EE 261 The Fourier Transform and its Applications
Fall 2007
Solutions to Midterm Exam

- There are 5 questions for a total of 110 points.
- Please write your answers in the exam booklet provided, and make sure that your answers stand out.
- Don't forget to write your name on your exam book!

1. (15 points) Let $f(x)$ be a real, periodic function of period 1. The autocorrelation of f with itself is the function

$$(f \star f)(x) = \int_0^1 f(y)f(y+x)\,dy\,.$$

 (a) Show that $f \star f$ is also periodic of period 1.

 (b) If

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx}$$

 show that the Fourier series of $(f \star f)(x)$ is

$$(f \star f)(x) = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 e^{2\pi inx}\,.$$

From the definition of autocorrelation

$$\begin{aligned}(f \star f)(x+1) &= \int_0^1 f(y)f(y+x+1)\,dy \\ &= \int_0^1 f(y)f(y+x)\,dy\end{aligned}$$

since $f(y+x+1) = f(y+x)$ by periodicity of f. This shows rhat $f \star f$ is periodic of period 1.

For part (b) we plug the Fourier series of f into the definition of autocorrelation:

$$\begin{aligned}(f \star f)(x) &= \int_0^1 f(y)f(y+x)\,dy \\ &= \int_0^1 \left(\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi iny}\right)\left(\sum_{m=-\infty}^{\infty} \hat{f}(m)e^{2\pi im(y+x)}\right)dy \\ &= \int_0^1 \left(\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi iny}\right)\left(\sum_{m=-\infty}^{\infty} \hat{f}(m)e^{2\pi imy}e^{2\pi imx}\right)dy \\ &= \int_0^1 \sum_{n,m=-\infty}^{\infty} \hat{f}(n)\hat{f}(m)e^{2\pi iny}e^{2\pi imy}e^{2\pi imx}\,dy\end{aligned}$$

Now swap the summation and integration

$$\begin{aligned}\int_0^1 \sum_{n,m=-\infty}^{\infty} \hat{f}(n)\hat{f}(m)e^{2\pi iny}e^{2\pi imy}e^{2\pi imx}\,dy &= \sum_{n,m=-\infty}^{\infty} \hat{f}(n)\hat{f}(m)e^{2\pi imx}\int_0^1 e^{2\pi iny}e^{2\pi imy}\,dy \\ &= \sum_{n,m=-\infty}^{\infty} \hat{f}(n)\hat{f}(m)e^{2\pi imx}\int_0^1 e^{2\pi i(n+m)y}\,dy\end{aligned}$$

We've seen that integral of exponentials. It will only be nonzero if $n+m=0$, i.e., if $m=-n$, in which case it integrates to 1. Thus what remains is

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{f}(-n)e^{2\pi inx}\,.$$

But now we use the symmetry property of Fourier coefficients,

$$\hat{f}(-n) = \overline{\hat{f}(n)}\,.$$

With this the sum becomes

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 e^{2\pi i n x}\,.$$

as we were asked to show.

2. (10 points each)

   (a) If $f(t) * g(t) = h(t)$ what is $f(t-1) * g(t+1)$ in terms of $h(t)$?

   Solution Take the Fourier transform. Convolution becomes multiplication and the result is:That gives

$$e^{-2\pi is}\mathcal{F}f(s)\, e^{2\pi is}\mathcal{F}g(s) = \mathcal{F}f(s)\mathcal{F}g(s) = \mathcal{F}(f * g)(s) = \mathcal{F}h(s)$$

   Thus we get back what we started with:

$$f(t-1) * g(t+1) = h(t).$$

   The next three parts are related.

   (b) Show that the following relation holds for any two functions u and v:

$$\int_{-\infty}^{\infty} u(t)v(-t)dt \quad = \quad \int_{-\infty}^{\infty} \mathcal{F}u(s)\mathcal{F}v(s)ds$$

   Solution Let $w(t) = (u * v)(t)$ then

$$w(t) = \int_{-\infty}^{\infty} u(\tau)v(t-\tau)d\tau$$

   We also know that $w = \mathcal{F}^{-1}(\mathcal{F}u \cdot \mathcal{F}v)$ by the convolution theorem. This means that

$$w(t) = \int_{-\infty}^{\infty} \mathcal{F}u(s)\mathcal{F}v(s)e^{2\pi ist}ds$$

   Hence

$$\int_{-\infty}^{\infty} u(\tau)v(t-\tau)d\tau = \int_{-\infty}^{\infty} \mathcal{F}u(s)\mathcal{F}v(s)e^{2\pi ist}ds$$

   Evaluating this equality at $t = 0$, we obtain the desired relation

$$\int_{-\infty}^{\infty} u(\tau)v(-\tau)d\tau = \int_{-\infty}^{\infty} \mathcal{F}u(s)\mathcal{F}v(s)ds$$

   (Replace the variable τ with the variable t.)

