EE 261 The Fourier Transform and its Applications
Fall 2006
Midterm Exam   Solutions

- There are six questions for a total of 100 points.
- Please write your answers in the exam booklet provided, and make sure that your answers stand out.
- Don't forget to write your name on your exam book!

1. (15 points) Let $f(t)$ be a periodic signal of period 1 and define the *averaging operator* depending on a parameter $h > 0$ by

$$\mathcal{A}_h f(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t)\, dt\,.$$

Thus $\mathcal{A}_h f(x)$ is a new signal.

(a) Show that $\mathcal{A}_h f(x)$ is periodic of period 1 as a function of x, i.e.,

$$\mathcal{A}_h f(x+1) = \mathcal{A}_h f(x)\,,$$

Hint:

$$\mathcal{A}_h f(x+1) = \frac{1}{2h} \int_{x+1-h}^{x+1+h} f(t)\, dt\,.$$

Now make a change of variable $t = u + 1$.

(b) Find the Fourier series for $\mathcal{A}_h f(x)$ in terms of the Fourier series for $f(t)$.

Solution:

To show that $\mathcal{A}_h f$ is periodic of period 1 we have to work directly with the integral,

$$\mathcal{A}_h f(x+1) = \frac{1}{2h} \int_{x+1-h}^{x+1+h} f(t)\, dt\,.$$

To bring the periodicity of $f(t)$ into the picture we make a change of variable $t = u + 1$ in the integrand. The limits of integration change according to $t = x + 1 - h \longrightarrow u = x - h$ and $t = x + 1 + h \longrightarrow u = x + h$. We find

$$\begin{aligned}
\mathcal{A}_h f(x+1) &= \frac{1}{2h} \int_{x+1-h}^{x+1+h} f(t)\, dt \\
&= \frac{1}{2h} \int_{x-h}^{x+h} f(u+1)\, du \\
&= \frac{1}{2h} \int_{x-h}^{x+h} f(u)\, du \quad \text{(since } f(t) \text{ is periodic of period 1)} \\
&= \mathcal{A}_h f(x)\,.
\end{aligned}$$

Next write the Fourier series for $f(t)$ as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t}\,.$$

We apply $\mathcal{A}_h$ to $f(t)$ by integrating the series term by term:

$$\mathcal{A}_h f(x) = \sum_{n=-\infty}^{\infty} c_n \frac{1}{2h} \int_{x-h}^{x+h} e^{2\pi i n t}\, dt$$

and it's the integral we have to evaluate. For this, if $n = 0$ then

$$\frac{1}{2h} \int_{x-h}^{x+h} 1\, dt = \frac{1}{2h}((x+h) - (x-h)) = \frac{2h}{2h} = 1\,.$$

If $n \neq 0$,

$$\begin{aligned}
\frac{1}{2h}\int_{x-h}^{x+h} e^{2\pi nit}\,dt &= \frac{1}{2h}\left[\frac{1}{2\pi in}e^{2\pi nit}\right]_{t=x-h}^{t=x+h} \\
&= \frac{1}{4\pi inh}(e^{2\pi in(x+h)} - e^{2\pi in(x-h)}) \\
&= \frac{1}{4\pi inh}e^{2\pi inx}(e^{2\pi inh} - e^{-2\pi inh}) \\
&= \frac{1}{2\pi nh}\frac{e^{2\pi inh} - e^{-2\pi inh}}{2i}e^{2\pi inx} \\
&= \frac{\sin 2\pi nh}{2\pi nh}e^{2\pi inx} \\
&= \operatorname{sinc}(2nh)e^{2\pi inx}\,.
\end{aligned}$$

Thus the Fourier series for $\mathcal{A}_h f(x)$ is

$$\mathcal{A}_h f(x) = \sum_{n=-\infty}^{\infty} c_n \operatorname{sinc}(2nh)e^{2\pi inx}\,.$$

Incidentally, the resulting expression obtained by integrating the Fourier series term by term in this way also shows that $\mathcal{A}_h f(x)$ is periodic of period 1.

