BOREL CLASSES OF UNIFORMIZATIONS OF SETS WITH LARGE SECTIONS

PETR HOLICKÝ

Abstract. We give several refinements of known theorems on Borel uniformizations of sets with "large sections". In particular, we show that a set $B \subset [0,1] \times [0,1]$ which belongs to $\mathbf{\Sigma}^0_\alpha$, $\alpha \geq 2$, and which has all "vertical" sections of positive Lebesgue measure, has a $\mathbf{\Pi}^0_\alpha$ uniformization which is the graph of a $\mathbf{\Sigma}^0_\alpha$-measurable mapping. We get a similar result for sets with nonmeager sections. As a corollary we derive an improvement of Srivastava's theorem on uniformizations for Borel sets with G_δ sections.

1. Introduction

We are going to answer a question posed by Piotr Borodulin-Nadzieja, namely what can be said about the Borel class of Borel measurable selections of Borel sets with sections of positive Lebesgue measure. The existence of a Borel uniformization in such a case was shown by Blackwell and Ryll-Nardzewski in [1]. The existence of Borel uniformizations of sets with large sections in the sense of category was proved by Sarbadhikari in [10]. An abstract version can be found in [4, Theorem 18.6]. Mauldin in [8] proved theorems on the existence of Borel parametrizations in the above cases. In fact, all the mentioned results deal with rather more general situations. Also we will deal with a slightly more general setting.

We get uniformization theorems for Borel sets $B \subset X \times Y$, for X and Y Polish, with sections $B_x = \{y \in Y : (x,y) \in B\}$ of positive probability $\mu(x, B_x)$ with respect to suitable probability kernels μ and for Borel sets with sections B_x nonmeager (i.e., not of the first category) in suitable Baire supersets $F(x) \subset Y$. In both cases we get some information about the Borel class of the corresponding selection. We also get a continuum of such pairwise disjoint uniformizations with their union parametrized by $X \times \{0,1\}^{\mathbb{N}}$ in a Borel isomorphic way. As a corollary of the "category case", we get a modification of the Srivastava selection theorem for G_δ-valued mappings (see [12, Theorem 4.1]). Thus we get information about the Borel class of the selection also in this case. Our proof is inspired by that of Kechris (see [4, Exercise 18.20(iv) and the hint to it] or [11, Theorem 5.9.2]). We get also a version of this theorem for selectors of partitions.

The main methods we need to get our refinements are well-known. The first of them concerns the finer description of Borel sets of a given class using a particular scheme of subsets (cf. [4, Theorem 22.21], or also [7, Theorem 2.2] for the nonseparable case). The other one is the method of getting a selection used in [6] which we only slightly modify. As is well-known the latter result gives the possibility to

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get some information about the class of the selection, so it is no surprise that we get our results in this way (cf. also [9] in this context).

Since the methods work almost without any further work also for X nonseparable, we formulate our results in this more general setting.

2. Projections along large sections

We replace the elegant proofs of the preservation of Borelness when projecting sets with large sections by modifications which preserve some information about the Borel class of the projection. In the case of large sections in the sense of category this is a result of Montgomery (see [4, Exercise 22.22]) which we improve, as a revision of the proof of the theorem by Montgomery and Novikov (see [4, Theorem 16.1]). Similarly, we get also a needed modification of the property "Borel on Borel" from [4, Definition 18.5] for the case of Borel probability kernels.

We use the standard notation $\mathcal{B}(Y)$ for the set of all Borel subsets of the topological space Y. We also denote by $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Delta}^0_\alpha$ the additive, multiplicative, and ambiguous classes for $\alpha \geq 1$. We write, e.g., $\mathbf{\Sigma}^0_\alpha(Y)$ for the family of all Borel sets of the additive class $\mathbf{\Sigma}^0_\alpha$ in Y. One may find this notation, e.g., in [4].

A family $\mathcal{D}$ of sets in a metric space X is *discrete* if every $x \in X$ has a neighbourhood intersecting at most one element of $\mathcal{D}$. The family is σ-discrete if it is the union of countably many discrete families. It is an easy, and well-known, observation that $\mathcal{D}_1 \wedge \dots \wedge \mathcal{D}_k := \{D_1 \cap \dots \cap D_k : D_j \in \mathcal{D}_j,\ j = 1, \dots, k\}$ is (σ-)discrete if every $\mathcal{D}_j$ is σ-discrete. We repeatedly use the fact that the union of a discrete family of sets of $\mathbf{\Sigma}^0_\alpha(X)$ in a metric space X is in $\mathbf{\Sigma}^0_\alpha$ for every countable ordinal $\alpha \geq 1$ (see [5, Section 30, Subsection 10, Theorems 3 and 4] for an explanation).

To point out what assumptions are needed, we formulate the following lemma in a more general setting than we need it below. Let us recall that a Borel measure μ on a topological space Y is *τ-additive* if $\sup\{\mu(A) : A \in \mathcal{A}\} = \mu(\bigcup \mathcal{A})$ for every family of open sets in Y which is directed upwards (cf. [2, Definition 2.3]). Let us recall that every Radon measure is τ-additive ([2, Proposition 6.9]) and that every finite Borel measure on a Polish space is Radon ([4, Theorem 17.11]).

Lemma 2.1. *Let X be a metrizable space, Y be a topological space, $\mu : X \times \mathcal{B}(Y) \to [0, 1]$ be such that*

(a) *$\mu(x, \cdot)$ is a Borel τ-additive probability on Y for every $x \in X$, and*
(b) *$\{x \in X : \mu(x, H) > r\}$ is $\mathbf{\Sigma}^0_{\alpha_0}$-measurable in X ($1 \leq \alpha_0 < \omega_1$) for every open $H \subset Y$ and $r \in \mathbb{R}$.*

Let $B \subset X \times Y$ be in $\mathbf{\Sigma}^0_\alpha(X \times Y)$, $1 \leq \alpha < \omega_1$.

Then the set $\{x \in X : \mu(x, B_x) > 0\}$ is in $\mathbf{\Sigma}^0_{\alpha^*}(X)$, where $\alpha^* = \alpha_0 + \alpha$ if $\alpha \geq \omega$ and $\alpha^* = \alpha_0 + (\alpha - 1)$ if $\alpha < \omega$.

Proof. We shall show that the set $\pi^*_\mu(B, r) := \{x \in X : \mu(x, B_x) > r\}$ is in $\mathbf{\Sigma}^0_{\alpha^*}(X)$ for every $r \in \mathbb{R}$ by induction with respect to α (with α_0 fixed).

Let B be an open subset of $X \times Y$. There is a σ-discrete base $\mathcal{U}$ of open subsets of X (see [5, Section 21, Subsection 16, Corollary 1a]). So $B = \bigcup\{U_a \times W_a : a \in I\}$, where each W_a is open in Y, and the family $\{U_a\}_{a \in I} \subset \mathcal{U}$ is σ-discrete in X. We use the τ-additivity of $\mu(x, \cdot)$ for each $x \in X$. We apply it to the upwards directed family of open sets $\mathcal{A}_x = \{W_{a_1} \cup \dots \cup W_{a_k} : x \in U_{a_1} \cap \dots \cap U_{a_k},\ a_1, \dots, a_k \in I,\ k \in \mathbb{N}\}$. Since $B_x = \bigcup \mathcal{A}_x$ by the equality $B = \bigcup\{U_a \times W_a : a \in I\}$, we have $\mu(x, B_x) = \sup\{\mu(x, A) : A \in \mathcal{A}_x\}$. Therefore $\mu(x, B_x) > r$ if and only if there are $a_1, \dots, a_k \in I$ such that $x \in U_{a_1} \cap \dots \cap U_{a_k}$ and $\mu(x, W_{a_1} \cup \dots \cup W_{a_k}) > r$. Put $U(a_1, \dots, a_k, r) = \{x \in U_{a_1} \cap \dots \cap U_{a_k} : \mu(x, W_{a_1} \cup \dots \cup W_{a_k}) > r\}$. The family $\{U(a_1, \dots, a_k, r) : k \in \mathbb{N},\ a_1, \dots, a_k \in I\}$ forms a σ-discrete cover of the set $\pi^*_\mu(B, r)$ by sets from $\mathbf{\Sigma}^0_{\alpha_0}(X)$ (this easy fact follows from the remark before the lemma). Thus $\pi^*_\mu(B, r)$ is in $\mathbf{\Sigma}^0_{\alpha_0}(X)$, which is our claim for $\alpha = 1$.