   (c) Using the result derived in the previous part (even if you couldn't derive it), show that the following holds for any two functions f and g:

$$\int_{-\infty}^{\infty} f(t)\mathcal{F}g(t)dt \quad = \quad \int_{-\infty}^{\infty} \mathcal{F}f(s)g(s)ds$$

   Solution Notice that $v^- = \mathcal{F}\mathcal{F}v$. Let $g = \mathcal{F}v$ and $f = u$ then $\mathcal{F}g = v^-$ and $\mathcal{F}f = \mathcal{F}u$. Therefore the relation derived in the previous part becomes

$$\int_{-\infty}^{\infty} f(t)\mathcal{F}g(t)dt = \int_{-\infty}^{\infty} \mathcal{F}f(s)g(s)ds$$

(d) Calculate the following integral:

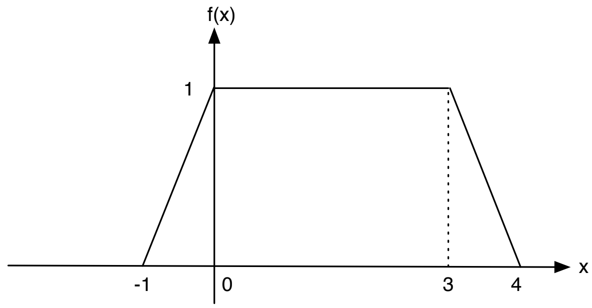
$$\int_{-\infty}^{\infty} \frac{e^{\pi i t}\text{sinc}(t)}{1+4\pi^2 t^2} dt$$

Solution Take $f(t) = e^{j\pi t}\text{sinc}(t)$ and $\mathcal{F}g(t) = \frac{1}{1+4\pi^2t^2}$. Then $\mathcal{F}f(s) = \Pi(s - \frac{1}{2})$ and $g(s) = \frac{1}{2}e^{-|s|}$.

Plugging into the expression derived in the previous part, we obtain the following result

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{j\pi t}\text{sinc}(t)}{1+4\pi^2 t^2} dt &= \frac{1}{2}\int_{-\infty}^{\infty} \Pi(s-\frac{1}{2})e^{-|s|}ds \\
&= \frac{1}{2}\int_0^1 e^{-s}ds \\
&= -\frac{1}{2}[e^{-s}]_0^1 \\
&= -\frac{1}{2}(e^{-1}-1) \\
&= \frac{1}{2}(1-\frac{1}{e})
\end{aligned}$$

3. (20 points) *Linearity and shifting properties of the Fourier transform*
Suppose we are given the following signal:

