takes its maximum, say at t_0 . Find T in terms of t_0 .

Solutions:

(a) The easiest way to find $\Pi \star \Pi$ is to use the relationship between correlation and convolution, established in a previous problem. For then, since Π is even,

$$\Pi \star \Pi = \Pi^- \ast \Pi = \Pi \ast \Pi = \Lambda.$$

Done.

(b) The Cauchy-Schwarz inequality reads, on squaring both sides (a little easier):

$$\left(\int_{-\infty}^{\infty} f(u)g(u) du \right)^2 \leq \int_{-\infty}^{\infty} f(u)^2 du \int_{-\infty}^{\infty} g(u)^2 du.$$

For $g(u)$ substitute $f(x + u)$ to write

$$\left(\int_{-\infty}^{\infty} f(u)f(u+x) du \right)^2 \leq \int_{-\infty}^{\infty} f(u)^2 du \int_{-\infty}^{\infty} f(u+x)^2 du.$$

which says

$$(f \star f)^2(x) \leq \int_{-\infty}^{\infty} f(u)^2 du \int_{-\infty}^{\infty} f(u+x)^2 du$$

But for any x ,

$$\int_{-\infty}^{\infty} f(u+x)^2 du = \int_{-\infty}^{\infty} f(u)^2 du,$$

and now we observe that

$$\int_{-\infty}^{\infty} f(u)^2 du = \int_{-\infty}^{\infty} f(u)f(u) du = (f \star f)(0).$$

Putting these inequalities together we have

$$(f \star f)^2(x) \leq (f \star f)^2(0),$$

and since $(f \star f)(0) \geq 0$ we conclude that

$$(f \star f)(x) \leq (f \star f)(0).$$

- (c) To show that $\mathcal{F}(f \star f) = |\mathcal{F}f|^2$ we appeal to the relationship between correlation and convolution, and apply the convolution theorem:

$$\begin{aligned} \mathcal{F}(f \star f) &= \mathcal{F}(f^- \star f) \\ &= (\mathcal{F}f^-)(\mathcal{F}f) \\ &= (\mathcal{F}f)^- (\mathcal{F}f) \quad (\text{using duality}). \end{aligned}$$

But because $f(t)$ is a real signal we have

$$(\mathcal{F}f)^- = \overline{\mathcal{F}f},$$

and hence, from the final line, above,

$$\mathcal{F}(f \star f) = (\mathcal{F}f)^- (\mathcal{F}f) = (\overline{\mathcal{F}f})(\mathcal{F}f) = |\mathcal{F}f|^2.$$

- (d) The transmitted signal is $f(t)$ and the received signal is $f_r(t) = \alpha f(t - 2T) + n(t) = \alpha(\tau_{2T}f)(t) + n(t)$. Take the cross-correlation of these two signals:

$$\begin{aligned} (f \star f_r) &= f \star (\alpha(\tau_{2T}f) + n) \\ &= f \star (\alpha\tau_{2T}f) + f \star n \\ &= \alpha(f \star (\tau_{2T}f)) + f \star n \\ &= \alpha(\tau_{2T}(f \star f)) + C, \end{aligned}$$

where we have used the shift property of correlation from earlier, $f \star (\tau_b g) = \tau_b(f \star g)$, and the assumption that $f \star n = C$. Now suppose, as we're told, that the maximum of $f \star f_r$ occurs at time t_0 . Since C is a constant, the function on the right hand side must also assume its maximum at t_0 :

$$\alpha(f \star f)(t_0 - 2T) + C = \max.$$

But the autocorrelation $f \star f$ assumes its maximum at 0, hence

$$t_0 - 2T = 0 \quad \text{or} \quad T = \frac{t_0}{2}.$$

3. (20 points) *Windowing functions* In signal analysis, it is not realistic to consider a signal $f(t)$ from $-\infty < t < \infty$. Instead, one considers a modified section of the signal, say from $t = -1/2$ to $t = 1/2$. The process is referred to as *windowing* and is achieved by multiplying the signal by a *windowing function* $w(t)$. By the Convolution Theorem (applied in the frequency domain), the Fourier transform of the windowed signal, $w(t)f(t)$ is the convolution of the Fourier transform of the original signal with the Fourier transform of the window function $\mathcal{F}w * \mathcal{F}f$. The ideal window is no window at all, i.e., it's just the constant function 1 from $-\infty$ to ∞ , which has no effect in either the time domain (multiplying by 1) or in the frequency domain (convolving with δ).