Let $\alpha > 1$ and the claim be valid for all Borel sets C of additive class β for every $1 \le \beta < \alpha$. Let $B \subset X \times Y$ be in $\mathbf{\Sigma}^0_\alpha(X \times Y)$. Thus there are $B_n \in \mathbf{\Pi}^0_{\beta_n}(X \times Y)$, $B_n \subset B_{n+1}$, $1 \le \beta_n < \alpha$ such that $B = \bigcup_{n\in\mathbb{N}} B_n$. Now

$$\begin{aligned}
\pi^*_\mu(B, r) &= \bigcup_{n=1}^{\infty} \pi^*_\mu(B_n, r) \\
&= \bigcup_{n=1}^{\infty} \{x \in X : \mu(x, (B^c_n)_x) < 1 - r\} \\
&= \bigcup_{n=1}^{\infty} \bigcup_{p=1}^{\infty} \{x \in X : \mu(x, (B^c_n)_x) \le 1 - r - \frac{1}{p}\} \\
&= \bigcup_{n=1}^{\infty} \bigcup_{p=1}^{\infty} X \setminus \{x \in X : \mu(x, (B^c_n)_x) > 1 - r - \frac{1}{p}\}.
\end{aligned}$$

By the induction hypothesis, $\{x \in X : \mu(x, (B^c_n)_x) > 1 - r - \frac{1}{p}\} \in \mathbf{\Sigma}^0_{\beta^*_n}(X)$. It follows that $\pi^*_\mu(B, r)$ is a countable union of sets of classes $\mathbf{\Pi}^0_{\beta^*_n}(X) \subset \mathbf{\Sigma}^0_{\alpha^*}(X)$. $\square$

We use the notion *Baire space* in the sense of [4, Definition 8.2], i.e., it is a topological space with no meager nonempty open subset. Thus the empty space is a Baire space and let us consider $\emptyset$ to be meager even in the empty space from formal reasons.

The symbol $\mathcal{P}(Y)$ denotes the power set of the set Y. The multivalued mapping $F : (X, \mathcal{A}) \to \mathcal{P}(Y)$, where Y is a topological space, is *lower $\mathcal{A}$-measurable* means that the sets $F^{-1}(H) := \{x \in X : F(x) \cap H \ne \emptyset\} \in \mathcal{A}$ for every open $H \subset Y$. Here $\mathcal{A}$ might be a family of subsets of X or a family of subsets of some superspace of X etc. Later on, we use also that F is *upper $\mathcal{A}$-measurable* if $F_{-1}(H) := \{x \in X : F(x) \subset H\} \in \mathcal{A}$ for every open $H \subset Y$. Similarly, we say that $f : X \to Y$ is $\mathcal{A}$-measurable if $f^{-1}(H) \in \mathcal{A}$ for every open subset H of Y ($H \in \mathbf{\Sigma}^0_1(Y)$). The set $\operatorname{graph} F := \{(x, y) \in X \times Y : y \in F(x)\}$ is the *graph* of the multivalued mapping F.

Lemma 2.2. *Let X be a metrizable space, Y be a separable metrizable space.*

Let $F : X \to \mathcal{P}(Y)$ be lower $\mathbf{\Sigma}^0_{\alpha_0}$-measurable ($1 \le \alpha_0 < \omega_1$), with $F(x)$ a Baire subspace of Y for every $x \in X$, and $B \subset \operatorname{graph} F$ of the type $\mathbf{\Sigma}^0_\alpha$ in $X \times Y$, $1 \le \alpha < \omega_1$.

Then the set $\{x \in X : B_x$ is not meager in $F(x)\}$ is in $\mathbf{\Sigma}^0_{\alpha^*}$, where α^* has the same meaning as in Lemma 2.1.

The appearance of F here is related to the modification of the proof of the Montgomery and Novikov Theorem suggested in the hint to [4, Exercise 18.20(iii)].

Proof. Let us show first the following claim.

Claim. $\pi^*_F(B, W) := \{x \in X : F(x) \cap B_x \cap W$ is not meager in $F(x) \cap W\}$ is in $\mathbf{\Sigma}^0_{\alpha^*}(X)$, if B is in $\mathbf{\Sigma}^0_\alpha(X \times Y)$, for every open set $W \subset Y$.

We proceed by induction over α. If $B = \bigcup\{U_a \times W_a : a \in I\}$, where U_a and W_a are open in X and Y, respectively, and if $\{U_a\}_{a\in I}$ is σ-discrete, then $\pi^*_F(B, W) = \bigcup\{\{x \in U_a : F(x) \cap W_a \cap W \neq \emptyset\} : a \in I\} = \bigcup\{U_a \cap F^{-1}(W \cap W_a) : a \in I\}$, which is in $\mathbf{\Sigma}^0_{\alpha_0}(X)$ by our assumptions. Here we used that every nonempty relatively open subset of $F(x)$ is not meager in $F(x)$. Thus our claim for $\alpha = 1$ is proved.

Let $\alpha > 1$ and the claim be valid for every $\beta \geq 1$ less than α. Let $B \subset X \times Y$ be in $\mathbf{\Sigma}^0_\alpha(X \times Y)$. Thus there are $B_n \in \mathbf{\Pi}^0_{\beta_n}(X \times Y)$, $\beta_n < \alpha$ such that $B = \bigcup_{n\in\mathbb{N}} B_n$. For a fixed open set $W \subset Y$ we get

$$
\begin{aligned}
&\pi^*_F(B, W) = \bigcup_{n\in\mathbb{N}} \pi^*_F(B_n, W) \\
= &\bigcup_{n\in\mathbb{N}} \bigcup_{\substack{W'\in\mathcal{W}\\ W'\subset W}} \{x \in X : F(x) \cap (B_n)_x \cap W' \text{ is residual in } F(x) \cap W' \neq \emptyset\} \\
= &\bigcup_{n\in\mathbb{N}} \bigcup_{\substack{W'\in\mathcal{W}\\ W'\subset W}} \{x \in X : F(x) \cap (B^c_n)_x \cap W' \text{ is meager in } F(x) \cap W'\} \cap F^{-1}(W'),
\end{aligned}
$$

where $\mathcal{W}$ is a countable base of Y consisting of nonempty open sets. Here we used the fact that a subset with the Baire property of a Baire space is nonmeager if and only if it is residual in some nonempty open subset (see, e.g., [4, Proposition 8.26]). Since the sets $F^{-1}(W')$ are in $\mathbf{\Sigma}^0_{\alpha_0}(X)$ and the sets

$$\pi^*_F(B^c_n, W') = X \setminus \{x \in X : F(x) \cap (B^c_n)_x \cap W' \text{ is meager in } F(x) \cap W'\}$$

are in $\mathbf{\Sigma}^0_{\beta^*_n}(X)$ by the induction assumption, $\pi^*_F(B, W)$ is in $\mathbf{\Sigma}^0_{\alpha^*}(X)$. □

3. Uniformizations of sets with large sections

We use in the proof of the uniformization theorems below the existence of a scheme of subsets of a Borel set in a complete metric space. Our requirements on it will be similar but weaker than that of a Luzin scheme in [4, Theorem 22.21] because they will be sufficient for our purpose and the existence of such a scheme follows from published results even in the case of $\alpha = 1$ for general Polish spaces X and Y, as well as in the case of nonseparable complete metric space X and $\alpha \geq 1$. We recall some notation first.