2. (15 points) *Energy of a bandlimited signal* The energy of a signal $g(t)$ is the integral

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt \, .$$

Suppose that $g(t)$ is bandlimited with

$$\mathcal{F}g(s) = 0 \, , \quad |s| \geq \frac{1}{2} \, .$$

Express the energy of $g(t)$, in terms of the sample values of $g(t)$ at the integers, $g(n)$, $n = 0, \pm 1, \pm 2, \ldots$. Hint: Use the Fourier series of the periodization of $\mathcal{F}g(s)$ of period 1.

Solution:

Write the Fourier series of the periodized version of $\mathcal{F}g(s)$ as

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n s} \, .$$

The coefficients are given by

$$\begin{aligned} c_n &= \int_{-1/2}^{1/2} e^{-2\pi i n s} \mathcal{F}g(s) \, ds \\ &= \int_{-\infty}^{\infty} e^{-2\pi i n s} \mathcal{F}g(s) \, ds \quad (\text{since } \mathcal{F}g(s) = 0 \text{ for } |s| \geq 1/2) \quad = \mathcal{F}^{-1}\mathcal{F}g(-n) \\ &= g(-n) \end{aligned}$$

Thus the Fourier series is

$$\sum_{n=-\infty}^{\infty} g(-n) e^{2\pi i n t}$$

By Parseval's identity

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt = \int_{-\infty}^{\infty} |\mathcal{F}f(s)|^2 \, ds = \int_{-1/2}^{1/2} |\mathcal{F}f(s)|^2 \, ds \, ,$$

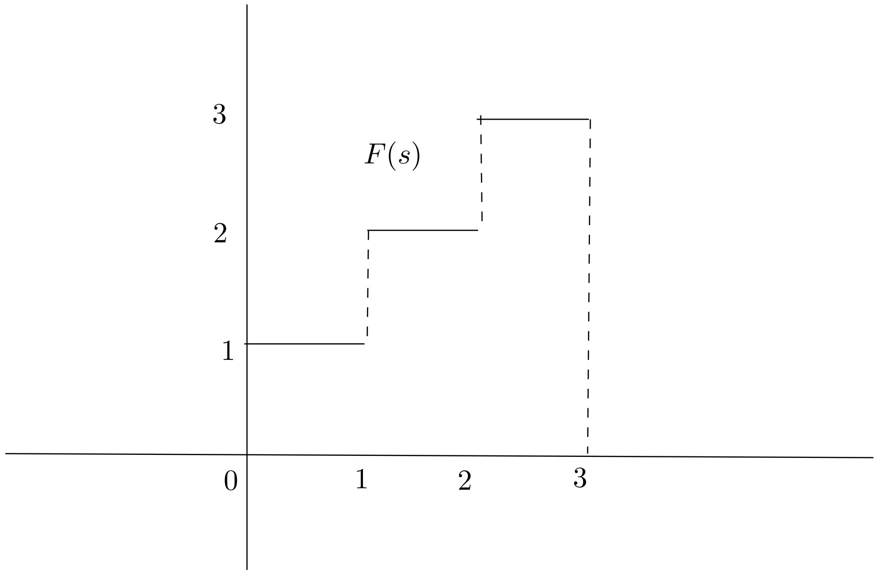
and by Rayleigh's identity

$$\int_{-1/2}^{1/2} |\mathcal{F}f(s)|^2 \, ds = \sum_{n=-\infty}^{\infty} |g(-n)|^2 = \sum_{n=-\infty}^{\infty} |g(n)|^2 \, .$$

This allows us to express the energy in terms of the samples:

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |g(n)|^2 \, .$$

3. (25 points) Let $f(t)$ be a signal. Its Fourier transform $F(s)$ is plotted below.

