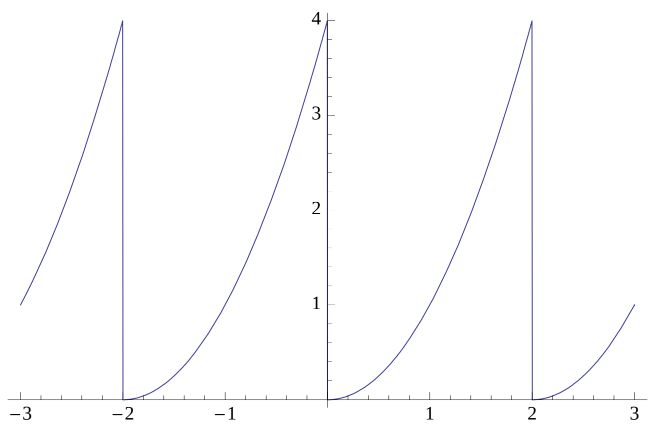
# EE 261 The Fourier Transform and its Applications
# Fall 2007
# Solutions to Problem Set Two

1. (25 points) *A periodic, quadratic function and some surprising applications*
   Let $f(t)$ be a function of period $T = 2$ with

$$f(t) = t^2 \quad if \quad 0 \le t < 2.$$

Here's the picture.




(a) Find the Fourier series coefficients, $c_n$, of $f(t)$.

(b) Using your result from part (a), obtain the following: *Hint: You might want to read section 1.14.3 on pages 54 - 57 of the course reader before trying this part.*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

*Solution:*

(a) The Fourier coefficients may be calculated using the following integral:

$$
\begin{aligned}
c_n &= \frac{1}{2}\int_0^2 f(t)\, e^{-2\pi i n t/2}\, dt \\
&= \frac{1}{2}\int_0^2 t^2 e^{-i\pi n t}\, dt \\
&= \frac{1}{2}\left\{\left[\frac{t^2 e^{-i\pi n t}}{-i\pi n}\right]_0^2 + 2\int_0^2 \frac{t\ e^{-i\pi n t}}{i\pi n}\, dt\right\} \qquad \text{(integration by parts)} \\
&= \frac{1}{2}\left\{\frac{-4}{i\pi n} + \frac{2}{i\pi n}\int_0^2 t\ e^{-i\pi n t}\, dt\right\} \\
&= \frac{2i}{\pi n} + \frac{1}{i\pi n}\int_0^2 t\ e^{-i\pi n t}\, dt \\
&= \frac{2i}{\pi n} + \frac{1}{i\pi n}\left(\frac{-2}{\pi i n}\right) \qquad \text{(integration by parts again)} \\
&= \frac{2\,(1 + i\pi n)}{\pi^2 n^2}, \qquad n \neq 0
\end{aligned}
$$

For the case of $n = 0$, we just have to find the average value of the function over a single period. Hence,

$$
\begin{aligned}
c_0 &= \frac{1}{2}\int_0^2 f(t)\, dt \\
&= \frac{1}{2}\int_0^2 t^2\, dt \\
&= \frac{4}{3}
\end{aligned}
$$

(b) We first write the Fourier series expansion of the function of interest

$$
\sum_{n=-\infty}^{\infty} c_n e^{i\pi n t} = t^2 \qquad \text{for } 0 \leq t < 2
$$

Let us evaluate both sides of the equation at $t = 0$. The left hand side of the equation would simply be $\sum_{n=-\infty}^{\infty} c_n$. We might be tempted to set the right hand side to zero. This is not true in this case since the periodic function has a discontinuity at $t = 0$. When such a case occurs, the value of $\sum_{n=-\infty}^{\infty} c_n$ would converge to the average of the left and right hand limits, which in this case is 2. Therefore, we have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} c_n &= 2 \\
\sum_{n=-\infty}^{\infty} \frac{2\left(1+i\pi n\right)}{\pi^2 n^2} &= 2 \\
\sum_{n=1}^{\infty} \frac{2\left(1+i\pi n\right)}{\pi^2 n^2} + \frac{4}{3} + \sum_{n=-\infty}^{-1} \frac{2\left(1+i\pi n\right)}{\pi^2 n^2} &= 2 \\
\sum_{n=1}^{\infty} \frac{2\left(1+i\pi n\right)}{\pi^2 n^2} + \sum_{n=1}^{\infty} \frac{2\left(1-i\pi n\right)}{\pi^2 n^2} &= \frac{2}{3} \\
\sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} &= \frac{2}{3} \\
\sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6}
\end{aligned}$$

For the second relation, we evaluate both sides of

$$\sum_{n=-\infty}^{\infty} c_n e^{i\pi n t} = t^2 \qquad \text{for } 0 \leq t < 2$$

at $t = 1$. This is the quickest and easiest way to introduce a $(-1)^n$ into the summation. The function $f(t)$ is continuous at $t = 1$, so the right hand side will just be $1^2 = 1$. Writing this out we have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} c_n (-1)^n &= 1 \\
\sum_{n=-\infty}^{\infty} \frac{2\left(1+i\pi n\right)}{\pi^2 n^2}(-1)^n &= 1 \\
\sum_{n=1}^{\infty} \frac{2\left(1+i\pi n\right)}{\pi^2 n^2}(-1)^n + \frac{4}{3} + \sum_{n=-\infty}^{-1} \frac{2\left(1+i\pi n\right)}{\pi^2 n^2}(-1)^n &= 1 \\
\sum_{n=1}^{\infty} \frac{2\left(1+i\pi n\right)}{\pi^2 n^2}(-1)^n + \sum_{n=1}^{\infty} \frac{2\left(1-i\pi n\right)}{\pi^2 n^2}(-1)^n &= -\frac{1}{3} \\
\sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}(-1)^n &= -\frac{1}{3}
\end{aligned}$$