If $\mathbb{D}$ is any set, $\mathbb{D}^{<\omega}$ denotes the set $\bigcup_{n=1}^\infty \mathbb{D}^n \cup \{\emptyset\}$ of finite sequences of elements of $\mathbb{D}$. For $s \in \mathbb{D}^{<\omega}$, the symbol $|s|$ stands for the length of s. We also write $s' \succ s$ if s' is a strict extension of s' (i.e., if s' extends s and $s' \neq s$). The abbreviation $\sigma|n$ stands for the sequence of first n members of any sequence, finite or infinite, σ of elements of $\mathbb{D}$ and we write $s^\wedge d$ for the sequence which begins by the finite sequence $s \in \mathbb{D}^{<\omega}$ followed by $d \in \mathbb{D}$.

Let us recall that we use the notation $\mathbb{N}$ for the set of positive integers and use the notation $C = \{0, 1\}^\mathbb{N}$ in what follows. We shall write C_i for the set $\{\iota \in C : \iota|n = i\}$, where $i \in \{0, 1\}^n$.

We denote by π_X, or π_Y, the projection mapping of $X \times Y$ to X, or Y, respectively.

Lemma 3.1. *Let B be a Borel subset of additive class $\mathbf{\Sigma}^0_\alpha$, $\alpha \geq 1$, in the product $Z = X \times Y$ of a complete metric space X and a Polish space Y. Then there are a set $\mathbb{D}$ (of sufficiently large cardinality) and sets B_s, $s \in \mathbb{D}^{<\omega}$, of the additive class $\mathbf{\Sigma}^0_\alpha$ in Z such that*

(a) $B_\emptyset = B$ *and* $\{B_{s^\wedge d} : d \in \mathbb{D}\}$ *is a (σ-discrete) cover of* B_s *for* $s \in \mathbb{D}^{<\omega}$ *such that the family* $\{\pi_X(B_{s^\wedge d}) : d \in \mathbb{D}\}$ *is σ-discrete;*
(b) $\bigcap\{\overline{B_{\sigma|n}} : n = 0, 1, \dots\} \subset B$ *for every* $\sigma \in \mathbb{D}^{\mathbb{N}}$*;*
(c) $\operatorname{diam} B_s \leq 2^{-|s|}$ *for* $s \in \mathbb{D}^{<\omega}$.

(We consider some fixed complete metrics on X *and* Y*, which are bounded by* 1*, and the corresponding maximum metric on* $X \times Y$*.)*

Proof. If X is separable, the existence of such a (countable) scheme of B follows from [4, Theorem 22.21] for $\alpha > 1$. In the general case of X and $\alpha \geq 1$, it follows from [7, Theorem 2.2], (a) implies (b), that there is a complete sequence of σ-discrete covers $\mathcal{C}_n$ of B by sets from $\mathbf{\Sigma}^0_\alpha(Z)$. (We need only to notice that B belongs to the multiplicative class $\alpha + 1$ in Z to get $B_s \in \mathbf{\Sigma}^0_\alpha(B)$ from the quoted result. Then $B_s \in \mathbf{\Sigma}^0_\alpha(Z)$ since $B \in \mathbf{\Sigma}^0_\alpha(Z)$.) Let us recall that the completeness means that each filter $\mathcal{F}$ in B, which fulfils $\mathcal{F} \cap \mathcal{C}_n \neq \emptyset$ for every $n \in \mathbb{N}$, has an accumulation point, i.e., $\bigcap\{\overline{F} : F \in \mathcal{F}\} \cap B \neq \emptyset$, where the closures may be understood in B or, equivalently, in Z. Replacing each cover by an arbitrary refinement we obtain a complete sequence of these refinements. Since the cover of the metric space Z by open balls of diameter at most 2^{-n} has a σ-discrete open refinement $\mathcal{R}_n$, the covers $\mathcal{C}_n \wedge \mathcal{R}_n$ are σ-discrete refinements of $\mathcal{C}_n$, they consist of sets of diameter at most 2^{-n}, and their elements belong to $\mathbf{\Sigma}^0_\alpha(Z)$. Due to [3, Lemma 2.1], we may also achieve that even the projections of elements of each cover form a σ-discrete family of sets of the same additive class. Let $\mathcal{C}^*_n$ be such refinements of covers $\mathcal{C}_n \wedge \mathcal{R}_n$. We may index each such cover by elements of a set $\mathbb{D}$ such that $\mathcal{C}^*_n = \{C^n_d : d \in \mathbb{D}\}$ (we may repeat some of the sets many times if the cardinality of $\mathcal{C}^*_n$ is smaller than that of $\mathbb{D}$). Put $B_{d_1,\dots,d_n} = C^1_{d_1} \cap \dots \cap C^n_{d_n}$. Note that the projections to X of the finite intersections $C^1_{d_1} \cap \dots \cap C^n_{d_n}$ are contained in the finite intersections $\pi_X(C^1_{d_1}) \cap \dots \cap \pi_X(C^n_{d_n})$, which form a σ-discrete family in X due to our remark before Lemma 2.1. Thus (a) holds. The condition (c) is obvious. The completeness of the sequence $(\mathcal{C}^*_n)$ implies the condition (b). The condition (b) means, due to (c), that if all the sets $B_{\sigma|n}$, $n \in \mathbb{N}$, are nonempty, then the intersection of their closures is a singleton in B. □

Note that our requirements above are weaker than those on a Luzin scheme in [4, Theorem 22.21], in particular we do not require the injectivity of the mapping $\sigma \in \mathbb{D}^{\mathbb{N}} \mapsto \bigcap_{n \in \mathbb{N}} \overline{B_{\sigma|n}}$.

We need the reduction theorem for families of Borel sets of an additive class in metric spaces.

Lemma 3.2. *Let* X_a*,* $a \in I$*, be a σ-discrete family of Borel sets of additive class* $\mathbf{\Sigma}^0_\alpha$*,* $\alpha > 1$*, in a metric space* X*. Then there are Borel sets* $X^*_a \subset X_a$ *for* $a \in I$ *of class* $\mathbf{\Sigma}^0_\alpha$ *in* X *which form a partition of* $\bigcup_{a \in I} X_a$*. The same holds for* $\alpha = 1$ *if* X *is* 0*-dimensional.*

Proof. For countable families it is proved in [5, Section 30, Subsection 7, Theorem 1 and Section 26, Subsection 2, Theorem 1]. If $I = \bigcup_{n \in \mathbb{N}} I_n$ with $\{X_a : a \in I_n\}$, discrete, we may consider the family of $X(n) = \bigcup\{X_a : a \in I_n\}$, which belong to $\mathbf{\Sigma}^0_\alpha(X)$, and choose a reduction $X^*(n) \subset X(n)$. Finally put $X^*_a = X_a \cap X^*(n)$ for $a \in I_n$. □

We are going to prove our main uniformization theorems now. The first one shows the existence of a selection, the next one shows that under a bit stronger assumptions there are continuum many disjoint uniformizations parametrized in a particular way. Although the proof of the first theorem can be understood from the second one, we prove the existence of one selection first to point out the main idea how we get our uniformization. The same procedure will be repeated in the proof of the subsequent theorem with some more technicalities. We point out that although we prove our uniformization results for sets of an additive class in a complete metric space, we get selections defined on a metric space which needs not to be complete in general. We need this observation in the proof of Corollary 3.10. We use $\widehat{X}$ to denote a completion of X in what follows.

Theorem 3.3. *Let X be a metric space and Y be a Polish space. Let $\mathcal{I}_x$, $x \in X$, be σ-ideals of subsets of Y such that*

$$\pi^*(A) := \{x \in X : A_x \notin \mathcal{I}_x\} \in \mathbf{\Sigma}^0_{\alpha^*}(X) \tag{1}$$

for $A \subset X \times Y$ in $\mathbf{\Sigma}^0_\alpha(X \times Y)$, where $2 \le \alpha \le \alpha^* < \omega_1$ are fixed. In case that X is 0-dimensional, we may assume $1 \le \alpha \le \alpha^* < \omega_1$.