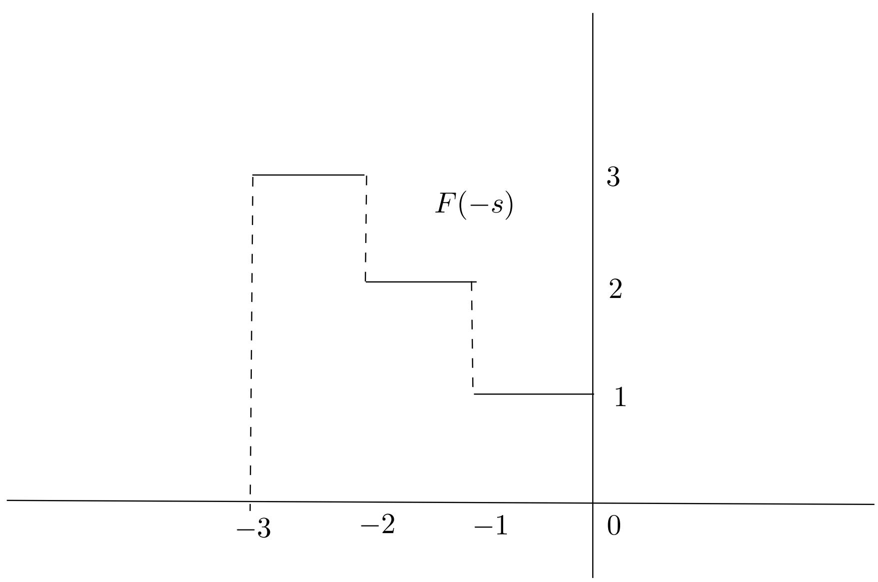


Sketch the graph of the Fourier transform of the signals:

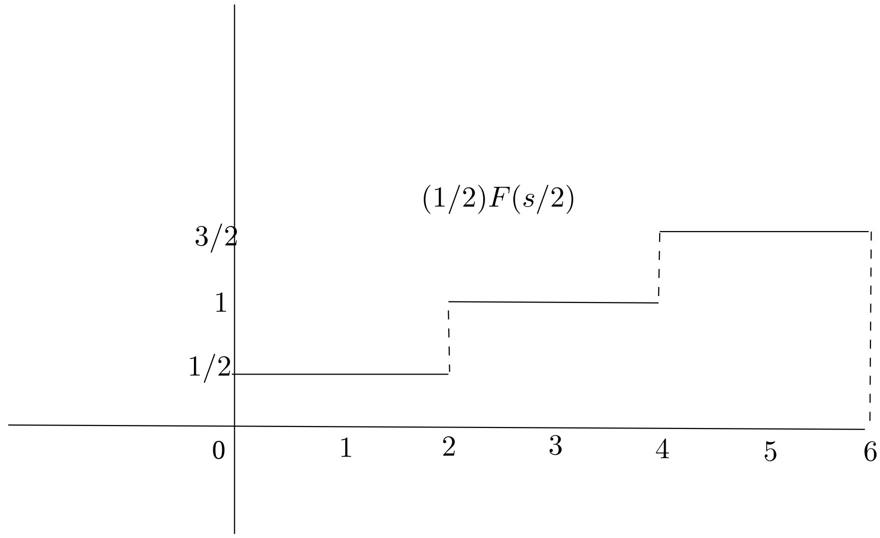
(a) $f(-x)$
(b) $f(2x)$
(c) $e^{4\pi i x} f(x)$
(d) $(f * f)(x)$
(e) $\frac{1}{2\pi i} f'(x)$

Solutions:

(a) One of the basic duality results tells us that $\mathcal{F}(f(-x)) = F(-s)$, the reverse of the Fourier transform. The graph looks like this.



