Interpretation of algebraic preconditioning as transformation of the discretization basis

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Standard scheme of numerical solution of PDE

Find \( x \in V : \langle Ax, v \rangle = \langle b, v \rangle \) for all \( v \in V \)

finite dimensional subspace \( V_h \subset V \)

Find \( x_h \in V_h : \langle Ax_h, v_h \rangle = \langle b, v_h \rangle \) for all \( v_h \in V_h \)

discretization basis \( \Phi \)

Solve algebraic system \( Ax = b \) and get \( x_h = \Phi x \)

algebraic preconditioner \( \hat{M} \)

Run preconditioned iterative method with \( Ax = b, \hat{M} \); compute approximation \( x_k \) and get \( x_h^{(k)} = \Phi x_k \)

Discretization and preconditioning are often considered separately.
Consider the standard FEM discretization of $V$ with piecewise polynomial discretization basis functions with local support:

- The locality of basis functions gives sparse $A$.
- However, the matrix $A^{-1}$ in $x = A^{-1}b$ is not sparse.
- Components of $x$ depend significantly on most of the components of $b$.
- Using the local information transfers in $Ab, A(Ab), A(A(Ab)), \ldots$ it takes many iterative steps for sparse $A$ to approximate $x$.

Preconditioning is needed in order to cure the computational inefficiency which the localized FEM discretization has contributed to.
Can we interpret algebraic preconditioning as transformation of the discretization basis functions?

**Ideas linking discretization and preconditioning**

Outline of the talk

1. Basic notation
2. CG in infinite dimensional Hilbert spaces
3. Finite dimensional subspace and matrix formulation
4. Preconditioning and transformation of the discretization basis
1. Functional formulation

Let $V$ be a real (infinite dimensional) Hilbert space with the inner product

$$(\cdot, \cdot)_V : V \times V \to \mathbb{R};$$

let $V^\#$ be the dual space of bounded linear functionals on $V$ with the duality pairing

$$\langle \cdot, \cdot \rangle : V^\# \times V \to \mathbb{R}.$$ 

Consider a PDE problem described in the form of the functional equation

$$\mathcal{A}x = b, \quad \mathcal{A} : V \to V^\#, \quad x \in V, \quad b \in V^\#,$$

where the linear, bounded, and coercive operator $\mathcal{A}$ is self-adjoint w.r.t. the duality pairing $\langle \cdot, \cdot \rangle$. 
For each $f \in V^\#$ there exists a unique $\tau f \in V$ such that

$$\langle f, v \rangle = (\tau f, v)_V \quad \text{for all } v \in V.$$ 

In this way the inner product $(\cdot, \cdot)_V$ determines the Riesz map $\tau : V^\# \to V$.

The transformation of the PDE problem using the Riesz map gives

$$\tau A x = \tau b, \quad \tau A : V \to V, \quad x \in V, \quad \tau b \in V,$$

which is called operator preconditioning.

References

[Steinbach and Wendland (1998)]; [McLean and Tran (1997)];
[Christiansen and Nédélec (2000, 2000)]; [Powell and Silvester (2003)];
[Zulehner (2011)]; . . . ; [Málek and Strakoš (2015)]
2. Krylov manifolds in Hilbert spaces

Using the Riesz map, one can form for \( x_0 \in V, \; r_0 = b - Ax_0 \in V^\# \) the Krylov subspace

\[
K_n \equiv \text{span} \left\{ \tau r_0, \tau A \tau r_0, (\tau A)^2 \tau r_0, \ldots, (\tau A)^{n-1} \tau r_0 \right\}
\]

and define Krylov subspace methods (here CG) in Hilbert space operator setting. The solution \( x = (\tau A)^{-1} \tau b \) is approximated by constructing the approximations \( x_n \in x_0 + K_n \).