Assume that

(2) *$B \subset X \times Y$ is in $\mathbf{\Sigma}^0_\alpha(\widehat{X} \times Y)$, and that $B_x \notin \mathcal{I}_x$ for every $x \in X^* := \pi_X(B)$.*

Then there is a $\mathbf{\Sigma}^0_{\alpha^*}(X)$-measurable selection $\xi : X^* \to Y$ of the mapping $x \mapsto B_x$. Its graph is a $\mathbf{\Pi}^0_{\alpha^*}$-measurable uniformization of B in $X \times Y$.

Proof. Let us consider some fixed complete metrics on $\widehat{X}$ and Y such that the diameters of X and Y are at most 1 for the corresponding metric and let us consider the maximum metric on $X \times Y$. Let B_s, $s \in \mathbb{D}^{<\omega}$, be a scheme of the set $B \subset \widehat{X} \times Y$ from Lemma 3.1 which exists due to the assumption (2).

We are now going to define a sequence of σ-discrete partitions $\mathcal{P}_{n-1} = \{X^*_s : s \in \mathbb{D}^{n-1}\}$, $n \in \mathbb{N}$, of X^* by induction. We require the following properties for every $n \in \mathbb{N}$.

(i) $\{X^*_{s^\wedge d} : d \in \mathbb{D}\}$ is a σ-discrete partition of X^*_s for $s \in \mathbb{D}^{n-1}$,
(ii) $\mathcal{P}_{n-1} \subset \mathbf{\Sigma}^0_{\alpha^*}(X)$,
(iii) $(B_s)_x \notin \mathcal{I}_x$ for $x \in X^*_s$ and $s \in \mathbb{D}^{n-1}$.

Put $\mathcal{P}_0 = \{X^*_\emptyset\}$, where $X^*_\emptyset = \pi^*(B) = \pi_X(B)$ due to (2). The property (i) is obviously fulfilled for $\mathcal{P}_0$, which also fulfils (ii) and (iii) due to assumptions (1) and (2).

Given the partition $\mathcal{P}_{n-1} = \{X^*_s : s \in \mathbb{D}^{n-1}\}$ of X^* fulfilling (i) to (iii) for some $n \in \mathbb{N}$, we consider a fixed $s \in \mathbb{D}^{n-1}$. We realize that the sets $X_{s^\wedge d} = X^*_s \cap \pi^*(B_{s^\wedge d})$, $d \in \mathbb{D}$, form a σ-discrete cover of X^*_s by $\mathbf{\Sigma}^0_{\alpha^*}(X)$ sets using that (iii) is fulfilled, each $\mathcal{I}_x$ is a σ-ideal, and the sets $(B_{s^\wedge d})_x$, $d \in \mathbb{D}$, form a countable cover of the separable set $(B_s)_x$ (since they form a σ-discrete family, only countably many of them have a nonempty intersection with $(B_s)_x$). Using Lemma 3.2, we find pairwise disjoint sets $X^*_{s^\wedge d} \subset X_{s^\wedge d}$, $d \in \mathbb{D}$, which cover X^*_s and put $\mathcal{P}_n = \{X^*_{s^\wedge d} : s \in \mathbb{D}^{n-1},\ d \in \mathbb{D}\}$. It is clearly σ-discrete. Thus the existence of $\mathcal{P}_{n-1}$'s fulfilling (i) to (iii) is proved.

Now, given $x \in X^*$, there is a uniquely determined $s^x \in \mathbb{D}^{\mathbb{N}}$ such that $x \in X^*_{s^x|n}$ for all $n \in \mathbb{N}$. Define $\xi(x) \in \bigcap\{\overline{(B_{s^x|n})_x} : n \in \mathbb{N}\}$ for $x \in X^*$. The definition is correct and $\xi(x) \in B_x$ for every $x \in X^*$. Indeed, the sets $\overline{(B_{s^x|n})_x}$, $n \in \mathbb{N}$, form a descreasing sequence of nonempty (by (iii)) closed sets in Y with diameter converging to zero by the property (c) of Lemma 3.1. By the completeness of the considered metric, the value $\xi(x)$ is uniquely defined. By the property (b) of the scheme B_s, $s \in \mathbb{D}^n$, we get $\xi(x) \in B_x$. To verify that ξ is $\mathbf{\Sigma}^0_{\alpha^*}(X)$-measurable, define ξ_n to be constant on each X^*_s, $s \in \mathbb{D}^n$. Put, e.g., $\xi_n(x)$ for $x \in X^*_s (= X^*_{s^x|n})$ to be an arbitrarily chosen element of the projection of $B_s (= B_{s^x|n})$ to Y. Then each ξ_n is $\mathbf{\Sigma}^0_{\alpha^*}(X)$-measurable and it converges uniformly to ξ due to the condition (c) on the diameters of the sets B_s from the scheme of Lemma 3.1. So ξ is $\mathbf{\Sigma}^0_{\alpha^*}(X)$-measurable as well (see [5, Section 31, Subsection 8, Theorem 2]). The graph of ξ is in $\mathbf{\Pi}^0_{\alpha^*}$ by [5, Section 31, Subsection 7, Theorem 1]. This concludes the proof. □

The next theorem is a strengthening of the previous one under the assumption that the σ-ideals $\mathcal{I}_x$ contain singletons. It is inspired by and might be compared with Mauldin's [8, Theorem 1.1].

Theorem 3.4. *Let X, Y, $\mathcal{I}_x$ for $x \in X$, and $B \subset X \times Y$ fulfil the assumptions of Theorem 3.3.*

If, moreover, $\mathcal{I}_x$ contains all singletons in B_x for each $x \in X$, then there is a Borel isomorphism Ξ of $X^* \times C$ onto $R \subset B$, with $X^* := \pi_X(B)$, which is of the form $\Xi(x, \iota) = (x, \Phi(x, \iota))$, such that

(a) *$\Phi(\cdot, \iota)$ is a $\mathbf{\Sigma}^0_{\alpha^*}(X)$-measurable selection of $x \in X^* \mapsto B_x$ for every $\iota \in C$;*
(b) *$\Phi(x, \cdot)$ is a homeomorphism of C onto R_x for every $x \in X^*$;*
(c) *$\Xi : X^* \times C \to R$ is $\mathbf{\Sigma}^0_{\alpha^*}(X \times Y)$-measurable and $\Xi^{-1} : R \to X \times C$ is $\mathbf{\Sigma}^0_{\alpha^*}(R)$-measurable;*
(d) *$F : x \in X^* \mapsto R_x$ is both upper and lower $\mathbf{\Sigma}^0_{\alpha^*}(X)$-measurable;*
(e) *$R = \Xi(X^* \times C)$ is in $\mathbf{\Pi}^0_{\alpha^*}(X \times Y)$.*

Proof. The notions of distance and diameter are related to the maximum metric, defined using complete metrics on $\widehat{X}$ and Y giving diameter less than one as in the proof of Theorem 3.3. Let B_s, $s \in \mathbb{D}^{<\omega}$, be a scheme of the set $B \subset X \times Y$ from Lemma 3.1 which exists due to the assumption (2).

We are now going to define a sequence of partitions of X^* by induction. The elements of the n-th partition will be indexed by the elements of the set $\mathcal{T}_n$ of mappings ("strategies") τ of $\{0,1\}^{\le n}$ to $\mathbb{D}^{<\omega}$ satisfying the properties

(A) $\tau(i^\wedge j) \succ \tau(i)$ for $i \in \{0,1\}^{<n}$ and $j \in \{0,1\}$ if $n \in \mathbb{N}$, and
(B) the sets $\overline{B_{\tau(i)}}$, $i \in \{0,1\}^n$, are pairwise disjoint.

We write $\mathcal{T}_0$ for the singleton which contains just the mapping $\tau_0 : \emptyset \mapsto \emptyset$. The elements of the n-th partition will be denoted by $X^*_n(\tau)$, where $\tau \in \mathcal{T}_n$. We denote by $\mathcal{T}_n(\tau)$ the set of elements of $\mathcal{T}_n$ which extend $\tau \in \mathcal{T}_{n-1}$, i.e.,

$$\mathcal{T}_n(\tau) := \{\tau' \in \mathcal{T}_n,\ \tau' \restriction \{0,1\}^{\le(n-1)} = \tau\}$$

for $n \in \mathbb{N}$. We require the following properties for every $\tau \in \mathcal{T}_n$ and $n = 0, 1, \ldots$.

