ON A QUESTION BY ALEXEY OSTROVSKY CONCERNING PRESERVATION OF COMPLETENESS

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ABSTRACT. Let $f : X \to Y$ be a surjection of a zero-dimensional metrizable X onto a metrizable Y which maps clopen sets in X to locally closed (or more generally, resolvable) sets in Y. We prove that if X is completely metrizable, or hereditarily Baire, then Y has also the respective property. This strengthens some recent results of A. Ostrovsky [0s] and provides an answer to his question.

In a recent paper [0s], Alexey Ostrovsky proved that if a continuous map $f : X \to Y$ from a G_δ-subspace of the irrationals onto a metrizable space takes any clopen set W in X to a set $f(W) = U \cup V$ where U is open and V is closed in Y, then Y is completely metrizable. Ostrovsky asked, if the union in the condition imposed upon $f(W)$ in this theorem can be changed to the intersection, or to another combination of closed and open sets in Y.

We give a positive answer to this question. After this note was submitted, we were kindly informed by Su Gao that he and Vincent Kieftenbeld obtained in [G-K] independently an answer to the Ostrovsky question (their approach is different from ours).

In fact, we shall establish a more general theorem, closely related to some results of E. Michael [Mi] linking the completeness with complete sieves formed by exhaustive covers, from which an answer to the Ostrovsky question follows readily. For the readers interested only in separable spaces, we give in Remark 7 a direct proof, avoiding some technicalities involved in the non-separable setting.

Let us recall that a set E in a metrizable space M is *resolvable* if for any nonempty closed set $F \subset M$, one of the sets $F \setminus \overline{(F \cap E)}$, $F \setminus \overline{(F \setminus E)}$ is nonempty, cf. [Ku], § 12, II and V (the characterization of resolvable sets which we use appears at the very end of V as a reformulation of 1. from the theorem there).

Resolvable sets are simultaneously F_σ and G_δ and the collection of resolvable sets in M is an algebra containing all *locally closed* sets, i.e., intersections of open and closed sets.

Given a metric space (X, d) we call $C \subset X$ *metrically discrete* if $\inf\{d(a, b) : a, b \in C, a \neq b\} > 0$.

Theorem 1. *Let $f : X \to Y$ be a continuous map of a complete metric space X onto a metrizable space Y such that for each countable metrically discrete C and its neighbourhood*

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V in X, there is L such that $C \subset L \subset V$ and $f(L)$ is resolvable. Then Y is completely metrizable.

A key element in our proof is Lemma 5, which is based on a variation of a reasoning of N. Ghoussoub and B. Maurey [G-M, Lemma I.1], cf. [Mi, Lemma 6.1]. With this lemma at hand, we are ready to use Theorem 1.6 from [Mi].

One can recover from Theorem 1 a theorem of E. Michael ([Mi, Corollary 1.7]), extending a result of N. Ghoussoub and B. Maurey [G-M] to non separable spaces. We have only to notice that every scattered set is clearly resolvable.

Corollary 2 (Michael; Ghoussoub and Maurey for separable spaces)**.** *Let $f : X \to Y$ be a continuous surjection of a complete metric space X onto a metrizable space Y. If f takes metrically discrete sets to scattered sets, then Y is completely metrizable.*

The following corollary answers (for separable X) a question at the end of [0s].

Corollary 3. *Let $f : X \to Y$ be a continuous map from a completely metrizable zero-dimensional space X onto a metrizable space Y. If f takes clopen sets in X to resolvable sets in Y, then Y is completely metrizable.*

Since every open neighbourhood of a metrically discrete set D in a zero-dimensional space contains a clopen neighbourhood of D, the corollary follows readily from Theorem 1.

Let us recall that X is *hereditarily Baire* (or F_{II} space, cf. [0s]) if each closed subspace of X is a Baire space. By Hurewicz's theorem [Hu], a metrizable space is hereditarily Baire if and only if it contains no closed homeomorphic copy of rational numbers. We shall also derive from Theorem 1 the following corollary, cf. [0s, Theorem 2].

Corollary 4. *Let $f : X \to Y$ be a continuous map from a zero-dimensional hereditarily Baire space onto a metrizable space. If f takes clopen sets in X to resolvable sets in Y, then Y is hereditarily Baire.*

Let us pass now to proofs of Theorem 1 and Corollary 4.

Given a metric d on X, $B(x, r) = \{y : d(x, y) < r\}$ is the open r-ball in X centered at x.