Multiplying both sides by -1 and rearranging the equation yields:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

The third relation is obvious when we write out the first two sums:

$$
\begin{aligned}
1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots &= \frac{\pi^2}{6} \\
1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots &= \frac{\pi^2}{12}
\end{aligned}
$$

Adding these sums together, we have

$$
\begin{aligned}
2\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) &= \frac{\pi^2}{4} \\
2\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} &= \frac{\pi^2}{4} \\
\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} &= \frac{\pi^2}{8}
\end{aligned}
$$

2. (10 points) *Whither Rayleigh?* What happens to Rayleigh's identity if $f(t)$ is periodic of period $T \neq 1$?

   *Solution:*

   For a function $f$ with period $T$, we have the expansion

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T}$$

   where

$$c_n = \frac{1}{T}\int_0^T e^{-2\pi i n t/T} f(t) dt.$$

   Define $g(t) = f(Tt)$. Then $g$ has period

   We shall derive Rayleigh's identity for $f$ from Rayleigh's identity for $g$. First observe that

$$
\begin{aligned}
\hat{g}(n) &= \int_0^1 e^{-2\pi i n t} g(t) dt \\
&= \int_0^1 e^{-2\pi i n t} f(Tt) dt \\
&= \int_0^T e^{-2\pi i n u/T} f(u) \frac{du}{T} \\
&= c_n. \qquad (1)
\end{aligned}
$$

   Next,

$$
\begin{aligned}
\int_0^1 |g(t)|^2 dt &= \int_0^1 |f(Tt)|^2 dt \\
&= \int_0^T |f(u)|^2 \frac{du}{T} \qquad (2)
\end{aligned}
$$

From Rayleigh's identity for a function with period 1,

$$\int_0^1 |g(t)|^2 dt = \sum_{n=-\infty}^{\infty} |\hat{g}(n)|^2 .$$

Therefore, from equations (**??**) and (**??**), we have

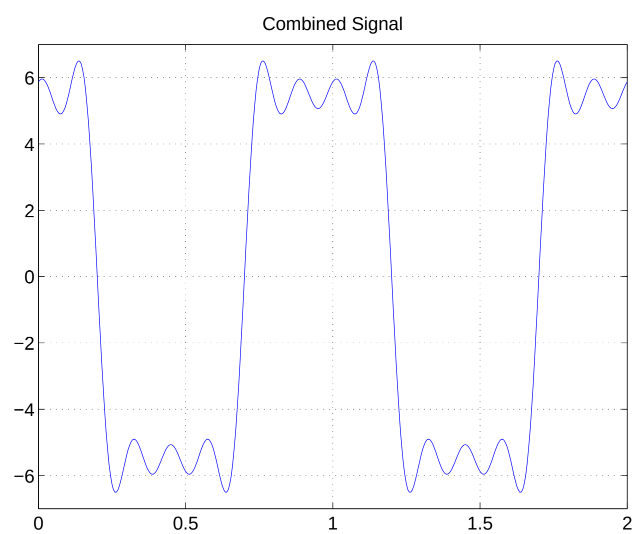
$$\frac{1}{T}\int_0^T |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2 ,$$

Rayleigh's identity for $f$.

3. (25 points) *Sinesum2, Square Wave, High Frequency Noise and AM Modulation.*

This problem is based on the Matlab application in the 'Sinesum2 Matlab Program' section of 'Handouts' on the course website. Go there to read the directions and to get the files. It's a tool to plot sums of sinusoids of the form

$$\sum_{n=1}^{N} A_n \sin(2\pi n t + \phi_n) .$$




(a) Using the sinesum2 application, generate (approximately) the signal plotted above. You should be able to do this with seven harmonics. Along with the overall shape, the things to get right are the approximate shift and orientation of the signal. To get started, observe that the signal looks like a square wave signal. Recall that the square wave signal is studied in Section 1.7 of the course reader. However, you'll see that additional flipping and shifting need to be done. This can be accomplished by changing the phases $\phi_n$ (do that in two steps, say first a flip and then a shift). Explain what you're doing at each stage.

(b) Additive high frequency noise is very common when signals go through various communications systems. In this case, we will assume that the previous signal goes through some communication system, that adds high frequency noise which can be approximated as: $0.5\sin(2\pi 50t)$. How does this change the original signal in the time domain? What happens if the amplitude of the noise increases from 0.5 to 2. Plot your results and explain what you observe.

(c) Amplitude Modulation (AM) is a technique used in communication systems for transmitting information. Typically, AM works by varying the amplitude of a carrier signal (simple sine signal of the form $A\sin(2\pi f_c t + \phi_c)$) in relation to the information signal that needs to be transmitted. For example, if we denote the information signal we want to trasmit by $m(t)$, $m(t)A\sin(2\pi f_c t + \phi_c)$ is one type of "AM signal" (double-sideband suppressed-carrier (DSBSC) AM signal to be exact!) that we could choose to transmit. Naturally, a lot more can be said about AM, but this is not the scope of this question.