(i) Each $X^*_n(\tau)$ is in $\mathbf{\Sigma}^0_{\alpha^*}(X)$;
(ii) $X^*_n(\tau) = \bigcup\{X^*_{n+1}(\tau') : \tau' \in \mathcal{T}_{n+1}(\tau)\}$;
(iii) $(B_{\tau(i)})_x \notin \mathcal{I}_x$ for $x \in X^*_n(\tau)$ and $i \in \{0,1\}^n$.

Put $X^*_0(\tau_0) = \pi^*(B)$ (cf. (1) from Theorem 3.3 for the notation). It is in $\mathbf{\Sigma}^0_{\alpha^*}(X)$ and it is equal to X^* by our assumptions (1) and (2). Thus the partition $\{X^*_0(\tau_0)\}$ of X^* fulfils (i) and (iii). Condition (ii) requires nothing for $n = 0$.

Given the partition $\{X^*_{n-1}(\tau) : \tau \in \mathcal{T}_{n-1}\}$ fulfilling (i) to (iii) for some $n \in \mathbb{N}$, we will consider a cover of each $X^*_{n-1}(\tau)$ with $\tau \in \mathcal{T}_{n-1}$ fixed. For every $\tau' \in \mathcal{T}_n(\tau)$, put

$$X_n(\tau') = X^*_{n-1}(\tau) \cap \bigcap\{\pi^*(B_{\tau'(i)}) : i \in \{0,1\}^n\}.$$

Observe that the family $\{X_n(\tau') : \tau' \in \mathcal{T}_n(\tau)\}$ is a cover of $X^*_{n-1}(\tau)$ for every $\tau \in \mathcal{T}_{n-1}$. Indeed, let $x \in X^*_{n-1}(\tau)$. So, by (iii) and (B), $(B_{\tau(i)})_x \notin \mathcal{I}_x$ for every $i \in \{0,1\}^{n-1}$ and the sets $\overline{B_{\tau(i)}}$, $i \in \{0,1\}^{n-1}$, are pairwise disjoint. Since $(B_{\tau(i)})_x \notin \mathcal{I}_x$, there are two distinct points $y_0(x,i), y_1(x,i) \in (B_{\tau(i)})_x$, for each fixed $i \in \{0,1\}^{n-1}$, such that $U \cap (B_{\tau(i)})_x \notin \mathcal{I}_x$ for every neighbourhood U of $y_0(x,i)$, and $y_1(x,i)$, respectively. (Otherwise, if there is at most one such point, there would be a cover of the separable metric space $(B_{\tau(i)})_x$ by countably many elements of $\mathcal{I}_x$, namely countably many neighbourhoods of some points in $(B_{\tau(i)})_x$ and at most one singleton in B_x, a contradiction with $(B_{\tau(i)})_x \notin \mathcal{I}_x$. Here we use the extra assumption on $\mathcal{I}_x$.) Let us choose open neighbourhoods $U_j(x,i)$ of $y_j(x,i)$ for $j = 0,1$ of diameters less than $\frac{1}{3}\mathrm{dist}\,(y_0(x,i), y_1(x,i))$. Since the sets $(B_{\tau(i)})_x \cap U_j(x,i)$, $j = 0,1$, are not in $\mathcal{I}_x$ and they are covered by the countably many sets $(B_s)_x \subset (B_{\tau(i)})_x$ with $s \succ \tau(i)$ for which $(B_s)_x \subset U_j(x,i)$ and $2^{-|s|} \le \frac{1}{3}\mathrm{dist}\,(y_0(x,i), y_1(x,i))$ by Lemma 3.1, (a) and (c), we find $\tau_x(i^\wedge j) \succ \tau(i)$ such that $2^{-|\tau_x(i^\wedge j)|} \le \frac{1}{3}\mathrm{dist}\,(y_0(x,i), y_1(x,i))$, $(B_{\tau_x(i^\wedge j)})_x \subset U_j(x,i)$, and $(B_{\tau_x(i^\wedge j)})_x \notin \mathcal{I}_x$ for $j = 0,1$. The sets $\overline{B_{\tau_x(i^\wedge j)}}$ are disjoint, since $\overline{B_{\tau_x(i^\wedge j)}} \subset \overline{B_{\tau(i)}}$, the sets $\overline{B_{\tau(i)}}$ form a pairwise disjoint family, and $\mathrm{dist}\,(\overline{B_{\tau_x(i^\wedge 0)}}, \overline{B_{\tau_x(i^\wedge 1)}}) \ge \frac{1}{3}\mathrm{dist}\,(y_0(x,i), y_1(x,i)) > 0$. Put $\tau_x \restriction \{0,1\}^{(n-1)} = \tau$. Now $x \in X_n(\tau_x)$ and $\tau_x \in \mathcal{T}_n(\tau)$.

Let us check that $X_n(\tau') \in \mathbf{\Sigma}^0_{\alpha^*}(X)$ for all $\tau' \in \mathcal{T}_n$. Having that $X^*_{n-1}(\tau) \in \mathbf{\Sigma}^0_{\alpha^*}(X)$ for $\tau \in \mathcal{T}_{n-1}$ by (i), we see that each $X_n(\tau') = X^*_{n-1}(\tau) \cap \bigcap\{\pi^*(B_{\tau'(i)}) : i \in \{0,1\}^n\}$ is in $\mathbf{\Sigma}^0_{\alpha^*}(X)$ as a finite intersection of elements of $\mathbf{\Sigma}^0_{\alpha^*}(X)$ by (1) for $\tau' \in \mathcal{T}_n(\tau)$.

By Lemma 3.1(a) the family $\mathcal{D} = \{\pi(B_s) : s \in \mathbb{D}^{<\omega}\}$ is σ-discrete (it is not difficult to show by induction that all $\{\pi(B_s) : s \in \mathbb{D}^n\}$ are σ-discrete using (a) and realize that $\mathcal{D}$ is the countable union of these families). Since $\pi^*(B_s) \subset \pi(B_s)$, also the family $\mathcal{D}^* = \{\pi^*(B_s) : s \in \mathbb{D}^{<\omega}\}$ is σ-discrete. Therefore also $\mathcal{D}_1 = \mathcal{D}^*$, $\mathcal{D}_2 = \mathcal{D}^* \wedge \mathcal{D}^*$,$\mathcal{D}_3 = \mathcal{D}^* \wedge \mathcal{D}^* \wedge \mathcal{D}^*$, ... are σ-discrete as well as their (countable) union $\mathcal{E} = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n$. The family $\{\bigcap\{\pi^*(B_{\tau'(i)}) : i \in \{0,1\}^n\} : \tau' \in \mathcal{T}_n(\tau)\}$ is σ-discrete as a subfamily of $\mathcal{E}$. Finally, the family $\{X^*_{n-1}(\tau) \cap \bigcap\{\pi^*(B_{\tau'(i)}) : i \in \{0,1\}^n\} : \tau' \in \mathcal{T}_n(\tau)\}$ is σ-discrete. We proved above that it is a cover of $X^*_{n-1}(\tau)$.

Hence we may find a σ-discrete partition $\{X^*_n(\tau') : \tau' \in \mathcal{T}_n(\tau)\}$ of $X^*_{n-1}(\tau)$ consisting of elements of $\mathbf{\Sigma}^0_{\alpha^*}(X)$ applying Lemma 3.2 to the family $\{X_n(\tau') : \tau' \in \mathcal{T}_n(\tau)\}$.

Thus we have for every $\tau \in \mathcal{T}_n$ an $X^*_n(\tau)$ such that the conditions (i)-(iii) are satisfied.