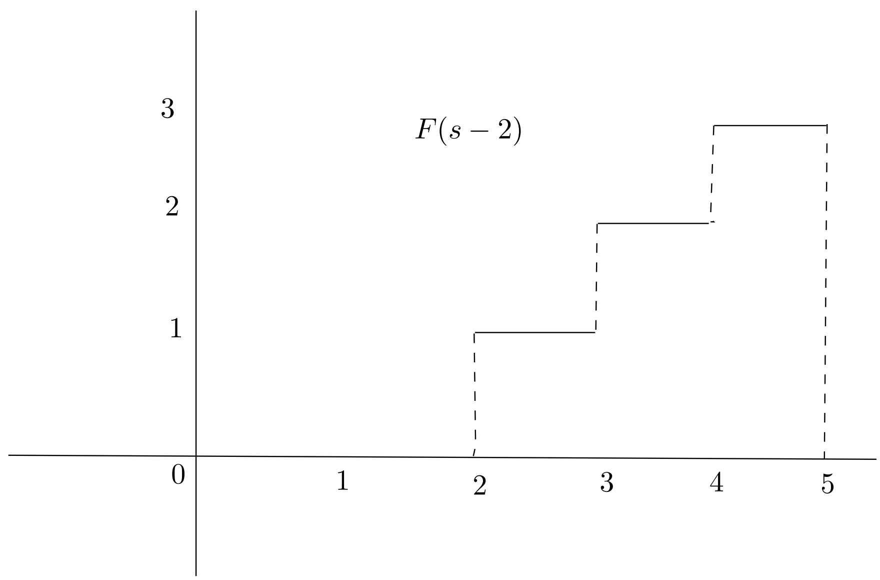
(b) According to the stretch theorem $\mathcal{F}(f(2x)) = (1/2)F(s/2)$. The plot is:



(c) A phase change in time corresponds to a shift in frequency,

$$\mathcal{F}(e^{4\pi i x} f(x)) = F(s-2)\,,$$

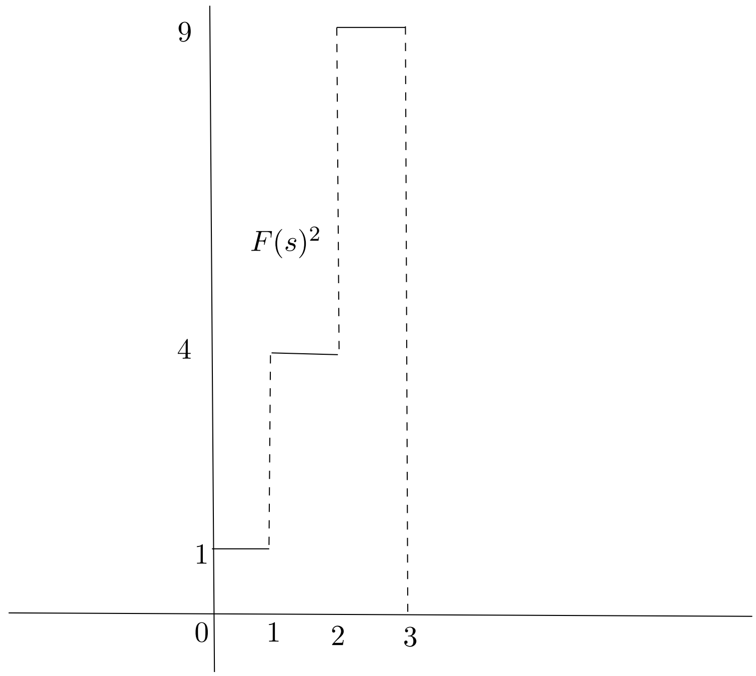
in this case a shift to the right by 2. The plot is:



(d) Under the Fourier transform convolution goes to multiplication, so

$$\mathcal{F}(f * f) = (\mathcal{F}f)^2\,.$$

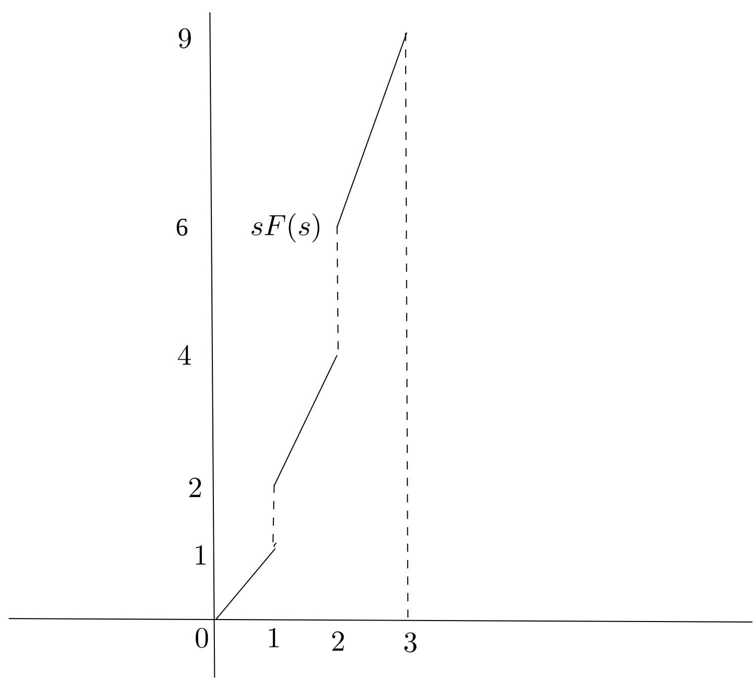
The graph is:



