Matematická analýza I

test MA I A

jméno: 20. prosince 2016

1. Vypočítejte následující limity posloupností.

   a) $$\lim_{n\to\infty} \frac{\sqrt[12]{n}}{n} \cdot \frac{\sqrt[3]{n^2+n+1} - \sqrt[3]{n^2+n-1}}{\sqrt[4]{(n+1)^3} - \sqrt[4]{(n-1)^3}} \cdot (2+4+6+8+\cdots+2n)$$

   b) $$\lim_{n\to\infty} \frac{7^{n+1} + n\ln n^5 + 8n^3e^n}{7^{n-1} + n^7e^{n+2} + n^3 + 10^{12}} \cdot \left(\frac{5n-2}{5n+2}\right)^{n\cdot\arcsin 1+1}$$

2. Vypočítejte následující limity funkcí (bez užití derivací a l'Hospitalova pravidla).

   a) $$\lim_{x\to 0} \left(\sin\left(x+\frac{\pi}{2}\right)\right)^{\operatorname{tg}\left(x+\frac{5\pi}{2}\right)} \cdot \left(\frac{1+\sin x}{1+\operatorname{tg} x}\right)^{\frac{1}{\operatorname{tg} x}}$$

   b) $$\lim_{x\to 0} \frac{(\ln\cos 3x)\cdot(\cos x - \cos 3x)}{\ln(3+x) - \ln 3} \cdot \frac{\operatorname{arccotg} e^x}{x\,\ln\cos 5x}$$

1a) $$\lim \frac{\sqrt[12]{n}}{n}\cdot\frac{\overset{a^3\ -\ b^3}{\sqrt[3]{n^2+n+1}-\sqrt[3]{n^2+n-1}}}{\underset{a^2\ -\ b^2}{\sqrt[4]{(n+1)^3}-\sqrt[4]{(n-1)^3}}}\cdot\underbrace{(2+4+6+\ldots+2n)} =$$

$\bullet\ 2\cdot(1+2+3+\ldots+n) = 2\cdot\frac{n}{2}(n+1) = n\cdot(n+1)$

nebo

$\bullet\ n\cdot\frac{2+2n}{2} = n\cdot(n+1)$

$$=\lim \frac{\sqrt[12]{n}}{n}\cdot\frac{\cancel{n^2}+\cancel{n}+1-(\cancel{n^2}+\cancel{n}-1)}{\underset{a^2\ -\ b^2}{\sqrt{(n+1)^3}-\sqrt{(n-1)^3}}}\cdot\frac{\left(\overbrace{\sqrt[4]{(n+1)^3}+\sqrt[4]{(n-1)^3}}^{\to 2} \ :\sqrt[4]{n^3}\right)\cdot\sqrt[4]{n^3}}{\underbrace{\left(\left(\sqrt[3]{n^2+n+1}\right)^2+\sqrt[3]{n^2+n+1}\cdot\sqrt[3]{n^2+n-1}+\left(\sqrt[3]{n^2+n-1}\right)^2 :\sqrt[3]{n^4}\right)}_{\to 3}\cdot\sqrt[3]{n^4}}\cdot n\cdot(n+1)=$$

$$=\lim \frac{\sqrt[12]{n}}{\cancel{n}}\cdot\frac{2}{\underset{a^2\ -\ b^2}{\sqrt{(n+1)^3}-\sqrt{(n-1)^3}}}\cdot\frac{2\cdot\sqrt[4]{n^3}}{3\cdot\sqrt[3]{n^4}}\cdot\cancel{n}(n+1) = \frac{4}{3}\lim \sqrt[12]{n}\cdot(n+1)\cdot n^{-\frac{7}{12}}\cdot\frac{\sqrt{(n+1)^3}+\sqrt{(n-1)^3}}{(n+1)^3-(n-1)^3}=$$

$n^{\frac{3}{4}-\frac{4}{3}} = n^{\frac{9-16}{12}} = n^{-\frac{7}{12}}$

$n^{\frac{1}{12}-\frac{7}{12}} = n^{-\frac{6}{12}} = n^{-\frac{1}{2}}$

$$=\frac{4}{3}\lim\frac{n+1}{\sqrt{n}}\cdot\frac{\overbrace{\frac{\sqrt{(n+1)^3}+\sqrt{(n-1)^3}}{\sqrt{n^3}}}^{2}\cdot\sqrt{n^3}}{\underbrace{\cancel{n^3}+3n^2+\cancel{3n}+1-(\cancel{n^3}-3n^2+\cancel{3n}-1)}_{6n^2+2}} = \frac{8}{3}\cdot\lim\frac{n+1}{\cancel{\sqrt{n}}}\cdot\frac{\cancel{\sqrt{n^3}}\ n}{2(3n^2+1)}=$$