Below are three possible windowing functions. Find the Fourier transform of each and plot the *log magnitude* of each from $-100 < s < 100$. Which window is closest to the ideal window?

Rectangle Window

$$w(t) = \Pi(t)$$

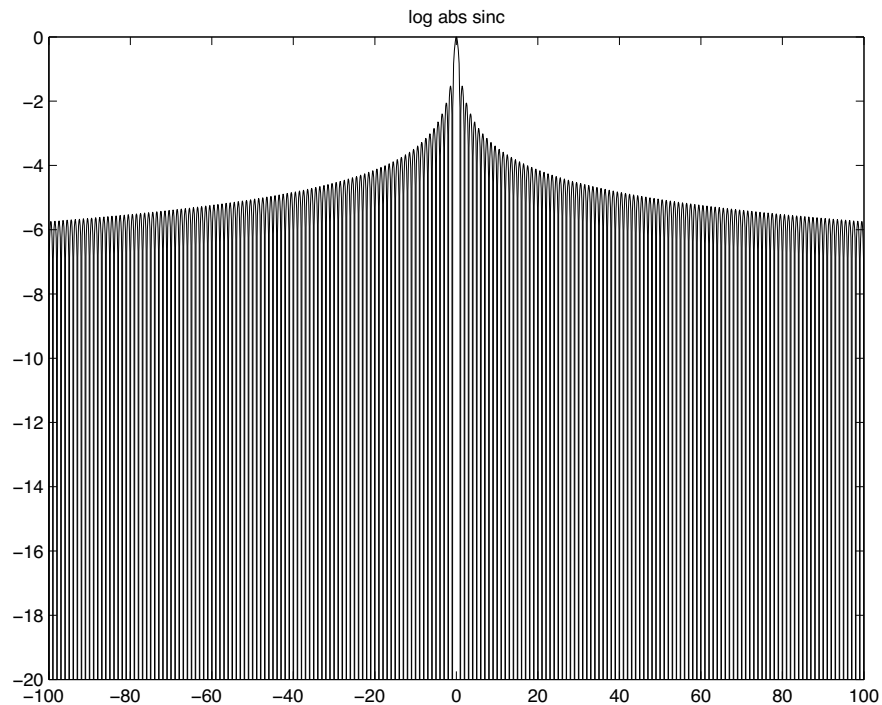
Triangular Window

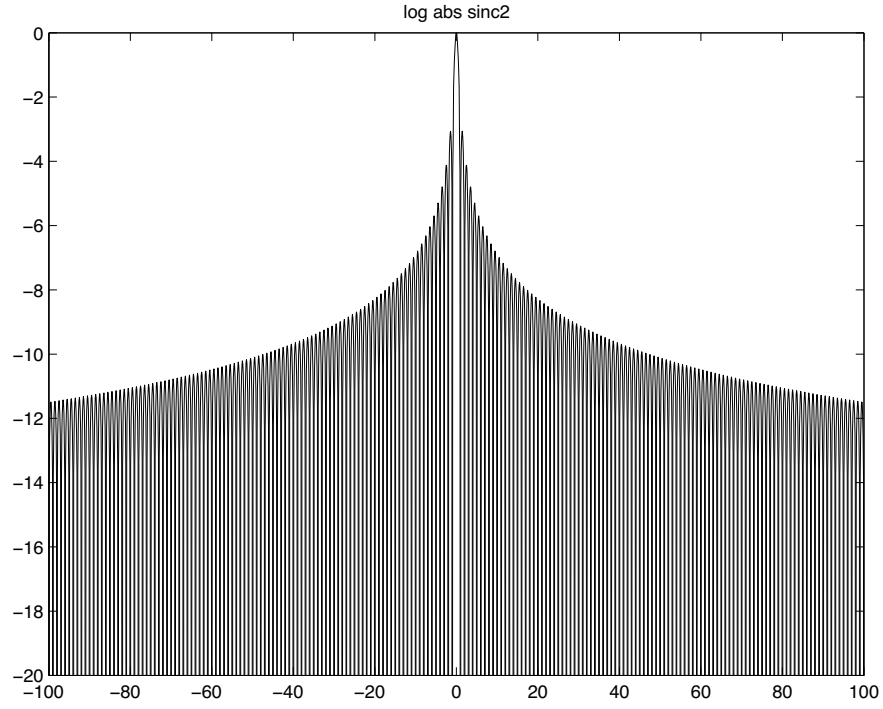
$$w(t) = 2\Lambda(2t)$$

Hamming Window

$$w(t) = \begin{cases} \cos^2(\pi t), & |t| < \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Solutions