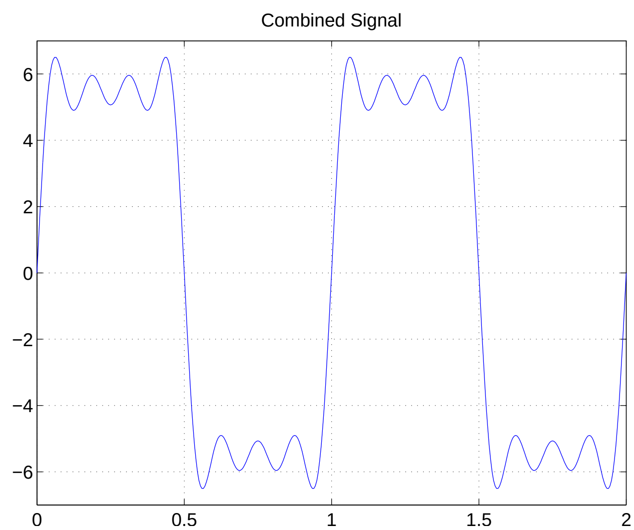
To examine an example of AM, we will assume that the signal from part (a) is multiplied by $\sin(2\pi 50t)$. Using sinesum2, plot the new signal, and explain how this can be done.

*Solution:*

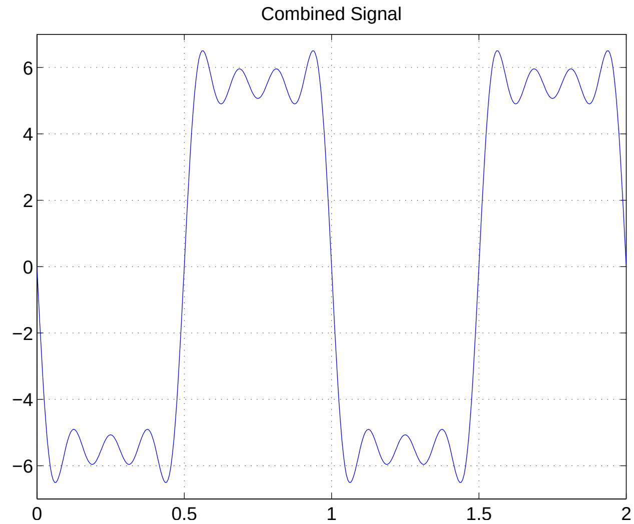
(a) (10points) We set up `sinesum` with 7 harmonics and we enter the following amplitudes, and no phase shifts:

$$A_1 = 7, \quad A_2 = 0, \quad A_3 = 2.3333, \quad A_4 = 0, \quad A_5 = 1.4, \quad A_6 = 0, \quad A_7 = 1\,.$$

The plot looks like



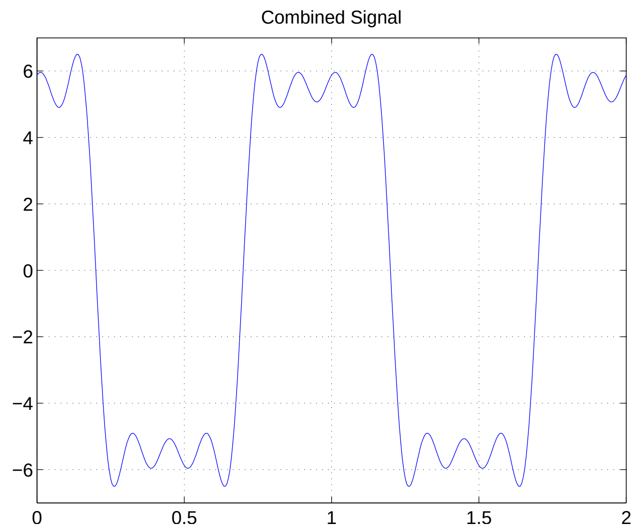

The shape looks pretty good, but it's upside down from what we want – it goes up where we want it to go down and vice versa. To flip the figure upside down we want to replace each $A_n$ by $-A_n$, and this can be accomplished by a phase shift of $\pi$ in *each* harmonic.



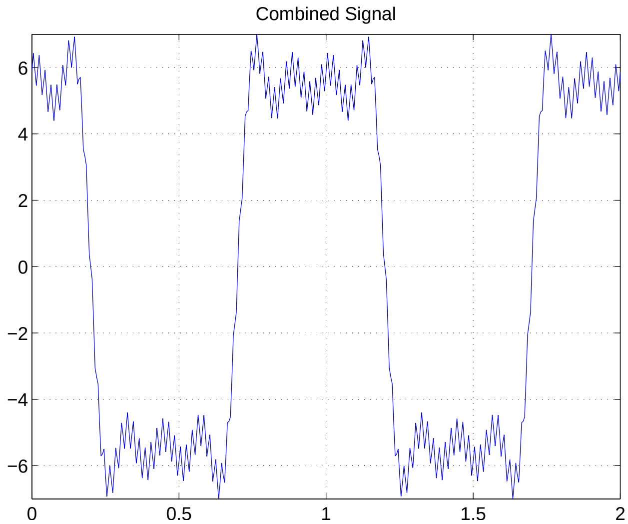

To obtain the signal that we want,we need to shift the whole signal to the left by about 0.3. This means that in each term we would replace $t$ by $t + 0.3$. Since the terms are of the form $\sin(2\pi nt + \phi)$, that means we want an *additional* phase shift of $n \times 0.3$ for the $n$'th term, $n = 1, \ldots, 5$. So the final phase shifts should be,