$$=\frac{8}{3}\cdot\frac{1}{6}=\frac{4}{9}$$

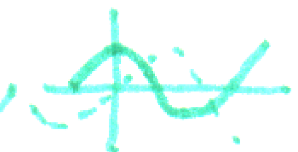
b) $$\lim\frac{7^{n+1}+n\cdot\ln n^5+8n^3e^n}{7^{n-1}+n^7e^{n+2}+n^3+10^{12}}\ \begin{matrix}:7^{n-1}\\ :7^{n-1}\end{matrix}\cdot\left(\frac{5n-2}{5n+2}\right)^{n\cdot\underbrace{\arcsin 1}_{\frac{\pi}{2}}+1}=$$

$$=\lim\frac{7^2+\underbrace{\frac{n\cdot\ln n^5}{7^{n-1}}}_{\to 0}+\underbrace{\frac{8n^3e^n}{7^{n-1}}}_{\to 0}}{1+\underbrace{\frac{n^7e^{n+2}}{7^{n-1}}}_{\to 0}+\underbrace{\frac{n^3}{7^{n-1}}}_{\to 0}+\underbrace{\frac{10^{12}}{7^{n-1}}}_{\to 0}}\cdot\left(\frac{5n+2-4}{5n+2}\right)^{n\cdot\frac{\pi}{2}+1}=7^2\cdot\lim\left(1+\frac{-4}{5n+2}\right)^{n\cdot\frac{\pi}{2}+1}=$$

$$=7^2\cdot\lim\Bigg[\underbrace{\left(1+\frac{1}{\frac{5n+2}{-4}}\right)^{\frac{5n+2}{-4}}}_{\to e}\Bigg]^{\underbrace{\frac{-4}{5n+2}\cdot\left(n\cdot\frac{\pi}{2}+1\right)}_{\to\frac{-4\frac{\pi}{2}}{5}}}=7^2\cdot e^{-\frac{2\pi}{5}}$$

2a) $$\lim_{x\to 0}\left[\sin\left(x+\frac{\pi}{2}\right)\right]^{\operatorname{tg}\left(x+\frac{5\pi}{2}\right)}\cdot\left(\frac{1+\sin x}{1+\operatorname{tg} x}\right)^{\frac{1}{\operatorname{tg} x}} =$$

$\sin\left(x+\frac{\pi}{2}\right) = \cos x$



$\underbrace{\sin x\cdot\cos\frac{\pi}{2}}_{0} + \cos x\cdot\underbrace{\sin\frac{\pi}{2}}_{1}$

$\operatorname{tg}\left(x+\frac{5\pi}{2}\right) = \operatorname{tg}\left(x+\frac{\pi}{2}+2\pi\right) = \operatorname{tg}\left(x+\frac{\pi}{2}\right) = -\operatorname{cotg} x$

tg π-per. fce

$$\frac{\sin\left(x+\frac{\pi}{2}\right)}{\underbrace{\cos\left(x+\frac{\pi}{2}\right)}_{\cos x\cdot\underbrace{\cos\frac{\pi}{2}}_{0} - \sin x\cdot\underbrace{\sin\frac{\pi}{2}}_{1}}} = \frac{\cos x}{-\sin x}$$

- $\lim_{x\to 0}\left[\sin\left(x+\frac{\pi}{2}\right)\right]^{\operatorname{tg}\left(x+\frac{\pi}{2}\right)} = \lim_{z\to\frac{\pi}{2}}(\sin x)^{\operatorname{tg} x} =$ jako na cv. $= e^0 = \underline{\underline{1}}$

$z = x+\frac{\pi}{2}$

nebo:

- $$\lim_{x\to 0}\left[\sin\left(x+\frac{\pi}{2}\right)\right]^{\operatorname{tg}\left(x+\frac{\pi}{2}\right)} = \lim_{x\to 0}(\cos x)^{\frac{\cos x}{\sin x}\to 1} = \lim_{x\to 0}(\cos x)^{\frac{-1}{\sin x}} = \lim_{x\to 0}(\cos^2 x)^{-\frac{1}{2\sin x}} =$$