We express w_H as

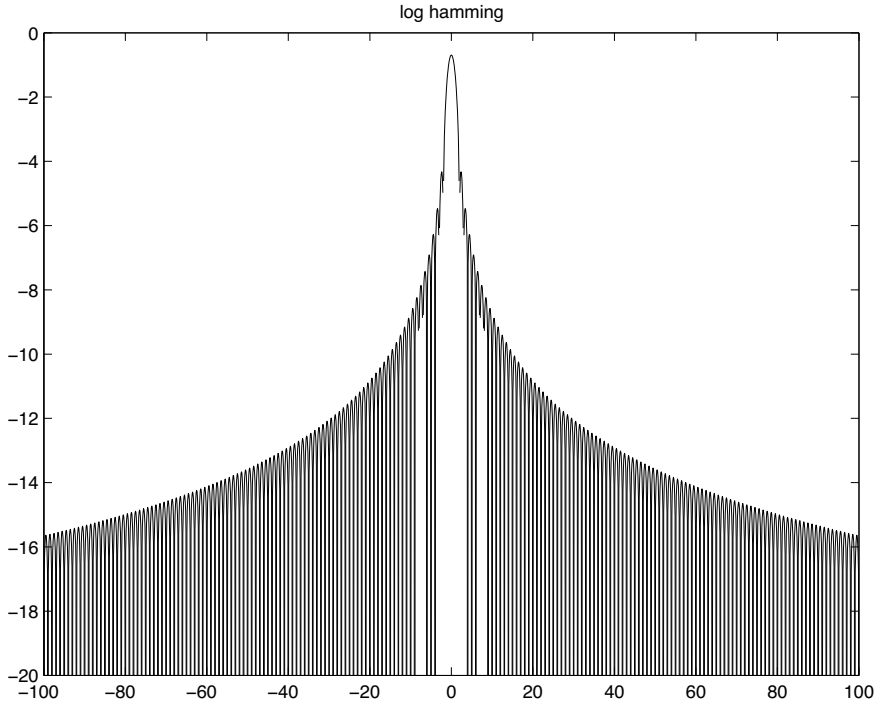
$$w_H(t) = \Pi(t) \cos^2(\pi t) = \Pi(t) \left(\frac{1}{2} + \frac{1}{2} \cos(2\pi t) \right)$$

so the transform is

$$\mathcal{F}w_H(s) = (\text{sinc} * (\frac{1}{2}\delta(s) + \frac{1}{4}\delta(s-1) + \frac{1}{4}\delta(s+1)))(s)$$

or

$$\mathcal{F}w_H(s) = \frac{1}{2} \text{sinc}(s) + \frac{1}{4} \text{sinc}(s+1) + \frac{1}{4} \text{sinc}(s-1)$$



4. (15 points) *Brownian motion:* Consider particles suspended in a liquid. Let $f(x, t)$ be the probability density function of finding a particle at a point x at time t . Using probabilistic arguments, Einstein showed that for some systems the evolution of $f(x, t)$ is governed by a diffusion equation,

$$f_t = D f_{xx},$$

where D is the *diffusion constant* and the subscripts indicate partial derivatives.

- Assuming that $f(x, 0) = \delta(x)$, find $f(x, t)$. (Your answer should depend on t).
- Find the mean and the variance of $f(x, t)$ (with respect to x).
- From his calculations, Einstein found that the diffusion constant is given by

$$D = \frac{kT}{6\pi\eta R},$$

where k is Boltzmann's constant, T is the temperature, η is the viscosity of the fluid, and R is the size of the particles. Given your results from part (b), what can you say about the diffusion of the particles (their evolution over time) as a function of their size and of the temperature?

Solution:

- Let us denote by $F(s, t)$ the Fourier transform of $f(x, t)$ (with respect to x), i.e. $F(s, t) = \mathcal{F}f(s, t)$.

From the derivative property of the Fourier transform, we know that:

$$\mathcal{F}\{f_{xx}\} = (2\pi is)^2 \mathcal{F}\{f\} = -4\pi^2 s^2 F$$

Since we are given that $f_t = Df_{xx}$ then $\frac{\partial \mathcal{F}f}{\partial t} = D\mathcal{F}\{f_{xx}\}$.