Lemma 5. *Let $f : X \to Y$ satisfy the assumptions of Theorem 1, except possibly for the completeness of X. Let $U \subset X$ be an open set in X, S a nonempty subset of $f(U)$ and $\varepsilon > 0$. There is an open set $M \subset U$, covered by finitely many ε-balls in X, such that $f(M) \cap S$ has a nonempty relative interior in S.*

Proof. Aiming at a contradiction assume that for some nonempty set $S \subset f(U)$ and $\varepsilon > 0$, there is no M satisfying the assertion of the lemma.

Let us begin with the following observation. Suppose that we have given a finite set $F \subset U$ and a nonempty relatively open set W in S. Then, using the assumption, we can pick $x \in U$ such that $f(x) \in W \setminus \bigcup_{c \in F} f(B(c, \varepsilon) \cap U)$.

Repeating this observation we can choose inductively $a_n \in U$, such that

(1) $f(a_n) \notin \bigcup_{j<n} f(B(a_j, \varepsilon) \cap U)$,

(2) for any n and p in $\mathbb{N}$ there is $k \in \mathbb{N}$ such that $\rho(f(a_n), f(a_k)) < \frac{1}{p}$,

where ρ is a fixed metric on Y generating the topology.

More specifically, let us fix a surjection $u : \mathbb{N} \to \mathbb{N}$ such that $u(n) < n$ for $n > 1$ and $u^{-1}(n)$ is infinite for $n \in \mathbb{N}$, cf. [Mi, proof of Lemma 6.1]. Choose $a_1 \in U$ arbitrarily. Then, at the n'th stage of the construction, we set $F = \{a_1, \ldots, a_{n-1}\}$, we let W be the $\frac{1}{n}$-ball in S centered at $a_{u(n)}$, and we use the observation to pick $a_n \in U$ with $f(a_n) \in W \setminus \bigcup_{j<n} f(B(a_j, \varepsilon) \cap U)$.

Having completed the inductive construction, we shall consider the metrically discrete set $A = \{a_n : n \in \mathbb{N}\}$ and we let

(3) $V_n = B(a_n, \frac{\varepsilon}{2}) \cap U \setminus \bigcup\{f^{-1}(a_j) : j < n\}$.

The set $Q = f(A)$ is homeomorphic to the set of rationals, being countable and infinite by (1), and dense-in-itself by (2). Let us take $C \subset A$ such that both $f(C)$ and $f(A \setminus C)$ are dense in Q, and let $V = \bigcup\{V_n : a_n \in C\}$, cf. (3). Notice that $f(C) \cap f(A \setminus C) = \emptyset$, cf. (1). By the assumptions, there is L such that $C \subset L \subset V$ and $f(L)$ is resolvable. But, by (1) and (3), $f(L) \cap Q = f(C)$, hence for the closure F of Q, $\overline{F \cap f(L)} = F = \overline{F \setminus f(L)}$, contradicting the resolvability of $f(L)$. □

Proof of Theorem 1. We shall apply Michael's Theorem 1.6 from [Mi]. Given an open set U in X, an $\varepsilon > 0$, and a nonempty subset S of $f(U)$, we have by Lemma 5 an open set $M \subset U$ and an open set $W \subset Y$ such that $\emptyset \neq W \cap S \subset f(M)$ and M is covered by finitely many $\frac{\varepsilon}{2}$-balls. Then $V = f^{-1}(W) \cap M \subset U$ is covered by finitely many sets of diameter $\leq \varepsilon$ and $f(V) \cap S = W \cap S$ is a nonempty relatively open set in S. Therefore, the assumptions of Michael's theorem are satisfied for the collection $\mathcal{U}$ of all open sets in X, and by the assertion of this theorem, Y is completely metrizable. □

Proof of Corollary 4. Aiming at a contradiction, assume that Y is not hereditarily Baire. Then, by Hurewicz's theorem [Hu], Y contains a closed copy Q of the rationals. The traces $B \cap f^{-1}(Q)$ of clopen sets in X on $f^{-1}(Q)$ form a basis for metrically discrete sets in $f^{-1}(Q)$, with resolvable images $f(B \cap f^{-1}(Q)) = f(B) \cap Q$. It follows that f restricted to $f^{-1}(Q)$ satisfies the assumptions of Theorem 1, except for the completeness, with respect to any metric on X. To simplify the notation we just set $X = f^{-1}(Q)$ and we shall consider in the sequel the surjection $f : X \to Q$.

Since X is hereditarily Baire and Q is countable, one can define by transfinite induction a sequence of closed sets $X_1 = X \supset X_2 \supset \ldots \supset X_\xi \supset \ldots \supset X_\lambda = \emptyset$, such that f is constant on each $E_\xi = X_\xi \setminus X_{\xi+1}$ and $X_\xi = \bigcap_{\eta<\xi} X_\eta$ for any limit ξ.