The requirement on $\tau \in \mathcal{T}_n$ to satisfy $x \in X^*_n(\tau)$ defines uniquely a $\tau = \tau^x_n \in \mathcal{T}_n$ for $x \in X^*$. Due to (ii) there is a uniquely determined $\tau^x : \{0,1\}^{<\omega} \to \mathbb{D}^{<\omega}$ with $\tau^x_n = \tau^x \restriction \{0,1\}^{\le n}$ for every $n = 0,1,\dots$. We define the required mapping $\Xi : X \times C \to B$ by the relation $\Xi(x,\iota) \in \bigcap_{n=0}^{\infty} \overline{B_{\tau^x(\iota|n)}}$. This defines indeed a mapping of $X \times C$ to B due to the fact that the sets $B_{\tau^x(\iota|n)}$ form a decreasing sequence of nonempty sets (by (iii)) with $\bigcap_{n=0}^{\infty} \overline{B_{\tau^x(\iota|n)}}$ a singleton in B by the properties (b) and (c) of our scheme from Lemma 3.1. Moreover, all the sets $B_{\tau^x(\iota|n)}$ have nonempty x-section $(B_{\tau^x(\iota|n)})_x$ by (iii), i.e., the intersection of $B_{\tau^x(\iota|n)}$ with the closed set $\{x\} \times Y \subset X \times Y$ is nonempty, and so $\bigcap_{n\in\mathbb{N}} \overline{B_{\tau^x(\iota|n)}} \cap (\{x\} \times Y) \ne \emptyset$. Thus this intersection is equal to the singleton $\{\Xi(x,\iota)\} = \bigcap_{n\in\mathbb{N}} \overline{B_{\tau^x(\iota|n)}}$, and $\Xi(x,\iota) \in \{x\} \times B_x$. So Ξ is of the form $\Xi(x,\iota) = (x, \Phi(x,\iota))$ with $\Phi : X \times C \to Y$ uniquely determined by $\Phi(x,\iota) \in \bigcap_{n\in\mathbb{N}} \overline{(B_{\tau^x(\iota|n)})_x} = (\bigcap_{n\in\mathbb{N}} \overline{B_{\tau^x(\iota|n)}})_x$.

It remains to show that Ξ has the required properties (a) - (e).

We prove (a) by constructing a sequence of $\mathbf{\Sigma}^0_{\alpha^*}$-measurable mappings Φ_n which converges uniformly to Φ. We define Φ_n to attain a constant value from $\pi_Y(B_{\tau(i)})$ on $X^*_n(\tau)\times C_i$ for every $\tau\in\mathcal{T}_n$ and $i\in\{0,1\}^n$. This is possible since each $(B_{\tau(i)})_x\notin\mathcal{I}_x$ if $x\in X^*_n(\tau)$ by (iii) and so $B_{\tau(i)}\neq\emptyset$.

Let W be open in Y. Then

$$\Phi_n^{-1}(W)=\bigcup\{X^*_n(\tau)\times C_i:\tau\in\mathcal{T}_n,\ i\in\{0,1\}^n,\ \Phi_n(X^*_n(\tau)\times C_i)\subset W\}$$

because the mapping Φ_n is constant on the sets $X^*_n(\tau)\times C_i$. Since the σ-discrete union of sets $X^*_n(\tau)\times C_i$ over any subset of pairs of $\tau\in\mathcal{T}_n$ and $i\in\{0,1\}^n$ is in $\mathbf{\Sigma}^0_{\alpha^*}(X\times C)$, the mapping Φ_n is $\mathbf{\Sigma}^0_{\alpha^*}(X\times C)$-measurable. The mappings Φ_n converge to Φ uniformly (in both x and ι) since the diameter of each $\pi_Y(B_{\tau(i)})$ is at most 2^{-n}. By [5, Section 31, Subsection 8, Theorem 2], Φ is also $\mathbf{\Sigma}^0_{\alpha^*}(X\times C)$-measurable. In particular, (a) is proved.

The mapping $\Phi(x,\cdot)$ is one-to-one and continuous since $\overline{B_{\tau^x(i)}}\cap\overline{B_{\tau^x(i')}}=\emptyset$ for $i\neq i'$, $i,i'\in\{0,1\}^n$, and since $\operatorname{diam}\pi_Y(B_{\tau^x(i)})\leq 2^{-n}$ for $i\in\{0,1\}^n$. Hence (b) follows by the compactness of C.

Let U be open in X and W be open in Y. Then $\Xi^{-1}(U\times W)=(U\times Y)\cap\Phi^{-1}(W)$. Thus, to prove that Ξ is $\mathbf{\Sigma}^0_{\alpha^*}(X\times C)$-measurable, it is enough to show that Φ is $\mathbf{\Sigma}^0_{\alpha^*}(X\times C)$-measurable. However, this was proved above. So the first claim of (c) is verified.

We have that Ξ is injective by (b). To show that Ξ^{-1} is $\mathbf{\Sigma}^0_{\alpha^*}(R)$-measurable, note that $\Xi(U\times C_i)=(U\times Y)\cap\Xi(X^*\times C_i)$ for U open in X and $i\in\{0,1\}^n$. Thus we need to prove that $\Xi(X^*\times C_i)$ is in $\mathbf{\Sigma}^0_{\alpha^*}$ in R for each $n\in\mathbb{N}$ and $i\in\{0,1\}^n$. The point $(x,y)\in R$ belongs to $\Xi(X^*\times C_i)$ if and only if $(x,y)\in\overline{B_{\tau^x(i)}}$ and $x\in X^*_n(\tau^x_n)$. Thus $\Xi(X^*\times C_i)=R\cap\bigcup\{(X^*_n(\tau))\times Y)\cap\overline{B_{\tau(i)}}:\tau\in\mathcal{T}_n\}$. This is a union of a σ-discrete family of elements of $\mathbf{\Sigma}^0_{\alpha^*}(X\times Y)$ intersected with R, and so an element of $\mathbf{\Sigma}^0_{\alpha^*}$ in R. This concludes the verification of (c).

Let $W\subset Y$ be open and nonempty. If $x\in F^{-1}(W)$ then there is a $y=\Phi(x,\iota)\in W$ for some $\iota\in C$. Thus the distance δ of y and W^c is positive and $\{y\}=\bigcap_{n\in\mathbb{N}}(\overline{B_{\tau^x(\iota|n)}})_x\subset W$. So there is an $n\in\mathbb{N}$ such that the diameter of $B_{\tau^x(\iota|n)}$ is less than δ and $(x,y)\in\overline{B_{\tau^x(\iota|n)}}\subset X\times W$. Having n and ι such that $\overline{B_{\tau^x(\iota|n)}}\subset X\times W$, we have $\{y\}=\bigcap_{n\in\mathbb{N}}(\overline{B_{\tau^x(\iota|n)}})_x\subset W$, and thus $y\in W$. Therefore $x\in F^{-1}(W)$ if and only if there are $n\in\mathbb{N}$ and $i\in\{0,1\}^n$ such that $\pi_Y(\overline{B_{\tau^x(i)}})\subset W$. Thus

$$F^{-1}(W)=\bigcup\{X^*_n(\tau):n\in\mathbb{N},\ \tau\in\mathcal{T}_n,\ i\in\{0,1\}^n,\ \pi_Y(\overline{B_{\tau(i)}})\subset W\}.$$

So F is lower $\mathbf{\Sigma}^0_{\alpha^*}(X)$-measurable.

Let the compact set $\Phi(x,C)$ be a subset of W. Thus it has a positive distance from W^c and so $\overline{B_{\tau^x(\iota|n)}}\subset W$ for sufficiently large $n\in\mathbb{N}$ and every $\iota\in C$. Obviously, if the latter condition holds, then $\Phi(x,C)\subset W$. Thus

$$F_{-1}(W)=\bigcup\{X^*_n(\tau):n\in\mathbb{N},\ \tau\in\mathcal{T}_n,\ \pi_Y(\overline{B_{\tau(i)}})\subset W\text{ for every }i\in\{0,1\}^n\}.$$

This is again a σ-discrete union of elements of $\mathbf{\Sigma}^0_{\alpha^*}(X)$ and so F is also upper $\mathbf{\Sigma}^0_{\alpha^*}(X)$-measurable, and (d) is proved.