Putting these relations together we obtain the following differential equation:

$$\frac{\partial F(s, t)}{\partial t} = -4\pi^2 D s^2 F(s, t)$$

This is an easy ODE whose solution is:

$$F(s, t) = F(s, 0)e^{-4\pi^2 D s^2 t}$$

Since we know that $f(x, 0) = \delta(x)$, then $F(s, 0) = 1$. Hence

$$F(s, t) = e^{-4\pi^2 D s^2 t}$$

Now we just need to take the inverse Fourier transform to find $f(x, t)$:

$$\begin{aligned} f(x, t) &= \mathcal{F}^{-1}\{F(s, t)\} \\ &= \mathcal{F}^{-1}\{e^{-4\pi^2 D s^2 t}\} \\ &= \mathcal{F}^{-1}\{e^{-2\pi^2 (2Dt) s^2}\} \\ &= \frac{1}{\sqrt{2D|t|}\sqrt{2\pi}} e^{-\frac{x^2}{2 \cdot (2D|t|)}} \\ &= \frac{1}{2\sqrt{\pi D t}} e^{-x^2/4Dt} \quad \text{for } t \geq 0 \end{aligned}$$

- (b) It is easy to see that the solution to the diffusion equation (as found in part (a)) is actually a Gaussian with zero mean and variance $2Dt$.

So we can immediately conclude that:

$$\begin{aligned} E[X; f(x)] &= 0 \\ E[X^2; f(x)] &= 2Dt \end{aligned}$$

- (c) The variance of the diffusion process was found to be equal to $2Dt$.

Since $D = \frac{k_b T}{6\pi\eta R}$, then if T increases then D increases linearly. Then the variance of the diffusion process will increase more quickly with t , which means that the particles will be diffused more quickly if the temperature is increased.

Similarly, if R increases then D decreases accordingly. Then the variance of the diffusion process will increase more slowly with t , which means that the particles will be diffused more slowly if their size increases.

5. (20 points) *Voice scrambling/descrambling* A simple scrambling circuit for voice communications works as follows. Consider the frequency band from 0 to 4 kHz. It is a known fact that the overwhelming majority of the power spectrum of human voice is concentrated in this frequency band. One way of scrambling this frequency band is to subdivide it into 4 equal sub-bands and interchange the sub-bands according to some pre-determined key. For example, let sub-band A correspond to frequencies between 0 and 1 kHz. Then, sub-band B corresponds to frequencies between 1 and 2 kHz, sub-band C corresponds to frequencies between 2 and 3 kHz, and sub-band D corresponds to frequencies between 3 and 4 kHz. The original order of the sub-bands is ABCD. A simple scrambling technique is to interchange this

order, i.e. reorder the sub-bands to BCDA or DCBA or CABD or any other pre-determined order. Call the resulting signal the scrambled signal. This scrambled signal is not comprehensible unless you know the key and can rearrange the sub-bands back into the original order.

The goal of this problem is for you to design a MATLAB program that will descramble a given voice signal. You may obtain the scrambled signal ('scramble.wav') from the course web site,

<http://see.stanford.edu/materials/Issoftae261/PS-4-scramble.wav>

in the 'Problem Sets' section of the Handouts. It has been scrambled in the above-described fashion by rearranging the bands ABCD into CBDA. Descramble this signal and transcribe the first and last 5 words. **Turn in your code as well as your transcription.** (No, this is not a test of your transcribing skills, we just want to make sure that your code works.)

Solution:

```
function [recov,fs,nbits] = descramble(filename);

[orig,fs,nbits] = wavread(filename); origfft = fft(orig);
% C B D A
% | | | |
% A B C D
if rem(length(origfft),2),
    highf = (length(origfft)+1)/2;
    C = origfft(2:floor(highf/4));
    B = origfft(floor(highf/4)+1:floor(highf/2));
    D = origfft(floor(highf/2)+1:floor(highf*0.75));
    A = origfft(floor(highf*0.75)+1:highf);
    recov = [origfft(1);A;B;C;D;flipud([conj(A);conj(B);conj(C);conj(D)])]);
else
    highf = length(origfft)/2 + 1;
    C = origfft(2:floor(highf/4));
    B = origfft(floor(highf/4)+1:floor(highf/2));
    D = origfft(floor(highf/2)+1:floor(highf*0.75));
    A = origfft(floor(highf*0.75)+1:highf-1);
    recov = [origfft(1);A;B;C;D;origfft(highf); ...
            flipud([conj(A);conj(B);conj(C);conj(D)])]);
end recov = real(ifft(recov));
```

Descrambled speech: "We shall fight in France... we shall fight in the hills, we shall never surrender".

Winston Churchill,

Address to the House of Commons

June 4, 1940