(e) The derivative theorem gives

$$\mathcal{F}\left(\frac{1}{2\pi i}f'(x)\right) = sF(s)\,.$$

The graph is:



4. (15 points) Let $f(t)$ be a signal with Fourier transform $F(s)$. Suppose we are given the following facts:

   (a) $f(t)$ is real.

   (b) $f(t) = 0$ for $t \leq 0$.

   (c) $\mathcal{F}^{-1}(\operatorname{Re} F(s)) = |t|e^{-|t|}$, where $\operatorname{Re} F(s)$ denotes the real part of $F(s)$.

   Find $f(t)$.

   Hint: Recall that $\operatorname{Re} F(s) = (F(s) + \overline{F(s)})/2$.

   Solution:

   From the hint,

$$\operatorname{Re} F(s) = \frac{F(s) + \overline{F(s)}}{2}$$

and since $f(t)$ is real its Fourier transform has the symmetry

$$\overline{F(s)} = F(-s)\,.$$

Thus

$$\operatorname{Re} F(s) = \frac{F(s) + F(-s)}{2}$$

and we have

$$\begin{aligned}
|t|e^{-|t|} &= \mathcal{F}^{-1}(\operatorname{Re} F(s)) \\
&= \frac{1}{2}(\mathcal{F}^{-1}(F(s)) + \mathcal{F}^{-1}(F(-s)) \\
&= \frac{1}{2}(f(t) + f(-t))\,.
\end{aligned}$$

Now, if $t > 0$ then $-t < 0$ and

$$te^{-t} = |t|e^{-|t|} = \frac{1}{2}(f(t) + f(-t)) = \frac{1}{2}f(t)\,.$$

We can then say

$$f(t) = \begin{cases} 2te^{-t}, & t > 0, \\ 0 \quad , & t \leq 0 \end{cases}$$

or simply

$$f(t)2te^{-t}H(t)\,,$$

where $H(t)$ is the unit step function.

5. (15 points) Stand a fixed distance from a wall, hold a laser horizontally, and point it at the wall at a random angle. The locations of the burn marks on the wall is a random variable X whose values are distributed according to

$$p(x) = \frac{1}{\pi(1+x^2)} .$$

This is a *Cauchy distribution* and arises in many applications.

(a) Find the distribution for the average of two (independent) series of such burn marks, i.e. $(1/2)(X_1 + X_2)$, where X_1 and X_2 are distributed as above. For this you'll likely need the Fourier transform of a two-sided decaying exponential (see the formula sheet), and the fact that if $p(x)$ is the distribution for a random variable X then $(1/a)X$ has distribution $ap(ax)$.

(b) Without any further work, what happens if we average N series of such burn marks?

(c) People's intuition is often that if they perform the same experiment many times and average the results then they're getting close to the 'actual' or 'ideal' answer. What do your results above say about this intuition?

Solutions:

(a) The distribution for the sum of two independent random variables is the convolution of their distributions. In our case the sum we want is

$$\frac{1}{2}X_1 + \frac{1}{2}X_2$$

and the distribution for each is

$$2p(2x) = \frac{2}{\pi(1+(2x)^2)} .$$

To find the convolution of this with itself we take the Fourier transform to convert to a product.