Let $\mathcal{W}$ be a countable base for the topology of Y. We may easily check that $(x,y)\notin R$ if and only if there is a $W\in\mathcal{W}$ such that $y\in W$, $F(x)\subset Y\setminus\overline{W}$ since $F(x)$

is closed. As F is upper $\mathbf{\Sigma}^0_{\alpha^*}(X)$-measurable, the set $R^c = \bigcup_{W\in\mathcal{W}} F_{-1}(Y\setminus\overline{W})\times W$ is in $\mathbf{\Sigma}^0_{\alpha^*}(X\times Y)$, and (e) is also proved. □

As corollaries of Theorems 3.3 and 3.4, and Lemmas 2.1 and 2.2 on the "generalized projections" π^*_μ and π^*_F from the previous section, we get the following results.

Theorem 3.5. *Let X be a metrizable space, Y be a Polish space, $\mu : X\times\mathcal{B}(Y)\to[0,1]$ be such that*

(a) *$\mu(x,\cdot)$ is a Borel probability on Y for every $x\in X$, and*
(b) *$\{x\in X : \mu(\cdot,H) > r\}$ is in $\mathbf{\Sigma}^0_{\alpha_0}(X)$ ($1\le\alpha_0<\omega_1$) for every open $H\subset Y$ and $r\in\mathbb{R}$.*

Let $B\subset X\times Y$ be in $\mathbf{\Sigma}^0_\alpha(\widehat{X}\times Y)$, $2\le\alpha<\omega_1$, or $1\le\alpha<\omega_1$ if X is 0-dimensional. Let $\mu(x,B_x)>0$ for every $x\in\pi_X(B)$.

Let α^* be as in Lemma 2.1. In particular, if $\mu(x,\cdot)=\mu$, then $\alpha^*=\alpha$.

Then there is a $\mathbf{\Sigma}^0_{\alpha^*}(X)$-measurable mapping $\xi:\pi_X(B)\to Y$ such that its $\mathbf{\Pi}^0_{\alpha^*}$-measurable graph is a uniformization of B.

If, moreover,

(c) *$\mu(x,\cdot)$ does not have atoms for every $x\in X$,*

then there is a mapping $\Xi:\pi_X(B)\times Y\to B$ as in Theorem 3.4.

Proof. Put $\mathcal{I}_x=\{N\subset B_x : \mu(x,N)=0\}$ for every $x\in X$. By Lemma 2.1, we have $\pi^*(A)=\pi^*_\mu(A,0)\in\mathbf{\Sigma}^0_{\alpha^*}(X)$ for every $A\in\mathbf{\Sigma}^0_\alpha(X\times Y)$, so we may apply Theorems 3.3 and 3.4. □

Theorem 3.6. *Let X be a metrizable space, Y be a Polish space.*

Let $F:X\to\mathcal{P}(Y)$ be such that

(a) *$F(x)$ is a Baire subspace of Y for every $x\in X$, and*
(b) *F is lower $\mathbf{\Sigma}^0_{\alpha_0}$-measurable ($1\le\alpha_0<\omega_1$).*

Let $B\subset$ graph F be in $\mathbf{\Sigma}^0_\alpha(\widehat{X}\times Y)$, $2\le\alpha<\omega_1$, or $1\le\alpha<\omega_1$ and X 0-dimensional. Let B_x be non-meager in $F(x)$ for $x\in\pi_X(B)$.

Let α^* be as in Lemma 2.1. In particular, if $F(x)=Y$, then $\alpha^*=\alpha$.

Then there is a $\mathbf{\Sigma}^0_{\alpha^*}(X)$-measurable mapping $\xi:\pi_X(B)\to Y$ such that its $\mathbf{\Pi}^0_{\alpha^*}$-measurable graph is a uniformization of B.

If moreover

(c) *$F(x)$ has no isolated point for every $x\in X$,*

then there is a mapping $\Xi:\pi_X(B)\times C\to B$ as in Theorem 3.4.

Proof. Put $\mathcal{I}_x=\{N\subset B_x : N \text{ is meager in } B_x\}$ for $x\in X$. Now $\pi^*(A)=\pi^*_F(A,Y)\in\mathbf{\Sigma}^0_{\alpha^*}(X)$ for every $A\in\mathbf{\Sigma}^0_\alpha(X\times Y)$ by Lemma 2.2. Finally, we apply Theorems 3.3 and 3.4. □

As a particular case, we point out a refinement of the theorem of Srivastava for uniformizations of Borel sets with G_δ sections ([12, Theorem 4.1]).

Corollary 3.7. *Let X be a metric and Y a Polish space. Assume that $B\subset X\times Y$ is a Borel subset of additive class $\mathbf{\Sigma}^0_\alpha$ in $\widehat{X}\times Y$, $2\le\alpha<\omega_1$, or $1\le\alpha<\omega_1$ if X is 0-dimensional. Let the sections B_x, $x\in X$, be G_δ in Y and the mapping $F:x\mapsto B_x$ be lower $\mathbf{\Sigma}^0_{\alpha_0}(X)$-measurable. Then there is a $\mathbf{\Sigma}^0_{\alpha^*}(X)$-measurable selection of F, where α^* is as in Lemma 2.1.*

Proof. As G_δ subsets of the Polish space Y, the sets B_x are Baire spaces and, in particular, they are not meager in themselves for $x \in \pi_X(B)$. So we may put $F(x) = B_x$ and apply Theorem 3.6. □

We show in the following example that the class of the selection mapping ξ in Theorems 3.5 and 3.6 cannot be improved, even its graph (the uniformization) cannot be of a lower multiplicative class for sets of nonlimit ambiguous class higher than 1.

Example 3.8. *There is a $\mathbf{\Delta}^0_{\alpha+1}$ subset B of $\mathbb{R}^2$ for every $\alpha \in [1, \omega_1)$ such that $\lambda(B_x) > 0$ and B_x is nonmeagre for every $x \in \mathbb{R}$, but there is no $\mathbf{\Pi}^0_\alpha$ uniformization of B.*

Proof. We use the notation $X = Y = Z = \mathbb{R}$. Let $\alpha \in [1, \omega_1)$.

Put $U \subset X \times (Y \times Z)$ to be a $\mathbf{\Pi}^0_\alpha$ universal set for $\mathbf{\Pi}^0_\alpha$ sets in $Y \times Z$ (e.g., $U \subset C \times (Y \times Z)$ in $\mathbf{\Pi}^0_\alpha(C \times Y \times Z)$ by Kechris, Theorem 22.3 is $\mathbf{\Pi}^0_\alpha$ in $X \times Y \times Z$ as well).

Let $A = \{(x,x) \in X \times Y : \lambda(U^c_{(x,x)}) > 0 \text{ and } U^c_{(x,x)} \text{ is nonmeagre in } \mathbb{R}\}$. It is $\mathbf{\Sigma}^0_\alpha$ in $D = \{(x,x) : x \in X\}$ by the lemmas on generalized projections. Put $B = ([A \times Z] \setminus U) \cup ([D \setminus A] \times Z)$.

B is in $\mathbf{\Delta}^0_{\alpha+1}(D \times Z)$ and $\lambda(B_{(x,x)}) > 0$ and B_x is nonmeagre for every $x \in \mathbb{R}$ by the definition of B.

Let G be a uniformization of B in $\mathbf{\Pi}^0_\alpha(D \times Z)$.

By the definition of a uniformization, $G_{(x,x)} \neq \emptyset$ for every $(x,x) \in D$ since $B_{(x,x)} \neq \emptyset$ for every $(x,x) \in D$.

Since U is universal, there is $x \in X$ such that $U_x = \pi_{Y\times Z}(G)$. So $U_{(x,y)} = G_{(y,y)}$ for every $y \in Y$, and in particular $U_{(x,x)} = G_{(x,x)}$ is a singleton. Thus $(x,x) \in A$. Therefore $G_{(x,x)} \subset B_{(x,x)} \subset U^c_{(x,x)}$.

It follows that $\emptyset \neq G_{(x,x)} = U_{(x,x)} \subset U^c_{(x,x)}$, a contradiction.