More specifically, if $X_\xi \neq \emptyset$ is already defined we proceed as follows. Using the fact that X_ξ is Baire and $\{f^{-1}(q) \cap X_\xi : q \in Q\}$ is a countable cover of X_ξ by closed sets, we find a nonempty relatively open set W in X_ξ contained in some $f^{-1}(q)$, and we let $X_{\xi+1} = X_\xi \setminus W$. If $X_{\xi+1} \neq \emptyset$ we continue, or we set $\lambda = \xi + 1$ and terminate the construction otherwise. If X_η are defined for $\eta < \xi$ and ξ is a limit ordinal, we consider $X_\xi = \bigcap_{\eta<\xi} X_\eta$ and we continue if $X_\xi \neq \emptyset$, or else we set $\lambda = \xi$ and we stop.

We can assume that Q is the set of rationals in the real line $\mathbb{R}$ and extend f to a continuous function $g : Z \to \mathbb{R}$ over a completely metrizable Z containing X. Let Z_ξ and F_ξ be the closures of X_ξ and E_ξ in Z, respectively. We let $H_\xi = F_\xi \cap (Z_\xi \setminus Z_{\xi+1})$ and $H = \bigcup_\xi H_\xi$. Then E_ξ is a dense subset of the G_δ-set H_ξ, and H is a G_δ-set in Z, cf. [Ku], §30, X. By continuity of g and the fact that f is constant on each E_ξ,

(4) $f(x) = g(y)$, whenever $x \in E_\xi$ and $y \in H_\xi$.

It follows that $g(H) = Q$. We shall check that the restriction $g \mid H : H \to Q$ satisfies the assumptions of Theorem 1 with respect to any complete metric d on H. Indeed, let $K \subset H$ be a countable set such that $d(y,z) > \delta > 0$ for any distinct $y, z \in K$, and let V be its open neighbourhood in H. For each $y \in K \cap H_\xi$ we pick $x(y) \in E_\xi \cap V$ with $d(y, x(y)) < \frac{\delta}{3}$. Then the set $C = \{x(y) : y \in K\} \subset V \cap X$ is metrically discrete and by (4), $f(C) = g(K)$. Let $L' \subset X$ be such that $C \subset L' \subset V$ and $f(L')$ is resolvable. In effect, setting $L = L' \cup K$, we obtain $K \subset L \subset V$ such that $g(L) = f(L')$ is resolvable. Since Q is not completely metrizable we reached a contradiction with Theorem 1. □

Remark 6. Any continuous surjection $f : X \to Y$ from a completely metrizable X onto a metrizable Y which is either open or closed satisfies the assumptions of Theorem 1. Therefore, Theorem 1 yields the invariance of complete metrizability under open maps (Hausdorff's theorem) and closed maps (Vainstein's theorem), cf. [Mi].

Remark 7. Let $g : G \to H$ be a continuous surjection from a G_δ-subspace of the irrationals onto a metrizable space H, taking clopen sets to resolvable sets. We indicate a direct argument to the effect that H is completely metrizable. As in [0s] (a comment following Theorem 1), from the fact that g takes open sets to Borel sets one infers that H is absolutely Borel. Aiming at a contradiction, assume that H is not an absolute G_δ-set and use the Hurewicz theorem to get a closed copy Y of the rationals in H, cf. [0s]. Setting $X = g^{-1}(Y)$ and $f = g \mid X$ we get a continuous map $f : X \to Y$ from a completely metrizable space onto rationals, satisfying the assumptions of Lemma 5. Indeed, if $C \subset X$ is a metrically discrete set and V is its neighbourhood in G, one can find a clopen set W in G such that $C \subset W \subset V$. Then, for $L = W \cap X$, $f(L) = g(W) \cap Y$ is a resolvable set in Y, the set Y being closed and $g(W)$ resolvable in H.

Let us list points of Y as $y_1, y_2, \ldots$ and let us fix a complete metric on X. We use Lemma 5 to get inductively nonempty open sets $U_0 \supset U_1 \supset U_2 \supset \ldots$ in X such that $U_0 = X$ and, for $n \geq 1$, U_n is covered by finitely many $\frac{1}{n}$-balls, $f(U_n)$ is open in Y and its closure misses y_n. Then, by the completeness, $\bigcap_n \overline{U_n} \neq \emptyset$, but on the other hand, $f(\bigcap_n \overline{U_n}) \subset \bigcap_n \overline{f(U_n)} \subset \bigcap_n (Y \setminus \{y_n\}) = \emptyset$, a contradiction.

Remark 8. We arrived at an answer to the question by Ostrovsky independently, and the present note is a result of our further discussion on the topic. The original setting of the first of the authors concerned non-metrizable spaces. This more general approach requires an explicite use of complete sequences of covers and will be presented separately.

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