Linear models of time series

MA(n): *The moving average sequence of order* n is defined by

$$X_t = b_0 Y_t + b_1 Y_{t-1} + \cdots + b_n Y_{t-n}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$ and $b_0, b_1, \ldots, b_n$ are real- or complex-valued constants, $b_0 \neq 0$, $b_n \neq 0$. It is a centered weakly stationary random sequence with the autocovariance function

$$R_X(t) = \begin{cases} \sigma^2 (b_t \overline{b_0} + \cdots + b_n \overline{b_{n-t}}) & \text{for } 0 \leq t \leq n, \\ \sigma^2 (b_0 \overline{b_{|t|}} + \cdots + b_{n-|t|} \overline{b_n}) & \text{for } -n \leq t \leq 0, \\ 0 & \text{for } |t| > n, \end{cases}$$

and the spectral density

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \left| \sum_{k=0}^{n} b_k \mathrm{e}^{-\mathrm{i}k\lambda} \right|^2, \quad \lambda \in [-\pi, \pi].$$

MA(∞): *The causal linear process* is a random sequence defined by

$$X_t = \sum_{j=0}^{\infty} c_j Y_{t-j}, \quad t \in \mathbb{Z}, \tag{1}$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise and $c_0, c_1, \ldots$ is a sequence of constants such that $\sum_{j=0}^{\infty} |c_j| < \infty$ (this condition implies the sum converges absolutely almost surely). $\{X_t, t \in \mathbb{Z}\}$ is a centered weakly stationary random sequence with the autocovariance function

$$R_X(t) = \begin{cases} \sigma^2 \sum_{k=0}^{\infty} c_{k+t} \overline{c_k} & \text{for } t \geq 0, \\ \sigma^2 \sum_{k=0}^{\infty} c_k \overline{c_{k+|t|}} & \text{for } t \leq 0, \end{cases} \tag{2}$$

and the spectral density

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \left| \sum_{k=0}^{\infty} c_k \mathrm{e}^{-\mathrm{i}k\lambda} \right|^2, \quad \lambda \in [-\pi, \pi].$$

AR(m): *The autoregressive sequence of order* m is defined by

$$X_t + a_1 X_{t-1} + \cdots + a_m X_{t-m} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise and $a_1, \ldots, a_m$ are real-valued constants, $a_m \neq 0$. If all the roots of the polynomial $1 + a_1 z + \cdots + a_m z^m$ lie outside the unit circle in $\mathbb{C}$ (which is equivalent to all the roots of $z^m + a_1 z^{m-1} + \cdots + a_m$ lying inside the unit circle) then $\{X_t, t \in \mathbb{Z}\}$ is a causal linear process (1) with coefficients c_j determined by

$$\sum_{j=0}^{\infty} c_j z^j = \frac{1}{1 + a_1 z + \cdots + a_m z^m}, \quad |z| \leq 1.$$

We may also get the coefficients c_j by solving the equations derived by plugging-in (1) into the defining relation and by comparing the coefficients by the respective terms Y_{t-j} on both sides. The autocovariance function is given by (2) and the spectral density is

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{|1 + a_1 \mathrm{e}^{-\mathrm{i}\lambda} + \cdots + a_m \mathrm{e}^{-\mathrm{i}m\lambda}|^2}, \quad \lambda \in [-\pi, \pi].$$

The autocovariance function may be also computed by means of the *Yule-Walker equations.*

ARMA(m,n): This model is defined by the equation

$$X_t + a_1 X_{t-1} + \cdots + a_m X_{t-m} = Y_t + b_1 Y_{t-1} + \cdots + b_n Y_{t-n}, \quad t \in \mathbb{Z}, \tag{3}$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise and $a_1, \ldots, a_m, b_1, \ldots, b_n$ are real-valued constants, $a_m \neq 0$, $b_n \neq 0$. Suppose that the polynomials $1 + a_1 z + \cdots + a_m z^m$ and $1 + b_1 z + \cdots + b_n z^n$ have no common roots and all the roots of the polynomial $1 + a_1 z + \cdots + a_m z^m$ are outside the unit circle. Then $\{X_t, t \in \mathbb{Z}\}$ is a causal linear process (1) with coefficients c_j given by

$$\sum_{j=0}^{\infty} c_j z^j = \frac{1 + b_1 z + \cdots + b_n z^n}{1 + a_1 z + \cdots + a_m z^m}, \quad |z| \leq 1.$$

