

L_2 -properties of stochastic processes

Definition 3.1: We say that a sequence of random variables X_n such that $\mathbb{E}|X_n|^2 < \infty$ converges in L_2 (or in the mean square) to a random variable X , if $\mathbb{E}|X_n - X|^2 \rightarrow 0$ for $n \rightarrow \infty$. In that case we write $X = \text{l.i.m. } X_n$.

Let $T \subset \mathbb{R}$ be an open interval and consider a stochastic process $\{X_t, t \in T\}$ with continuous time and finite second moments.

Definition 3.2: We call the process $\{X_t, t \in T\}$ L_2 -continuous (mean square continuous) at the point $t_0 \in T$ if $\mathbb{E}|X_t - X_{t_0}|^2 \rightarrow 0$ for $t \rightarrow t_0$. The process is L_2 -continuous if it is L_2 -continuous at all points $t \in T$.

Theorem 3.1: A stochastic process $\{X_t, t \in T\}$ is L_2 -continuous if and only if its mean value $\mathbb{E}X_t$ is a continuous function on T and its autocovariance function $R_X(s, t)$ is continuous at points $[s, t]$ for which $s = t$.

Corollary 3.1: Centered weakly stationary process is L_2 -continuous if and only if its autocovariance function $R(t)$ is continuous at point 0.

Definition 3.4: Let $T = [a, b]$ be a bounded closed interval. For any $n \in \mathbb{N}$ let $D_n = \{t_{n,0}, \dots, t_{n,n}\}$ be a division of the interval $[a, b]$ where $a = t_{n,0} < t_{n,1} < \dots < t_{n,n} = b$. We define the partial sums I_n of the centered stochastic process $\{X_t, t \in T\}$ by the formula

$$I_n = \sum_{i=0}^{n-1} X_{t_{n,i}}(t_{n,i+1} - t_{n,i}), \quad n \in \mathbb{N}.$$

If there is a random variable I such that $\mathbb{E}|I_n - I|^2 \rightarrow 0$ for $n \rightarrow \infty$ and for each division of the interval $[a, b]$ such that $\max_{0 \leq i \leq n-1} (t_{n,i+1} - t_{n,i}) \rightarrow 0$ we call it the *Riemann integral* of the process $\{X_t, t \in T\}$ and denote it by $I = \int_a^b X_t dt$. For a non-centered process with the mean value $\mathbb{E}X_t$ we define the Riemann integral as

$$\int_a^b X_t dt = \int_a^b (X_t - \mathbb{E}X_t) dt + \int_a^b \mathbb{E}X_t dt,$$

if the centered process $\{X_t - \mathbb{E}X_t, t \in T\}$ has a Riemann integral and the Riemann integral $\int_a^b \mathbb{E}X_t dt$ exists and is finite.

Theorem 3.3: A stochastic process $\{X_t, t \in [a, b]\}$ where $[a, b]$ is a bounded closed interval is Riemann-integrable if the Riemann integrals $\int_a^b \mathbb{E}X_t dt$ and $\int_a^b \int_a^b R_X(s, t) ds dt$ exist and are finite.

Theorem 3.4: Let $M \subset \mathbb{R}^n$ be a bounded set, f be a real function on \mathbb{R}^n , bounded on M . Then the Riemann integral $\int_M f(x) dx$ exists if and only if both following conditions are fulfilled:

- a) the boundary of M has Lebesgue measure 0,
- b) the set of inner points of M in which f is not continuous has Lebesgue measure 0.

Theorem 3.5: Let $M \subset \mathbb{R}^n$ be a bounded set and let the Riemann integral $\int_M f(x) dx$ exist. Then also the Lebesgue integral $\int_M f(x) dx$ exists and both integrals are equal.

Exercise 3.1: Consider a stochastic process $X_t = \cos(t + B)$, $t \in \mathbb{R}$, where B is a random variable with the uniform distribution on the interval $(0, 2\pi)$. Is this process L_2 -continuous? Is it Riemann-integrable on a bounded closed interval $[a, b]$?

Further question How does the $\int_1^2 X_t dt$ from the previous exercise look like?

Exercise 3.3: *Integrated Wiener process* is defined as

$$X_t = \int_0^t W_\tau d\tau, \quad t \geq 0.$$

Using the properties of the Wiener process and L_2 -convergence prove that $X_t \sim N(0, v_t^2)$ for all $t \geq 0$ where $v_t^2 = \frac{1}{3}\sigma^2 t^3$ and σ^2 is the parameter of the Wiener process W_t . Use the fact that the L_2 -limit of a sequence of Gaussian random variables is a Gaussian random variable.

Exercise 3.4: Let $\{X_t, t \in \mathbb{R}\}$ be a centered, weakly stationary stochastic process with the autocovariance function

$$R_X(t) = \exp\{\lambda(e^{it} - 1)\}, \quad t \in \mathbb{R},$$

where $\lambda > 0$. Is this process L_2 -continuous? Is it Riemann-integrable on a bounded closed interval $[a, b]$?

Exercise 3.6: Let $\{X_t, t \in \mathbb{R}\}$ be a process of independent identically distributed random variables with a mean value μ and a finite variance $\sigma^2 > 0$. Is this process L_2 -continuous? Is it Riemann-integrable on a bounded closed interval $[a, b]$?

Further question How does the $\int_1^5 X_t dt$ from the previous exercise look like?