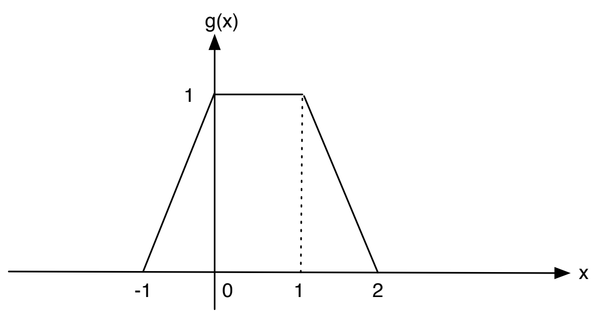


and we are told that its Fourier transform is

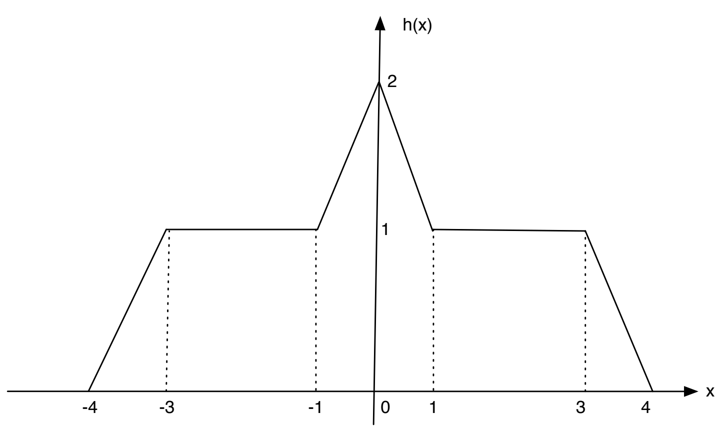
$$\mathcal{F}f(s) = 4\,\mathrm{sinc}^2(s)\cos(\pi s)\cos(2\pi s)e^{-i3\pi s}.$$

Using *ONLY* this information, find the Fourier transform of the following signals:

(a) $g(x)$



(b) $h(x)$



Solution:

(a) It can be seen that $f(x) = g(x) + g(x-2)$. Taking the Fourier transform on both sides:

$$
\begin{aligned}
\mathcal{F}f(s) &= \mathcal{F}g(s) + \mathcal{F}g(s)e^{-i4\pi s} \\
4\text{sinc}^2(s)\cos(\pi s)\cos(2\pi s)e^{-i3\pi s} &= \mathcal{F}g(s)\left(1 + e^{-i4\pi s}\right)
\end{aligned}
$$

Rearranging the equation, we have

$$
\begin{aligned}
\mathcal{F}g(s) &= \frac{4\text{sinc}^2(s)\cos(\pi s)\cos(2\pi s)e^{-i3\pi s}}{(1 + e^{-i4\pi s})} \\
&= \frac{4\text{sinc}^2(s)\cos(\pi s)\cos(2\pi s)e^{-i3\pi s}}{2e^{-i2\pi s}\cos(2\pi s)} \\
&= 2\,\text{sinc}^2(s)\cos(\pi s)e^{-i\pi s}
\end{aligned}
$$

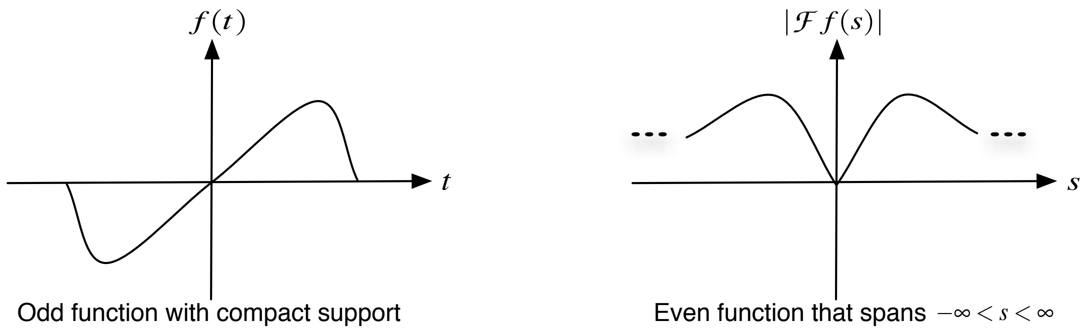
(b) In this part, $h(x) = f(x) + f(-x)$. Taking the Fourier transform gives us:

$$
\begin{aligned}
\mathcal{F}h(s) &= \mathcal{F}f(s) + \mathcal{F}f(-s) \\
&= 4\,\text{sinc}^2(s)\cos(\pi s)\cos(2\pi s)e^{-i3\pi s} + 4\,\text{sinc}^2(s)\cos(\pi s)\cos(2\pi s)e^{i3\pi s} \\
&= 4\,\text{sinc}^2(s)\cos(\pi s)\cos(2\pi s)\left(e^{-i3\pi s} + e^{i3\pi s}\right) \\
&= 8\,\text{sinc}^2(s)\cos(\pi s)\cos(2\pi s)\cos(3\pi s)
\end{aligned}
$$

4. (20 points) *How well do you know your transform?*
In this question, the figure on the left is the real signal $f(t)$ and the figure on the right shows either the phase of $\mathcal{F}f(s)$, denoted by $\angle\mathcal{F}f(s)$; or the magnitude of $\mathcal{F}f(s)$, denoted by $|\mathcal{F}f(s)|$.