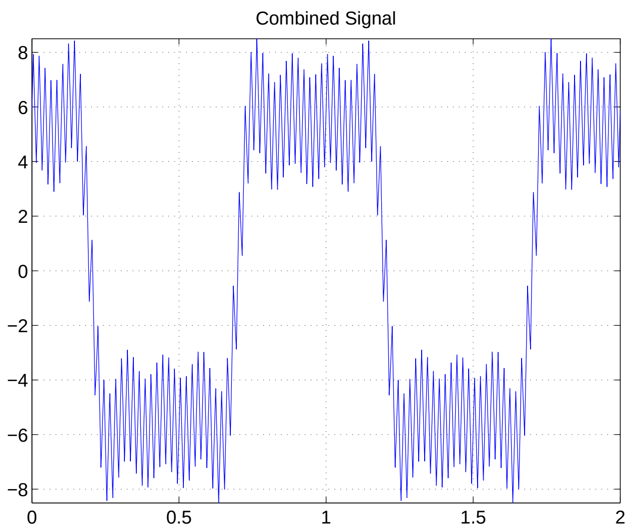
$$\phi_1 = 0.6\pi, \quad \phi_2 = 0 \quad \phi_3 = 1.8\pi, \quad \phi_4 = 0, \quad \phi_5 = \pi, \quad \phi_6 = 0, \quad \phi_7 = 0.2\pi\ .$$

Perfect match!




(b) (5 points) Clearly, the signal distortion increases as the amplitude of the noise increases.



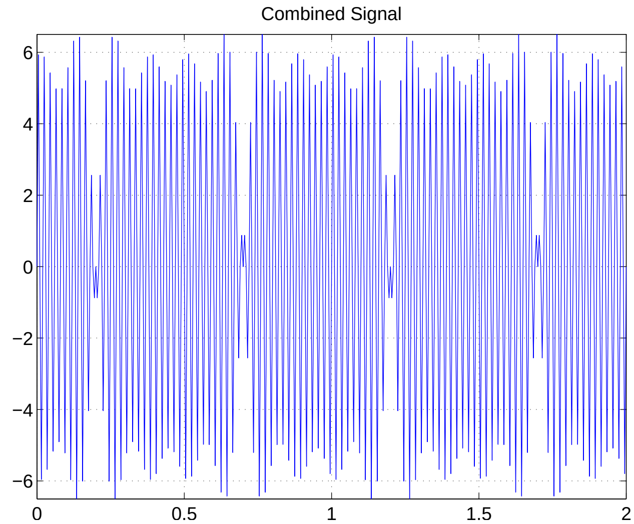

(c) (10 points) Using the identity:

$$\begin{aligned}\sin(2\pi 50t)\sin(2\pi ft+\phi) &= 0.5\cos(2\pi(50-f)t-\phi) - 0.5\cos(2\pi(50+f)t+\phi)\\ &= 0.5\cos(2\pi(50-f)t-\phi) + 0.5\cos(2\pi(50+f)t+\phi+\pi)\\ &= 0.5\sin(2\pi(50-f)t-\phi+\pi/2) + 0.5\sin(2\pi(50+f)t+\phi+3\pi/2)\end{aligned}$$

we can immediately see that the new signal can be written as one withe the following amplitudes and phase shifts:

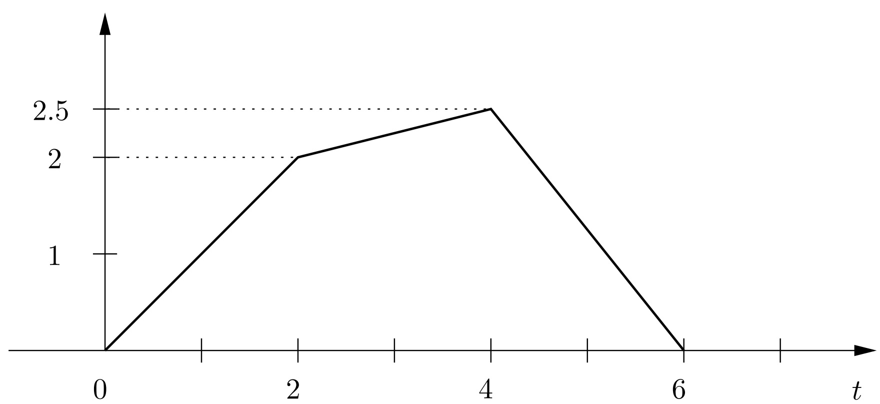
$$A_{49} = 3.5, \quad A_{51} = 3.5, \quad A_{47} = 1.1667, \quad A_{53} = 1.1667,$$
$$A_{45} = 0.7, \quad A_{55} = 0.7, \quad A_{43} = 0.5, \quad A_{57} = 0.5$$

$$\phi_{49} = -0.6\pi + \pi/2, \quad \phi_{51} = 0.6\pi + 3\pi/2, \quad \phi_{47} = -1.8\pi + \pi/2, \quad \phi_{53} = 1.8\pi + 3\pi/2,$$
$$\phi_{45} = -\pi + \pi/2, \quad \phi_{55} = \pi + 3\pi/2, \quad \phi_{43} = -0.2\pi + \pi/2, \quad \phi_{57} = 0.2\pi + 3\pi/2$$