We may also get the coefficients c_j by solving the equations derived by plugging-in (1) into the defining relation and by comparing the coefficients by the respective terms Y_{t-j} on both sides. The autocovariance function is given by (2) and the spectral density is

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{|1 + b_1 \mathrm{e}^{-\mathrm{i}\lambda} + \cdots + b_n \mathrm{e}^{-\mathrm{i}n\lambda}|^2}{|1 + a_1 \mathrm{e}^{-\mathrm{i}\lambda} + \cdots + a_m \mathrm{e}^{-\mathrm{i}m\lambda}|^2}, \quad \lambda \in [-\pi, \pi].$$

The autocovariance function may be also computed by means of the *Yule-Walker equations.*

Definition 6.1: Let $\{X_t, t \in \mathbb{Z}\}$ be a stationary ARMA(m,n) random sequence defined by (3). If there exists a sequence of constants $\{d_j, j \in \mathbb{N}_0\}$ such that $\sum_{j=0}^{\infty} |d_j| < \infty$ and

$$Y_t = \sum_{j=0}^{\infty} d_j X_{t-j}, \quad t \in \mathbb{Z},$$

then $\{X_t, t \in \mathbb{Z}\}$ is called *invertible* (it has an AR(∞) representation).

Theorem 6.1: Let $\{X_t, t \in \mathbb{Z}\}$ be a stationary ARMA(m,n) random sequence. Let the polynomials $a(z) = 1 + a_1 z + \cdots + a_m z^m$ and $b(z) = 1 + b_1 z + \cdots + b_n z^n$ have no common roots and the polynomial $b(z) = 1 + b_1 z + \cdots + b_n z^n$ have all the roots outside the unit circle. Then $\{X_t, t \in \mathbb{Z}\}$ is invertible and the coefficients d_j are given by

$$\sum_{j=0}^{\infty} d_j z^j = \frac{1 + a_1 z + \cdots + a_m z^m}{1 + b_1 z + \cdots + b_n z^n}, \quad |z| \leq 1.$$

Remark: We may obtain the coefficients d_j by solving the equations we get by plugging the equality $Y_t = \sum_{j=0}^{\infty} d_j X_{t-j}$ into the defining formula of the ARMA sequence and comparing the coefficients on both sides.

Exercise 6.1: Determine the autocovariance function and the spectral density of the sequence

$$X_t = Y_t + \theta Y_{t-2}, \quad t \in \mathbb{Z},$$

where $\theta \in \mathbb{C}$ a $\{Y_t, t \in \mathbb{Z}\}$ is a white noise WN$(0, \sigma^2)$.

Exercise 6.4: Solve the Yule-Walker equations and determine the autocovariance function of the random sequence $\{X_t, t \in \mathbb{Z}\}$ defined by

$$X_t - 0.4 X_{t-1} + 0.04 X_{t-2} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$.

Exercise 6.6: Let $\{X_t, t \in \mathbb{Z}\}$ be an ARMA(2,1) random sequence defined by

$$X_t - X_{t-1} + \frac{1}{4} X_{t-2} = Y_t + Y_{t-1}, \quad t \in \mathbb{Z}, \tag{4}$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$. Determine the coefficients of the $\mathrm{MA}(\infty)$ representation of X_t and compute its autocovariance function and spectral density. Is the process invertible?

Exercise 6.10: The random sequence $\{X_t, t \in \mathbb{Z}\}$ is defined by

$$X_t - \frac{2}{15} X_{t-1} - \frac{1}{15} X_{t-2} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$. Express the random sequence $\{X_t, t \in \mathbb{Z}\}$ as a causal linear process and compute its autocovariance function and spectral density.