Looking for the approximate solution minimizing energy leads to

\[
\|x - x_n\|_a = \min_{z \in x_0 + K_n} \|x - z\|_a, \quad \|z\|_a^2 = \langle Az, z \rangle,
\]

which is equivalent to the Galerkin orthogonality condition

\[
\langle b - Ax_n, w \rangle = \langle r_n, w \rangle = 0 \quad \text{for all } w \in K_n.
\]
2. Preconditioned CG in Hilbert spaces

Given \( r_0 = b - Ax_0 \in V^\# \), set \( p_0 = \tau r_0 \in V \)

For \( n = 1, 2, \ldots, n_{\text{max}} \)

\[
\alpha_{n-1} = \frac{\langle r_{n-1}, \tau r_{n-1} \rangle}{\langle \mathcal{A} p_{n-1}, p_{n-1} \rangle} = \frac{(\tau r_{n-1}, \tau r_{n-1})_V}{(\tau \mathcal{A} p_{n-1}, p_{n-1})_V},
\]

\[
x_n = x_{n-1} + \alpha_{n-1} p_{n-1}, \quad \text{stop when the stop. criterion is satisfied}
\]

\[
r_n = r_{n-1} - \alpha_{n-1} \mathcal{A} p_{n-1},
\]

\[
\beta_n = \frac{\langle r_n, \tau r_n \rangle}{\langle r_{n-1}, \tau r_{n-1} \rangle} = \frac{(\tau r_n, \tau r_n)_V}{(\tau r_{n-1}, \tau r_{n-1})_V}
\]

\[
p_n = \tau r_n + \beta_n p_{n-1},
\]

End

3. Finite dimensional CG and matrix formulation

Consider finite dimensional subspace $V_h \subset V$ and denote by

$$\Phi_h = (\phi_1, \ldots, \phi_N)$$

the discretization basis of $V_h$, and

$$\Phi_h^\# = (\phi_1^\#, \ldots, \phi_N^\#)$$

the canonical basis of its dual $V_h^\#$.

$$\langle \phi_j^\#, \phi_i \rangle = \delta_{ij}, \text{ i.e., } \Phi_h^\# \Phi_h = I.$$

Using the coordinates in $\Phi_h$ and in $\Phi_h^\#$,

$$\langle f, v \rangle = v^* f,$$

$$(u, v)_V = v^* M u,$$

$$Au = A\Phi_h u = \Phi_h^\# A u,$$

$$\tau f = \tau \Phi_h^\# f = \Phi_h M^{-1} f,$$

where

$$M = [M_{ij}] = [(\phi_j, \phi_i)_{V}]_{i,j},$$

$$A = [A_{ij}] = [\langle A\phi_j, \phi_i \rangle]_{i,j}, \quad i, j = 1, \ldots, N.$$
3. Preconditioned algebraic CG

With \( b = \Phi_h^# b \), \( x_n = \Phi_h x_n \), \( p_n = \Phi_h p_n \), \( r_n = \Phi_h^# r_n \) we get the preconditioned algebraic CG with the preconditioner \( M \):

Given \( r_0 = b - Ax_0 \), solve \( Mz_0 = r_0 \), set \( p_0 = z_0 \)

For \( n = 1, 2, \ldots, n_{max} \)

\[
\alpha_{n-1} = \frac{z_{n-1}^* r_{n-1}}{p_{n-1}^* A p_{n-1}},
\]

\( x_n = x_{n-1} + \alpha_{n-1} p_{n-1} \), stop when the stop. criterion is satisfied

\( r_n = r_{n-1} - \alpha_{n-1} A p_{n-1} \),

\( Mz_n = r_n \) solve for \( z_n \),

\[
\beta_n = \frac{z_n^* r_n}{z_{n-1}^* r_{n-1}},
\]

\( p_n = z_n + \beta_n p_{n-1} \),

End
3. Preconditioning as the basis orthogonalization

- Unpreconditioned CG, i.e., $M = I$, corresponds to the basis $\Phi$ orthonormal w.r.t. the inner product $(\cdot, \cdot)_V$ on $V_h$.
- The transformed basis $\Phi_t = \Phi L^{-*}$ where $M = LL^*$ is orthonormal w.r.t. the inner product $(\cdot, \cdot)_V$ on $V_h$.
- Operator preconditioning on the discrete space can be interpreted as orthogonalization of the discretization basis.