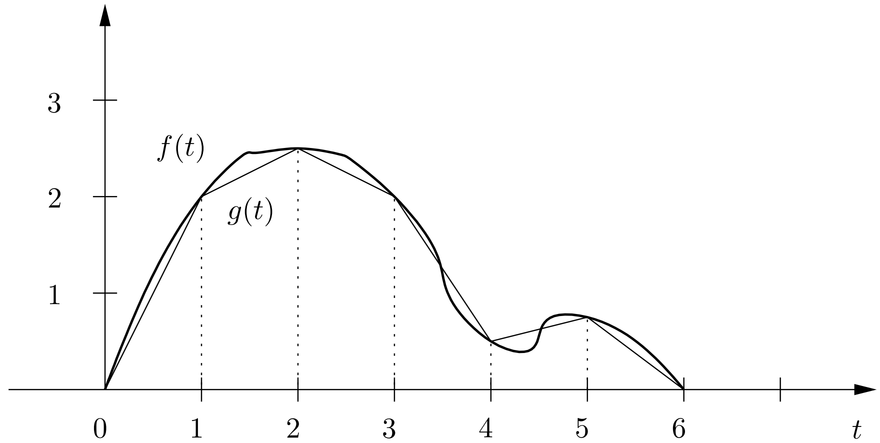

4. (20 points) *Piecewise linear approximations and Fourier transforms.*

(a) Find the Fourier transform of the following signal.




Hint: Think $\Lambda$'s.

(b) Consider a signal $f(t)$ defined on an interval from 0 to $D$ with $f(0) = 0$ and $f(D) = 0$. We get a uniform, piecewise linear approximation to $f(t)$ by dividing the interval into $n$ equal subintervals of length $T = D/n$, and then joining the values $0 = f(0), f(T), f(2T), \ldots, f(nT) = f(D) = 0$ by consecutive line segments. Let $g(t)$ be the linear approximation of a signal $f(t)$, obtained in this manner, as illustrated in the following figure where $T = 1$ and $D = 6$.




Find $\mathcal{F}g(s)$ for the general problem (*not* for the example given in the figure above) using any necessary information about the signal $f(t)$ or its Fourier transform $\mathcal{F}f(s)$. Think $\Lambda$'s, again.

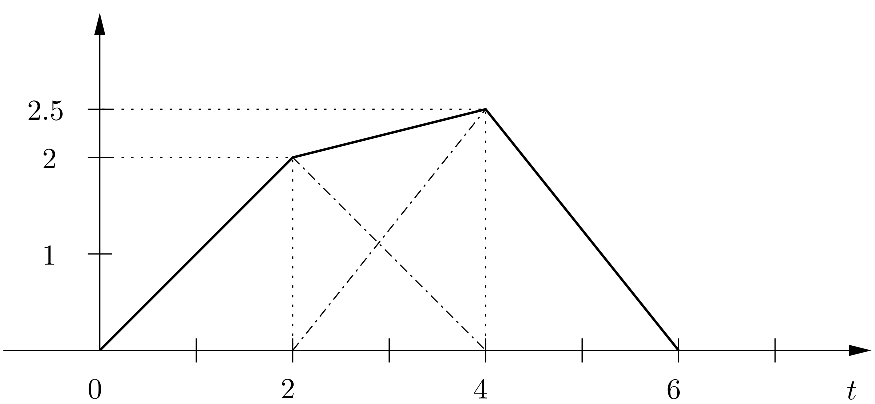
*Solution:*

(a) The function is given by the sum of two scaled and shifted triangle functions. Recall from the first problem set the triangle function with a parameter $a > 0$ is

$$\Lambda_a(t) = \Lambda(t/a) = \begin{cases} \frac{a-|t|}{a}, & |t| \le a \\ 0, & |t| > a \end{cases}$$

The function in part (a) is given by

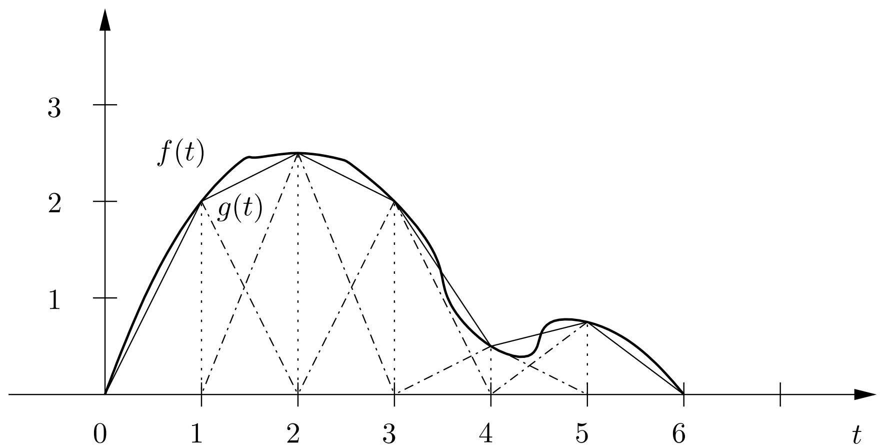
$$2\Lambda_2(t-2) + 2.5\Lambda_2(t-4).$$