For this we need the formula

$$\mathcal{F}e^{-a|t|} = \frac{2a}{a^2 + 4\pi^2 s^2} ,$$

where by duality we have

$$\mathcal{F}\left(\frac{2a}{a^2 + 4\pi^2 s^2}\right) = e^{-a|t|} .$$

To get this to match with our distribution $2p(2x)$ we take $a = \pi$,

$$\frac{2\pi}{\pi^2 + 4\pi^2 x^2} = \frac{2}{\pi(1+(2x)^2)} .$$

Then

$$\mathcal{F}\left\{\left(\frac{2\pi}{\pi^2 + 4\pi^2 x^2}\right) * \left(\frac{2\pi}{\pi^2 + 4\pi^2 x^2}\right)\right\} = (e^{-\pi|s|})(e^{-\pi|s|}) = e^{-2\pi|s|} .$$

The inverse Fourier transform of this is

$$\frac{4\pi}{4\pi^2 + 4\pi^2 x^2} = \frac{1}{\pi(1+x^2)} = p(x) .$$

We're back to $p(x)$!

(b) Thus we see that the distribution of the average of two Cauchy random variables is the same Cauchy random variable. From this, the average of N Cauchy Random variables is the same as that of the first Cauchy random variable.

(c) However many times we repeat the experiment, on averaging the results we'll still find the same distribution. There is no improvement.

6. (15 points) The three TA's are out having dinner one night and talk EE261 and sampling theory.

Liz: So I want to sample this signal at 3 samples per second, but I only have samplers that operate at 1 sample per second. I ought to be able to start them a third of a second apart and add the results together to make three samples a second. That will allow me to reconstruct a signal with bandwidth 1.5 Hz.

Thomas: Well that sounds good, but if you take the Fourier Transform of the 1 second samplers you will get three Shah functions, and if you add those up you will get the sum of impulse trains each spaced 1 hertz apart. Won't you? And that's only going to allow you to reconstruct a signal with bandwidth 0.5 Hz.

Paul: What is going on here? Who is right, and why is the other person wrong? And who is picking up the check?

Solution

Liz is correct, combining three shifted samples one third second apart will give an impulse train with impulses 1/3 seconds apart.

$$Ш(t) + Ш(t - \frac{1}{3}) + Ш(t - \frac{2}{3}) = Ш_{\frac{1}{3}}(t)\,.$$

The Fourier transform of the right hand side is

$$\mathcal{F}Ш_{1/3} = 3Ш_3\,.$$

Now here's what happens in Thomas's objection. The Fourier Transform of the sum of the shifted Ш's is

$$\begin{aligned}
Ш(s) + e^{-2\pi i \frac{1}{3}s}Ш(s) + e^{-2\pi i \frac{2}{3}s}Ш(s) &= Ш(s)(1 + e^{-2\pi i \frac{1}{3}s} + e^{-2\pi i \frac{2}{3}s}) \\
&= \frac{1 - (e^{-2\pi i s/3})^3}{1 - e^{-2\pi i s/3}}Ш(s) \quad \text{(using the sum of a geometric series)} \\
&= \frac{1 - e^{-2\pi i s}}{1 - e^{-2\pi i s/3}}Ш(s) \\
&= e^{-2\pi i s/3}\frac{\sin \pi s}{\sin(\pi s/3)}Ш(s)
\end{aligned}$$

Multiplying the terms in Ш by the factor out front gives

$$e^{-2\pi i s/3}\frac{\sin \pi s}{\sin(\pi s/3)}\delta(s-n) = e^{-2\pi i n/3}\frac{\sin \pi n}{\sin(\pi n/3)}\delta(s-n)\,.$$

If n is an integer *not* a multiple of 3 then

$$e^{-2\pi i n/3}\frac{\sin \pi n}{\sin(\pi n/3)} = 0$$

so those terms are killed off. If n *is* a multiple of 3 then

$$e^{-2\pi i n/3} = 1 \quad \text{and} \quad \frac{\sin \pi n}{\sin(\pi n/3)} = 3\,.$$

(To see this last point, it's like $\lim_{x\to 0} \dfrac{\sin x}{\sin(x/3)} = 1/(1/3) = 3$.) Thus we only get every third impulse, multiplied by 3:

$$e^{-2\pi i s/3} \frac{\sin \pi s}{\sin(\pi s/3)} Ш(s) = 3Ш_3(s)$$

just as before.