\[
\text{PCG: } Ax = b, \quad \tau \text{ given by } (\cdot, \cdot)_V
\]

\[
\Phi \quad \Phi_t = \Phi L^{-*}
\]

\[
\text{PCG: } Ax = b, \quad M \quad \rightarrow \quad \text{CG: } A_t x_t = b_t
\]

\[
A_t = L^{-1} A L^{-*}, \quad x_t = L^* x, \quad b_t = L^{-1} b
\]
4. For any algebraic preconditioner

Consider the standard approach of the FEM discretization on $V_h$:

\[ \mathcal{A}, b, \Phi \rightarrow A, b \rightarrow \text{alg. preconditioner } \hat{M} \rightarrow \text{PCG} : Ax = b, \hat{M} \]

- The standard discretization FEM basis $\Phi$ and the matrix $\hat{M}$ determine the new inner product $(\cdot, \cdot)_{\hat{V}_h}$ on $V_h$:

\[ (u, v)_{\hat{V}_h} = (\hat{\Phi}_t \hat{u}, \hat{\Phi}_t \hat{v})_{\hat{V}_h} \equiv \hat{v}^* \hat{u} = v^* \hat{M} u; \]

- As before, the transformed basis $\hat{\Phi}_t = \Phi \hat{L}^{-*}$ where $\hat{M} = \hat{L} \hat{L}^*$ is orthonormal w.r.t. the new inner product $(\cdot, \cdot)_{\hat{V}_h}$ on $V_h$.

PCG: $Ax = b$, $(\cdot, \cdot)_{\hat{V}_h}$

$\Phi$

$\hat{\Phi}_t = \Phi \hat{L}^{-*}$

PCG: $Ax = b$, $\hat{M}$

CG: $\hat{A}_t \hat{x}_t = \hat{b}_t$; $\hat{A}_t = \hat{L}^{-1} \hat{A} \hat{L}^{-*}$
Consider a problem with the inhomogeneous diffusion tensor

\[-\nabla \cdot (S \nabla u) = 0 \quad \text{in } \Omega , \]

\[u = u_D \quad \text{on } \partial \Omega , \]

\[\Omega \equiv (-1, 1) \times (-1, 1) , \]

\[S = s_i I \quad \text{in } \Omega_i , \]

\[s_1 = s_3 = 5 , \quad s_2 = s_4 = 1 , \]

with the exact solution \( u \) given in each sub-domain \( \Omega_i \) in polar coordinates \((r, \theta)\) by

\[u(r, \theta)|_{\Omega_i} = r^{\alpha}(a_i \sin(\alpha \theta) + b_i \cos(\alpha \theta))\]

with the coefficients \( \alpha, a_i, b_i \) as in, e.g., [Luce, Wohlmuth (2004)], [Jiránek, Strakoš, Vohralík (2010)].
4. Numerical illustration

Squared energy norm of the algebraic error

- P1 FEM; cond = 2.57e+03
- P1 FEM ichol; cond = 2.67e+01
- P1 FEM lapl; cond = 1.00e+02
- P1 FEM ichol(1e−02); cond = 1.72e+00

P1 FEM ichol; nnz = 5

P1 FEM lapl; nnz = 225

P1 FEM ichol(1e−02); nnz = 214
Concluding remarks

- Choice of the preconditioner (algebraic or operator) determines (implicitly or explicitly) the inner product in the corresponding Hilbert space operator setting.

- Preconditioning can be interpreted as transformation of the discretization basis such that the resulting basis functions are orthogonal w.r.t. the given inner product.

- Preconditioning can be interpreted in part as addressing the difficulty related to the sparsity given by the locality of the supports of the discretization basis functions.

- CG in Hilbert space provides straightforward discretization of the problem on the finite dimensional Krylov subspace.

- Can Krylov subspace of functions be cheaply computed?
Reference


Thank you for your attention!

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