Exercise 6.15: The random sequence $\{X_t, t \in \mathbb{Z}\}$ is defined by the equation

$$X_t - (a + b) X_{t-1} + ab X_{t-2} = Y_t - a Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$ and $a \neq 0, b \neq 0$ are real constants. For which values of a, b is the process causal? For which values of a, b is the process invertible? Derive the causal ($\mathrm{MA}(\infty)$) and inverted ($\mathrm{AR}(\infty)$) representation. Compute the autocovariance function of $\{X_t, t \in \mathbb{Z}\}$.

Exercise 6.16: Consider the ARMA(2,1) model defined by

$$X_t - 0.5 X_{t-1} + 0.04 X_{t-2} = Y_t + 0.25 Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $\mathrm{WN}(0, \sigma^2)$. Determine the coefficients of the $\mathrm{AR}(\infty)$ representation.

Definition 6.2: Let $\{Y_t, t \in \mathbb{Z}\}$ be a centered weakly stationary sequence. Let $\{c_j, j \in \mathbb{Z}\}$ be a sequence of (complex-valued) numbers such that $\sum_{j=-\infty}^{\infty} |c_j| < \infty$.
We say that a random sequence $\{X_t, t \in \mathbb{Z}\}$ is obtained by filtration of the sequence $\{Y_t, t \in \mathbb{Z}\}$ if

$$X_t = \sum_{j=-\infty}^{\infty} c_j Y_{t-j}, \quad t \in \mathbb{Z}.$$

The sequence $\{c_j, j \in \mathbb{Z}\}$ is called *time-invariant linear filter.*
Provided that $c_j = 0$ for all $j < 0$, we say that the filter $\{c_j, j \in \mathbb{Z}\}$ is *causal.*

Theorem 6.2: Let $\{Y_t, t \in \mathbb{Z}\}$ be a centered weakly stationary sequence with an autocovariance function R_Y and spectral density f_Y and let $\{c_k, k \in \mathbb{Z}\}$ be a linear filter such that $\sum_{k=-\infty}^{\infty} |c_k| < \infty$. Then $\{X_t, t \in \mathbb{Z}\}$, where $X_t = \sum_{k=-\infty}^{\infty} c_k Y_{t-k}$, is a centered weakly stationary sequence with the autocovariance function

$$R_X(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_j \overline{c_k} R_Y(t - j + k), \quad t \in \mathbb{Z},$$

and spectral density

$$f_X(\lambda) = |\Psi(\lambda)|^2 f_Y(\lambda), \quad \lambda \in [-\pi, \pi],$$

where

$$\Psi(\lambda) = \sum_{k=-\infty}^{\infty} c_k e^{-ik\lambda}, \quad \lambda \in [-\pi, \pi],$$

is called *the transfer function of the filter.*

Exercise 6.18: Let $\{Y_t, t \in \mathbb{Z}\}$ be a white noise $\mathrm{WN}(0, \sigma^2)$. Let it be transformed by a linear filter to $\{X_t, t \in \mathbb{Z}\}$ so that

$$X_t - 2X_{t-1} = Y_t, \quad t \in \mathbb{Z}, \tag{5}$$

holds. Determine the coefficients of the linear filter, the transfer function of the filter and compute the autocovariance function and the spectral density of $\{X_t, t \in \mathbb{Z}\}$.

Exercise 6.20: Consider a random sequence given by the formula

$$X_t - \frac{1}{3} X_{t-1} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a centered real-valued white noise with positive finite variance σ^2. Let $\{Z_t, t \in \mathbb{Z}\}$ be a process obtained by the filtration

$$Z_t = X_t - \frac{1}{2} X_{t-1}, \quad t \in \mathbb{Z}.$$

Derive the transfer function of the filter and compute the spectral density of $\{Z_t, t \in \mathbb{Z}\}$. Compute the autocovariance function of $\{Z_t, t \in \mathbb{Z}\}$.