L_2-properties of stochastic processes

Definition 3.1: We say that a sequence of random variables X_n such that $\mathbb{E}|X_n|^2 < \infty$ *converges in L_2 (or in the mean square)* to a random variable X, if $\mathbb{E}|X_n - X|^2 \to 0$ for $n \to \infty$. In that case we write $X = \text{l.i.m.}\, X_n$.

Let $T \subset \mathbb{R}$ be an open interval and consider a stochastic process $\{X_t, t \in T\}$ with continuous time and finite second moments.

Definition 3.2: We call the process $\{X_t, t \in T\}$ *L_2-continuous (mean square continuous)* at the point $t_0 \in T$ if $\mathbb{E}|X_t - X_{t_0}|^2 \to 0$ for $t \to t_0$. The process is L_2-continuous if it is L_2-continuous at all points $t \in T$.

Theorem 3.1: A stochastic process $\{X_t, t \in T\}$ is L_2-continuous if and only if its mean value $\mathbb{E}X_t$ is a continuous function on T and its autocovariance function $R_X(s,t)$ is continuous at points $[s,t]$ for which $s = t$.

Corollary 3.1: Centered weakly stationary process is L_2-continuous if and only if its autocovariance function $R(t)$ is continuous at point 0.

Definition 3.3: We call the process $\{X_t, t \in T\}$ *L_2-differentiable (mean square differentiable)* at the point $t_0 \in T$ if there is a random variable X'_{t_0} such that

$$\lim_{h \to 0} \mathbb{E} \left| \frac{X_{t_0+h} - X_{t_0}}{h} - X'_{t_0} \right|^2 = 0.$$

The random variable X'_{t_0} is called the *derivative in the L_2 (mean square) sense* of the process $\{X_t, t \in T\}$ at the point t_0. The process is L_2-differentiable if it is L_2-differentiable at all points $t \in T$.

Theorem 3.2: A stochastic process $\{X_t, t \in T\}$ is L_2-differentiable if and only if its mean value $\mathbb{E}X_t$ is differentiable and the second-order generalized partial derivative of the autocovariance function $R(s,t)$ exists and is finite at points $[s,t]$ for which $s = t$, i.e. there is a finite limit

$$\lim_{h,h' \to 0} \frac{1}{hh'} \big[R_X(t+h, t+h') - R_X(t, t+h') - R_X(t+h, t) + R_X(t,t)\big].$$

Remark: A sufficient condition for the existence of the second-order generalized partial derivative is the existence and continuity of the second-order partial derivatives $\frac{\partial^2 R(s,t)}{\partial s \partial t}$ and $\frac{\partial^2 R(s,t)}{\partial t \partial s}$.

Remark: Any L_2-differentiable process is also L_2-continuous.

Definition 3.4: Let $T = [a,b]$ be a bounded closed interval. For any $n \in \mathbb{N}$ let $D_n = \{t_{n,0}, \ldots, t_{n,n}\}$ be a division of the interval $[a,b]$ where $a = t_{n,0} < t_{n,1} < \ldots < t_{n,n} = b$. We define the partial sums I_n of the centered stochastic process $\{X_t, t \in T\}$ by the formula

$$I_n = \sum_{i=0}^{n-1} X_{t_{n,i}} (t_{n,i+1} - t_{n,i}), \quad n \in \mathbb{N}.$$

If there is a random variable I such that $\mathbb{E}|I_n - I|^2 \to 0$ for $n \to \infty$ and for each division of the interval $[a, b]$ such that $\max_{0 \le i \le n-1}(t_{n,i+1} - t_{n,i}) \to 0$ we call it the *Riemann integral* of the process $\{X_t, t \in T\}$ and denote it by $I = \int_a^b X_t \, \mathrm{d}t$. For a non-centered process with the mean value $\mathbb{E}X_t$ we define the Riemann integral as

$$\int_a^b X_t \, \mathrm{d}t = \int_a^b (X_t - \mathbb{E}X_t) \, \mathrm{d}t + \int_a^b \mathbb{E}X_t \, \mathrm{d}t,$$

if the centered process $\{X_t - \mathbb{E}X_t, t \in T\}$ has a Riemann integral and the Riemann integral $\int_a^b \mathbb{E}X_t \, \mathrm{d}t$ exists and is finite.

Theorem 3.3: A stochastic process $\{X_t, t \in [a, b]\}$ where $[a, b]$ is a bounded closed interval is Riemann-integrable if the Riemann integrals $\int_a^b \mathbb{E}X_t \, \mathrm{d}t$ and $\int_a^b \int_a^b R_X(s, t) \, \mathrm{d}s \, \mathrm{d}t$ exist and are finite.

Theorem 3.4: Let $M \subset \mathbb{R}^n$ be a bounded set, f be a real function on $\mathbb{R}^n$, bounded on M. Then the Riemann integral $\int_M f(x) \, \mathrm{d}x$ exists if and only if both following conditions are fulfilled:
a) the boundary of M has Lebesgue measure 0,
b) the set of inner points of M in which f is not continuous has Lebesgue measure 0.

Theorem 3.5: Let $M \subset \mathbb{R}^n$ be a bounded set and let the Riemann integral $\int_M f(x) \, \mathrm{d}x$ exist. Then also the Lebesgue integral $\int_M f(x) \, \mathrm{d}x$ exists and both integrals are equal.

Exercise 3.1: Consider a stochastic process $X_t = \cos(t + B)$, $t \in \mathbb{R}$, where B is a random variable with the uniform distribution on the interval $(0, 2\pi)$. Is this process L_2-continuous and L_2-differentiable? Is it Riemann-integrable on a bounded closed interval $[a, b]$?

Further question How does the L_2-derivative and the $\int_1^2 X_t \mathrm{d}t$ from the previous exercise look like?

Exercise 3.3: *Integrated Wiener process* is defined as

$$X_t = \int_0^t W_\tau \, \mathrm{d}\tau, \quad t \ge 0.$$

Using the properties of the Wiener process and L_2-convergence prove that $X_t \sim N(0, v_t^2)$ for all $t \ge 0$ where $v_t^2 = \frac{1}{3}\sigma^2 t^3$ and σ^2 is the parameter of the Wiener process W_t. Use the fact that the L_2-limit of a sequence of Gaussian random variables is a Gaussian random variable.

Exercise 3.6: Let $\{X_t, t \in \mathbb{R}\}$ be a process of independent identically distributed random variables with a mean value μ and a finite variance $\sigma^2 > 0$. What are the L_2 properties of such a process (including Riemann-integrability)?

Further question How does the $\int_1^5 X_t \mathrm{d}t$ from the previous exercise look like?