Thus, using the shift and stretch theorems we find the following Fourier transform,

$$\begin{aligned} 2\Lambda_2(t-2) + 2.5\Lambda_2(t-4) &\rightleftharpoons 2e^{-2\pi is(2)}2\text{sinc}^2(2s) + 2.5e^{-2\pi is(4)}2\text{sinc}^2(2s) \\ &\rightleftharpoons \text{sinc}^2(2s)\left(4e^{-4\pi is} + 5e^{-8\pi is}\right). \end{aligned}$$

(b) This is an extension of the previous part (a). The piecewise linear approximation $g(t)$ is the sum of shifted and stretched triangles that are scaled by values of function $f(t)$ at their center points:




We compute $\mathcal{F}f(s)$ by

$$
\begin{aligned}
\mathcal{F}f(s) &= \mathcal{F}\left(\sum_{k=1}^{n-1} f(kT)\Lambda_T(t-kT)\right) \\
&= \sum_{k=1}^{n-1} f(kT)\mathcal{F}\Lambda_T(t-kT) \\
&= \sum_{k=1}^{n-1} f(kT)T\operatorname{sinc}^2(Ts)e^{-2\pi i s(kT)} \\
&= T\operatorname{sinc}^2(Ts)\sum_{k=1}^{n-1} f(kT)e^{-2\pi i s kT}.
\end{aligned}
$$

5. (10 points) *The modulation property of the Fourier transform.*

   (a) Let $f(t)$ be a signal, $s_0$ a number, and define

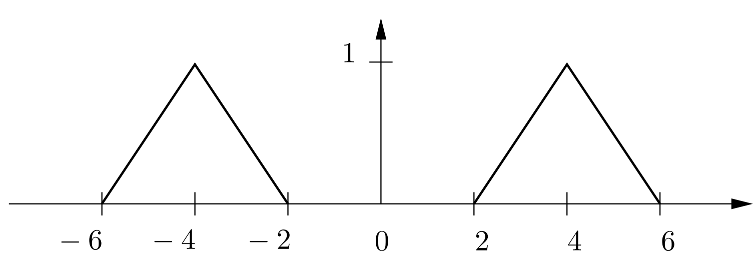
   $$g(t) = f(t)\cos(2\pi s_0 t)$$

   Show that

   $$\mathcal{F}g(s) = \frac{1}{2}\mathcal{F}f(s-s_0) + \frac{1}{2}\mathcal{F}f(s+s_0)$$

   (No delta functions, please, for those who know about them.)

   (b) Find the signal (in the time domain) whose Fourier transform is pictured, below.




*Solution:*

(a) We appeal directly to the definition of the Fourier transform:

$$\begin{aligned}\mathcal{F}g(s) &= \int_{-\infty}^{\infty} g(t)e^{-2\pi ist}dt \\ &= \int_{-\infty}^{\infty} f(t)\cos(2\pi s_0 t)e^{-2\pi ist}dt \\ &= \int_{-\infty}^{\infty} f(t)\frac{1}{2}(e^{2\pi is_0t} + e^{-2\pi is_0t})e^{-2\pi ist}dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} f(t)e^{-2\pi i(s-s_0)t}dt + \frac{1}{2}\int_{-\infty}^{\infty} f(t)e^{-2\pi i(s+s_0)t}dt \\ &= \frac{1}{2}\mathcal{F}f(s-s_0) + \frac{1}{2}\mathcal{F}f(s+s_0)\end{aligned}$$

(b) With an eye toward using part (a), the function illustrated in this part can be written as

$$\frac{1}{2}(2\Lambda_2(s-4)) + \frac{1}{2}(2\Lambda_2(s+4)) = \Lambda_2(s-4) + \Lambda_2(s+4)\,.$$

So it looks like we want to find a function $f(t)$ whose Fourier transform is $2\Lambda_2(s)$, for if we then multiply it by $\cos(2\pi \cdot 4 \cdot t)$ the modulation property gives us what we want for the Fourier transform:

$$\mathcal{F}(f(t)\cos 8\pi t) = \Lambda_2(s+4) + \Lambda_2(s-4)\,.$$

Now, since $\mathcal{F}\Lambda_a(s) = a\,\mathrm{sinc}^2(as)$, we obtain by duality

$$\mathcal{F}(a\,\mathrm{sinc}^2(as)) = \Lambda_a^- = \Lambda_a\,,$$

using also that $\Lambda_a$ is even. If we thus set

$$f(t) = 4\,\mathrm{sinc}^2(2t)$$

we have

$$4\,\mathrm{sinc}^2(2t) \rightleftharpoons 2\Lambda_2(s)$$

as desired. Finally, we then take

$$g(t) = 4\,\mathrm{sinc}^2(2t)\cos(8\pi t)\,.$$

6. (10 points) *Fourier transforms and Fourier coefficients* Suppose the function $f(t)$ is zero outside the interval $-1/2 \le t \le 1/2$. We form a function $g(t)$ which is a periodic version of $f(t)$ with period 1 by the formula

$$g(t) = \sum_{k=-\infty}^{\infty} f(t-k)\,.$$

The Fourier series representation of $g(t)$ is given by

$$g(t) = \sum_{k=-\infty}^{\infty} \hat{g}(n)e^{2\pi int}\,.$$

Find the relationship between the Fourier transform $\mathcal{F}f(s)$ and the Fourier series coefficients $\hat{g}(n)$.