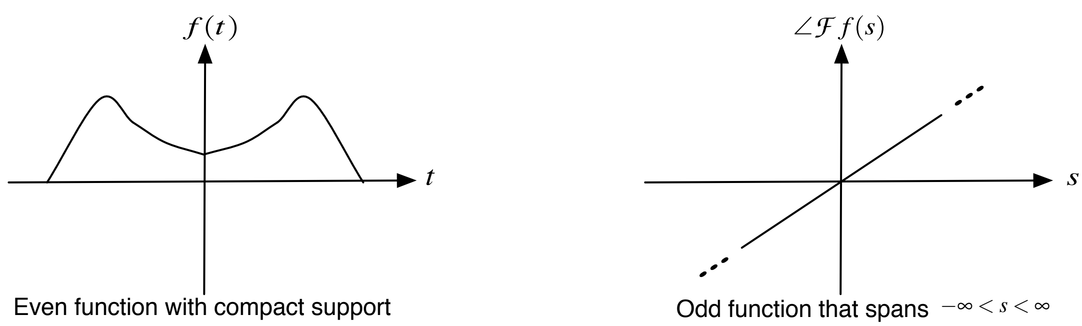
State if each of the given Fourier transform pairs is possible.
Justify your results.

(a) Is this Fourier transform pair possible?



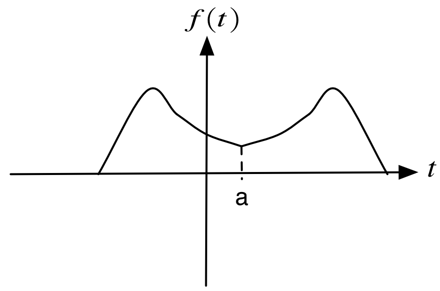
Solution:
Since $f(t)$ is a real and odd function, we would expect $|\mathcal{F}f(s)|$ to be even and the value $|\mathcal{F}f(0)|$ to be zero. This is what is observed and hence, this is a possible pair.
To come to a definite conclusion, we should expect the phase profile, $\angle\mathcal{F}(s)$, to be odd and take on only values $\pm\frac{\pi}{2}$

(b) Is this Fourier transform pair possible?

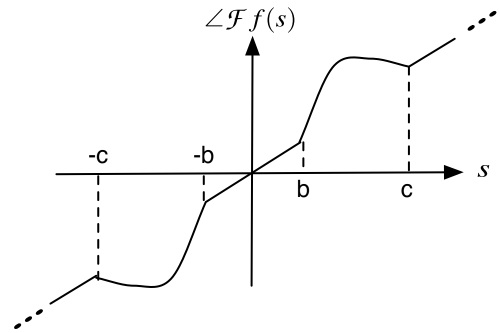


Solution:
Since $f(t)$ is a real and even function, we would expect its Fourier transform to be real and even as well. A real function can only take on a phase of 0 or $\pm\pi$. This is not the case for $\angle\mathcal{F}(s)$ and hence this pair cannot be possible.

(c) Is this Fourier transform pair possible?



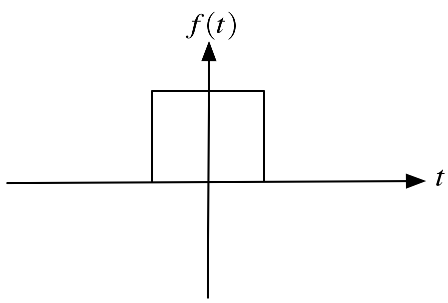
Shifted even function with compact support

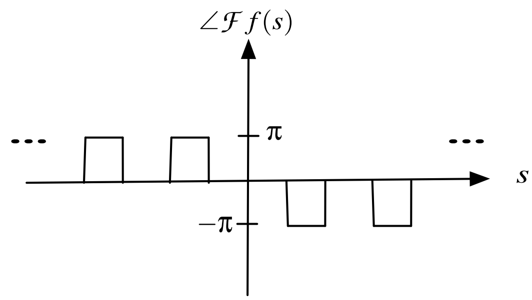


Odd function that spans $-\infty < s < \infty$

Solution:
This is similar to the previous part. The shift induces a linear phase term in $\mathcal{F}f(s)$. However, since $f(t+a)$ is a real and even function, where a is the shift, we would expect its Fourier transform to take on a phase of 0 or $\pm\pi$. This is not the case for $\angle\mathcal{F}(s)$ because we see a linear phase term added to a continuum of phases between $b \leq |s| \leq c$. Hence this pair cannot be possible.