As D can be identified with $\mathbb{R}$, B is our example. □

We give still a trivial example showing that we cannot get a selection with $\mathbf{\Pi}^0_1$ measurable graph for all sets of the multiplicative class $\mathbf{\Pi}^0_1$ which have large sections neither in the case of μ being the normalized Lebesgue measure on $Y := [-2, 2]$, nor in the case of sections of second category in Y, for 0-dimensional X. So, to get a uniformization of B in $\mathbf{\Pi}^0_1$, the assumption that B is of class $\mathbf{\Sigma}^0_1$ seems to be the optimal one.

Example 3.9. *Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, $Y = [-2, 2]$. The set $B = \{(\frac{1}{n}, y) \in X \times Y : n \in \mathbb{N},\ y \in [(-1)^n - \frac{1}{n}, (-1)^n + \frac{1}{n}]\} \cup (\{0\} \times [-1, 1]$, is closed, the space X is 0-dimensional, and there is no closed uniformization. The sets B_x are intervals in Y, so they are of positive measure μ and nonmeager in Y.*

Our result gives also an estimate, probably not the best possible one, on the Borel class for selectors of partitions to "relatively large sets". The first assumption (a) gives an improvement of Srivastava's theorem on selectors for partitions to G_δ sets (see [12, Theorem 5.1]).

Corollary 3.10. *Let Y be a Polish space and $\mathcal{P}$ be a partition of Y such that either*

(a) *P is a nonempty set which is residual in $\overline{P}$ for every $P \in \mathcal{P}$, or*
(b) *$\mu(P) > \frac{1}{2}\mu(\overline{P})$ for every $P \in \mathcal{P}$, with respect to some fixed Borel probability on Y*

such that the set $W^* := \bigcup\{P \in \mathcal{P} : P \cap W \neq \emptyset\}$ *is in* $\mathbf{\Delta}^0_\alpha$ *for every* $W \in \mathcal{W}$*, where* $\mathcal{W}$ *is a countable open base of* Y *and* $1 \leq \alpha < \omega_1$ *is fixed.*

Then there is a selector mapping $s : Y \to Y$ $(s(P) = \{y\} \subset P$*, for every* $P \in \mathcal{P})$ *which is* $\mathbf{\Sigma}^0_{\alpha+\alpha+1}$*-measurable.*

There is also a selector set $S \subset Y$ *(the cardinality of each* $S \cap P$ *for* $P \in \mathcal{P}$ *is one) of class* $\mathbf{\Pi}^0_{\alpha+\alpha+1}$*.*

Proof. Let $\mathcal{W} = \{W_n : n \in \mathbb{N}\}$. Let $f : Y \to C = \{0,1\}^{\mathbb{N}}$ be the characteristic function of the family $(W^*_n : n \in \mathbb{N})$, i.e., $f(y) = (i_1, i_2, \dots)$, where $i_n = 1$ if and only if $y \in W^*_n$. This mapping is clearly $\mathbf{\Sigma}^0_\alpha$-measurable, so its graph is in $\mathbf{\Pi}^0_\alpha(Y \times C)$, and thus also in $\mathbf{\Sigma}^0_{\alpha+1}(Y \times C)$. We observe that the following claim holds.

Claim. The equality $f(y) = f(y')$ is equivalent to y and y' lying in the same element of $\mathcal{P}$.

Indeed, if $y, y' \in P \in \mathcal{P}$, then $f(y) = f(y')$. If $y \in P \in \mathcal{P}$ and $y' \in P' \in \mathcal{P}$, where $P \neq P'$, then $P' \setminus \overline{P} \neq \emptyset$ or $P \setminus \overline{P'} \neq \emptyset$ since otherwise P or P' is meager in $\overline{P} = \overline{P'}$ in the case (a), and P or P' is of measure at most $\frac{1}{2}\mu(\overline{P}) = \frac{1}{2}\mu(\overline{P'})$ in the other case (b), which is a contradiction with (a), or (b), respectively. Thus there is a $W_n \in \mathcal{W}$ such that $P' \subset W^*_n$ and $P \cap W^*_n = \emptyset$, or vice versa. This implies that $f(y)$ and $f(y')$ have distinct n-th coordinates.

To prove the case (a) we consider the inverse $f^{-1} : f(Y) \to \mathcal{P}(Y)$. It is a multivalued mapping which is lower semicontinuous (lower $\mathbf{\Sigma}^0_1$-measurable) on $X := f(Y)$ because $f(W_n) = f(W^*_n) = \{\iota \in f(Y) : \iota_n = 1\}$ for every $n \in \mathbb{N}$. This follows easily by the previous Claim. Thus the multivalued mapping $F : X \to \mathcal{P}(Y)$ defined by $F(x) = \overline{f^{-1}(x)}$ is also lower semicontinuous on X. The subspaces $F(x)$ are closed (and nonempty) in the Polish space Y for $x \in X$, so they are Baire spaces. Let $\mathcal{I}_x$ denote the σ-ideal of meager subsets of $F(x)$ for every $x \in X$. Lemma 2.2 gives that the condition (1) of Theorem 3.3 holds for every $1 \leq \alpha = \alpha^* < \omega_1$.

We put $B = \operatorname{graph} f^{-1}$. It is in $\mathbf{\Sigma}^0_{\alpha+1}(C \times Y)$ since $\operatorname{graph} f \in \mathbf{\Sigma}^0_{\alpha+1}(Y \times C)$. The sets $B_x = f^{-1}(x)$ are nonempty residual in $F(x)$ for every $x \in X$ by (a). So the assumption (2) of Theorem 3.3 is fulfilled with $\alpha + 1$ instead of α.

To prove the case (b) put $B = \operatorname{graph} f^{-1} \in \mathbf{\Sigma}^0_{\alpha+1}(C \times Y)$ again and $\mathcal{I}_x = \mathcal{I}$ to be the σ-ideal of $\mu = \mu(x, \cdot)$ null sets for $x \in X = f(Y)$. Using Lemma 2.1, we get (1) of Theorem 3.3 for all $1 \leq \alpha = \alpha^* < \omega_1$ again. Each B_x is of positive measure $\mu = \mu(x, \cdot)$ for $x \in X = f(Y)$ by (b). Thus (2) of Theorem 3.3 is also satisfied, and we may use its conclusion also in this case with $\alpha + 1$ instead of α.

In both cases, due to Theorem 3.3, there is a selection ξ for $f^{-1}(x) = B_x$ which is $\mathbf{\Sigma}^0_{\alpha+1}(X)$-measurable. The mapping $s : y \mapsto \xi(f(y))$ is $\mathbf{\Sigma}^0_{\alpha+\alpha+1}$-measurable by [5, Section 31, Subsection 3, Theorem 2].

The point $s(y)$ is an element of the set $P \in \mathcal{P}$ which contains y by the above Claim. Thus s is a selector for the partition and the set $S = \{y \in Y : y = s(y)\}$ is a selector set for the partition $\mathcal{P}$ which is of class $\mathbf{\Pi}^0_{\alpha+\alpha+1}$ in Y. Indeed, the mapping $(y, z) \in Y \times Y \mapsto (s(y), z) \in Y \times Y$ is $\mathbf{\Sigma}^0_{\alpha+\alpha+1}(Y \times Y)$-measurable (it suffices to check it on the preimages of sets of the form $W_n \times W_{n'}$) and S is the preimage of the closed diagonal $\{(y, y) : y \in Y\}$ under it. □

In the case $\alpha = 2$ and for partitions to G_δ sets, a theorem by Miller [9, Theorem 1] gives a finer result, namely the existence of a $\mathbf{\Sigma}^0_2$-measurable selector $s : Y \to Y$ of the partition, i.e., a mapping such that $s(P)$ is a singleton in P. This indicates that our last corollary might not give the optimal estimates on the classes of selector mappings and selector sets even for higher classes α.

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Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Sokolovská 83, 186 75 Prague 8, Czech Republic

E-mail address: holicky@karlin.mff.cuni.cz