$a^{\cos x} \to a^1 = a$

$$= \lim_{x\to 0}(1-\sin^2 x)^{-\frac{1}{2\sin x}} = \lim_{x\to 0}\Bigg[\underbrace{\left(1+(-\sin^2 x)\right)^{\frac{1}{-\sin^2 x}}}_{\to e}\Bigg]^{\underbrace{\frac{\sin x}{2}}_{\to 0}} = e^0 = \underline{\underline{1}}$$

- $$\lim_{x\to 0}\left(\frac{1+\sin x}{1+\operatorname{tg} x}\right)^{\frac{1}{\operatorname{tg} x}} = \lim_{x\to 0}\left(\frac{1+\operatorname{tg} x+\sin x-\operatorname{tg} x}{1+\operatorname{tg} x}\right)^{1/\operatorname{tg} x} = \lim_{x\to 0}\Bigg[\underbrace{\left(1+\frac{\sin x-\operatorname{tg} x}{1+\operatorname{tg} x}\right)^{\frac{1+\operatorname{tg} x}{\sin x-\operatorname{tg} x}}}_{\to e}\Bigg]^{\frac{\sin x-\operatorname{tg} x}{1+\operatorname{tg} x}\cdot\frac{1}{\operatorname{tg} x}} =$$

$$= e^0 = \underline{\underline{1}}$$

exponent:

$$\lim_{x\to 0}\frac{\sin x-\operatorname{tg} x}{\underbrace{1+\operatorname{tg} x}_{\to 1}}\cdot\frac{1}{\operatorname{tg} x} = \lim_{x\to 0}\frac{\cancel{\sin x}\cdot\left(1-\frac{1}{\cos x}\right)^{\to 0}}{\frac{\cancel{\sin x}}{\cos x \to 1}} = 0$$

celkový výsledek: $e^0\cdot e^0 = \underline{\underline{1}}$

2b) $$\lim_{x\to 0} \frac{(\ln\cos 3x)\cdot(\cos x-\cos 3x)}{\ln(3+x)-\ln 3}\cdot\frac{\operatorname{arccotg} e^x}{x\cdot\ln\cos 5x} =$$

$\to \operatorname{arccotg} 1 = \frac{\pi}{4}$

$$= \lim_{x\to 0} \frac{\frac{1}{2}\ln\cos^2 3x\cdot(\cos x-1+1-\cos 3x)}{\ln\frac{3+x}{3}}\cdot\frac{\frac{\pi}{4}}{x\cdot\frac{1}{2}\ln\cos^2 5x} =$$

$$= \lim_{x\to 0} \frac{\ln(1+(-\sin^2 3x))\cdot\left[\frac{1-\cos^2 3x}{1+\cos 3x}-\frac{1-\cos^2 x}{1+\cos x}\right]}{\ln\left(1+\frac{x}{3}\right)\cdot x\cdot\ln(1+(-\sin^2 5x))}\cdot\frac{\pi}{4} =$$

$$= \frac{\pi}{4}\cdot\lim_{x\to 0} \frac{\underbrace{\frac{\ln(1+(-\sin^2 3x))}{-\sin^2 3x}}_{\to 1}\cdot\underbrace{\frac{-\sin^2 3x}{(3x)^2}}_{\to -1}\cdot(3x)^2\cdot\left[\underbrace{\frac{\sin^2 3x}{(1+\cos 3x)\cdot(3x)^2}}_{\to\frac{1}{2}}\cdot(3x)^2-\underbrace{\frac{\sin^2 x}{(1+\cos x)\cdot x^2}}_{\to\frac{1}{2}}\cdot x^2\right]}{\underbrace{\frac{\ln\left(1+\frac{x}{3}\right)}{\frac{x}{3}}}_{\to 1}\cdot\frac{x}{3}\cdot x\cdot\underbrace{\frac{\ln(1+(-\sin^2 5x))}{-\sin^2 5x}}_{\to 1}\cdot\underbrace{\frac{-\sin^2 5x}{(5x)^2}}_{\to -1}\cdot(5x)^2} =$$

$$= \frac{\pi}{4}\cdot\lim_{x\to 0} \frac{-9\cancel{x^2}\cdot\left[\frac{1}{2}\cdot 9x^2-\frac{1}{2}x^2\right]}{\frac{\cancel{x^2}}{3}\cdot(-25x^2)} = \frac{3\pi}{4}\cdot\lim_{x\to 0}\frac{9\cdot\frac{1}{2}\cancel{x^2}[9-1]}{25\cancel{x^2}} = \frac{3\pi}{\cancel{4}}\cdot\frac{9\cdot\cancel{4}}{25} =$$

$$= \underline{\underline{\frac{27\pi}{25}}}$$