(d) Is this Fourier transform pair possible?





Solution:
This is possible. The phase takes on the values 0 where $\mathcal{F}f(s)$ is positive and $\pm\pi$ where $\mathcal{F}f(s)$ is negative. Moreover, $\angle\mathcal{F}(s)$ is an odd function.

5. (15 points) Let $f(x)$ be a signal and for $h > 0$ let $A_h f(x)$ be the averaging operator,

$$A_h f(x) = \frac{1}{2h}\int_{x-h}^{x+h} f(y)\,dy = \frac{1}{2h}\int_{-h}^{h} f(x+y)\,dy = \frac{1}{2h}\int_{-h}^{h} f(x-y)\,dy\,.$$

(a) Show that we should define $A_h T$ for a distribution T by

$$\langle A_h T\,,\,\varphi\rangle = \langle T\,,\,A_h\varphi\rangle\,.$$

(b) Assuming the result in part (a) (even if you didn't derive it), what is $A_h\delta$?

Solutions: Suppose ψ is a smooth function. Then the pairing $\langle A_h\psi\,,\,\varphi\rangle$ with a test function φ is given by integration, and

$$\begin{aligned}
\langle A_h\psi\,,\,\varphi\rangle &= \int_{-\infty}^{\infty} A_h\psi(x)\varphi(x)\,dx \\
&= \int_{-\infty}^{\infty}\left(\frac{1}{2h}\int_{-h}^{h}\psi(x+y)\,dy\right)\varphi(x)\,dx \\
&= \frac{1}{2h}\int_{-h}^{h}\int_{-\infty}^{\infty}\psi(x+y)\varphi(x)\,dxdy
\end{aligned}$$

Now make the change of variable $u = x+y$ in the inner integral,

$$\int_{-\infty}^{\infty}\psi(x+y)\varphi(x)\,dx = \int_{-\infty}^{\infty}\psi(u)\varphi(u-y)\,du\,,$$

leading to

$$\begin{aligned}
\frac{1}{2h}\int_{-h}^{h}\int_{-\infty}^{\infty}\psi(x+y)\varphi(x)\,dxdy &= \frac{1}{2h}\int_{-h}^{h}\int_{-\infty}^{\infty}\psi(u)\varphi(u-y)\,dudy \\
&= \int_{-\infty}^{\infty}\psi(u)\left(\frac{1}{2h}\int_{-h}^{h}\varphi(u-y)\,dy\right)du \\
&= \int_{-\infty}^{\infty}\psi(u)\left(\frac{1}{2h}\int_{-h}^{h}\varphi(u-y)\,dy\right)du \\
&= \int_{-\infty}^{\infty}\psi(u)\left(\frac{1}{2h}\int_{-h}^{h}\varphi(u+y)\,dy\right)du \\
&= \int_{-\infty}^{\infty}\psi(u)A_h\varphi(u)\,du \\
&= \langle\psi\,,\,A_h\varphi\rangle
\end{aligned}$$

Thus, for a general distribution T we define

$$\langle A_h T\,,\,\varphi\rangle = \langle T\,,\,A_h\varphi\rangle\,.$$

Next, to find $A_h\delta$ we have for any test function φ,

$$
\begin{aligned}
\langle A_h\delta\,,\,\varphi\rangle &= \langle \delta\,,\,A_h\varphi\rangle \\
&= A_h\varphi(0) \\
&= \frac{1}{2h}\int_{-h}^{h}\varphi(y)\,dy \\
&= \frac{1}{2h}\int_{-\infty}^{\infty}\Pi_{2h}(y)\varphi(y)\,dy \\
&= \langle\frac{1}{2h}\Pi_{2h}\,,\,\varphi\rangle\,.
\end{aligned}
$$

We conclude that

$$
A_h\delta = \frac{1}{2h}\Pi_{2h}\,.
$$