*Solution* Recall that to calculate the Fourier series coefficient, we can take the integral over any period of the signal. We can choose the period $(-\frac{1}{2}, \frac{1}{2})$. Then, the Fourier coefficients of $g(t)$ are

$$\hat{g}(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t) \exp^{2\pi i n t} \, dt$$

but since $f(t)$ is zero outside of the region $(-\frac{1}{2}, \frac{1}{2})$ and $g(t)$ is simply the replicas of $f(t)$ with period 1, both functions are equal in the region $(-\frac{1}{2}, \frac{1}{2})$. Thus,

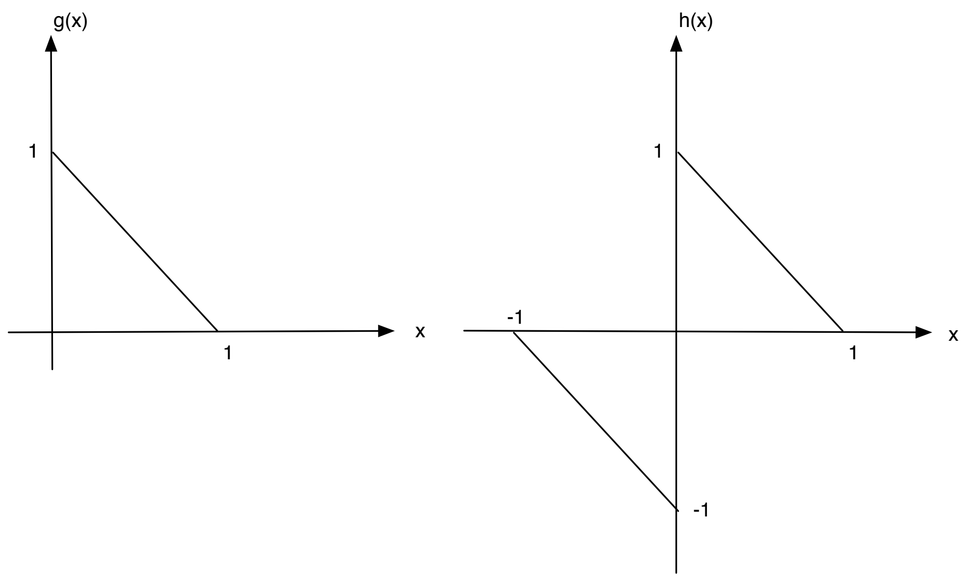
$$\begin{aligned} \hat{g}(n) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) \exp^{2\pi i n t} \, dt \\ &= \int_{-\infty}^{\infty} f(t) \exp^{2\pi i n t} \, dt \end{aligned}$$

which we immediately recognize as the Fourier Transform of $f(t)$. Therefore, we get

$$\hat{g}(n) = \hat{f}(s)|_{s=n}$$

The Fourier coefficients of the periodic function $g(t)$ are simply the samples of the Fourier Transform of $f(t)$, taken at all the integers.

7. (50 points) Consider the functions $g(x)$ and $h(x)$, shown below




Denote the Fourier transforms by $\mathcal{F}g(s)$ and $\mathcal{F}h(s)$, respectively.

(a) (5 points) Lykomidis says that the imaginary part of $\mathcal{F}g(s)$ is $(\sin(2\pi s) - 2\pi s) / \left(4\pi^2 s^2\right)$. Brad, however, expresses concerns about Lykomidis' work. He is not sure that this can be the imaginary part of $g$. "Why would it have a singularity at $s = 0$?" Brad says, as a general fact, if a function is integrable, as $g(t)$ is, then its Fourier transform is continuous. Lykomidis, asks Brad whether he is willing to buy him coffee if he (Lykomidis) can prove him (Brad) wrong. Brad feels very confident and quickly accepts. Is Lykomidis getting free coffee?

(b) (5 points) What are the two possible values of $\angle \mathcal{F}h(s)$, i.e., the phase of $\mathcal{F}h(s)$? Express your answer in radians.

(c) (5 points) Evaluate $\int_{-\infty}^{\infty} \mathcal{F}g(s)\cos(\pi s)\, ds$.

(d) (5 points) Evaluate $\int_{-\infty}^{\infty} \mathcal{F}h(s)e^{i4\pi s}\, ds$.

(e) (10 points) Without performing any integration, what is the real part of $\mathcal{F}g(s)$? Explain your reasoning.

(f) (10 points) Without performing any integration, what is $\mathcal{F}h(s)$? Explain your reasoning.

(g) (10 points) Suppose $h(x)$ is periodized to have period $T = 2$. Without performing any integration, what are the Fourier coefficents, $c_k$, of this periodic signal?

*Solution:*

(a) Lykomidis is getting free coffee because

$$\begin{aligned}\sin 2\pi s &= 2\pi s - \frac{(2\pi s)^3}{3!} + O(s^5)\\ &= 2\pi s - \frac{8\pi^3 s^3}{6} + O(s^5)\end{aligned}$$

so

$$\frac{1}{4\pi^2 s^2}\sin 2\pi s = \frac{1}{2\pi s} + O(s)\,.$$

Then

$$\frac{1}{4\pi^2 s^2}\sin 2\pi s - \frac{1}{2\pi s}\,;$$

the singularities cancel and what remains is continuous at $s = 0$.

(b) Since $h(x)$ is a real and odd function, the Fourier transform $\mathcal{F}h(s)$ would be purely imaginary and odd. The phase of a purely imaginary function is either $\pi/2$ or $-\pi/2$ radians.

(c) There are a few methods to do this. The most straightforward approach would be to write the cosine function in terms of complex exponentials.

$$
\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{F}g(s)\cos(\pi s)\,ds &= \int_{-\infty}^{\infty} \mathcal{F}g(s)\left(\frac{1}{2}e^{i\pi s}+\frac{1}{2}e^{-i\pi s}\right)\,ds \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \mathcal{F}g(s)e^{i\pi s}\,ds + \frac{1}{2}\int_{-\infty}^{\infty} \mathcal{F}g(s)e^{-i\pi s}\,ds \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \mathcal{F}g(s)e^{i2\pi s(\frac{1}{2})}\,ds + \frac{1}{2}\int_{-\infty}^{\infty} \mathcal{F}g(s)e^{i2\pi s(-\frac{1}{2})}\,ds \\
&= \frac{1}{2}\mathcal{F}^{-1}\mathcal{F}g\left(\frac{1}{2}\right) + \frac{1}{2}\mathcal{F}^{-1}\mathcal{F}g\left(-\frac{1}{2}\right) \\
&= \frac{1}{2}g\left(\frac{1}{2}\right) + \frac{1}{2}g\left(-\frac{1}{2}\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}(0) \\
&= \frac{1}{4}.
\end{aligned}
$$

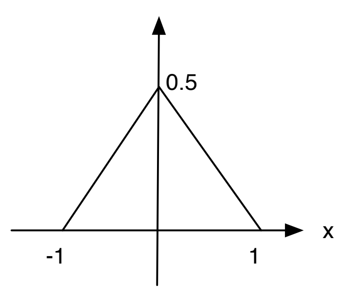
(d) This part is similar to part(b). We can write the integral as:

$$
\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{F}h(s)e^{i4\pi s}\,ds &= \int_{-\infty}^{\infty} \mathcal{F}h(s)e^{i2\pi s(2)}\,ds \\
&= \mathcal{F}^{-1}\mathcal{F}h(2) \\
&= h(2) \\
&= 0.
\end{aligned}
$$

(e) For this part, we appeal to the even and odd decomposition of the real function $g(x)$. Note that the values on the vertical axis are real numbers, so you can state with absolute certainty that $g(x)$ is real. This decomposition is:

$$g(x) = g_e(x) + g_o(x),$$

where $g_e(x) = \frac{1}{2}\left[g(x)+g(-x)\right]$ and $g_o(x) = \frac{1}{2}\left[g(x)-g(-x)\right]$.
Since $g_e(x)$ gives rise to a transform that is real and even while $g_o(x)$ results in one that is purely imaginary and odd, it follows that the real part of $\mathcal{F}g(s)$ will be the Fourier transform of $g_e(x)$. If we assume the value of $g(x)$ at $x=0$ to be $\frac{1}{2}$, $g_e(x)$ is:




Notice that the shape is nothing more than $\frac{1}{2}\Lambda(x)$. The Fourier transform of $\frac{1}{2}\Lambda(x)$, and hence the real part of $\mathcal{F}g(s)$, will therefore be $\frac{1}{2}\,\text{sinc}^2(s)$.

Some students may assume that the value of $g(x)$ at $x = 0$ is 1. In this case, $g_e(x) = \Lambda(x)$ and hence $\mathcal{F}g(s)$ would be $\text{sinc}^2(s)$.

Both solutions are acceptable.

(f) If you got the previous part, this part should be obvious. Notice that $h(x) = 2g_o(x)$ no matter what value you assume $g(x)$ to be at $x = 0$. Since the Fourier transform of $g_o(x)$ is "i" multiplied by the imaginary part of $\mathcal{F}g(s)$, it follows that $\mathcal{F}h(s) = i2.\left[\frac{\sin(2\pi s)-2\pi s}{4\pi^2 s^2}\right] = i\left[\frac{\sin(2\pi s)-2\pi s}{2\pi^2 s^2}\right]$.

(g) Consider $h_p(x)$ which is obtained by periodizing a function $h(x)$, i.e.,

$$h_p(x) = \sum_{n=-\infty}^{\infty} h(x - nT) = \sum_{k=-\infty}^{\infty} c_k e^{i\frac{2\pi kt}{T}}.$$

It was mentioned in the lectures, that $c_k = \frac{1}{T}\mathcal{F}h\left(\frac{k}{T}\right)$. With $T = 2$ and $\mathcal{F}h(s)$ from the previous part, the Fourier coefficients would be:

$$c_k = \frac{1}{2}\mathcal{F}h\left(\frac{k}{2}\right) = i2\left(\frac{\sin(\pi k) - \pi k}{2\pi^2 k^2}\right) = -\frac{i}{\pi k}$$

since $\sin(\pi k) = 0$ for all $k = 0, \pm 1, \pm 2, \ldots.$ As a check, $h(x)$ is odd, so the Fourier coefficients of its periodized version would be purely imaginary and odd, which is the case.