NMSA405: exercise 1 – space of sequences of real numbers

Definition 1.3: For sequences of real numbers $x = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}}$ and $y = (y_1, y_2, \dots) \in \mathbb{R}^{\mathbb{N}}$ we define

$$d(x,y) = \sum_{j=1}^{\infty} \frac{\min\{|x_j - y_j|, 1\}}{2^j}.$$

Exercise 1.1: (Proposition 1.2)

a) Show that d defines a metric on $\mathbb{R}^{\mathbb{N}}$.
b) Let $x^n = (x_1^n, x_2^n, \dots)$ be sequences of real numbers for $n \in \mathbb{N}$ and $x = (x_1, x_2, \dots)$. Prove that

$$d(x^n, x) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{if and only if} \quad |x_j^n - x_j| \underset{n\to\infty}{\longrightarrow} 0 \ \text{ for all } j \in \mathbb{N}.$$

c) Prove that $(\mathbb{R}^{\mathbb{N}}, d)$ is a complete separable metric space.

Definition 1.5: Mapping $p : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is called *a finite permutation (of order n)*, if there is $n \in \mathbb{N}$ and a permutation $(k_1, \dots, k_n)$ of the elements of the set $\{1, \dots, n\}$ such that

$$p(x_1, \dots, x_n, x_{n+1}, \dots) = (x_{k_1}, \dots, x_{k_n}, x_{n+1}, \dots), \quad (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}}.$$

Exercise 1.2: (Proposition 1.5a) Prove that any finite permutation p is a homeomorphism.

Definition 1.6: Mapping $s : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ defined by

$$s(x_1, x_2, \dots) = (x_2, x_3, \dots), \quad (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}},$$

is called *shift*.

Exercise 1.3: (Proposition 1.5b) Prove that the shift s is a continuous mapping.

Definition 1.7: A set $T \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ is called *terminal* if the following implication holds:

$$x = (x_1, x_2, \dots) \in T, y = (y_1, y_2, \dots) \in \mathbb{R}^{\mathbb{N}} : y_k = x_k \text{ for all } k \in \mathbb{N} \text{ except of finitely many } \Rightarrow y \in T.$$

We call $T \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ *n-terminal* if

$$x = (x_1, x_2, \dots) \in T, y = (y_1, y_2, \dots) \in \mathbb{R}^{\mathbb{N}} : y_k = x_k \text{ for } k > n \ \Rightarrow y \in T.$$

Exercise 1.4: (Proposition 1.5c) Prove that $T \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ is n-terminal if and only if there is a $T_n \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ such that $T = \mathbb{R}^n \times T_n$.

Definition 1.8: We use a particular notation for the following systems of sets:

- *n-symmetric sets*: $\mathcal{S}_n = \{S \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) : p(S) = S \text{ for any finite permutation } p \text{ of order } n\}$,
- *symmetric sets*: $\mathcal{S} = \{S \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) : p(S) = S \text{ for any finite permutation } p\}$,
- *shift invariant sets*: $\mathcal{I} = \{I \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) : s^{-1}I = I\}$,
- *n-terminal sets*: $\mathcal{T}_n = \{T \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) : T \ n\text{-terminal}\}$,
- *terminal sets*: $\mathcal{T} = \{T \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) : T \text{ terminal}\}$.

Exercise 1.5: (Proposition 1.5d)

a) Show that $\mathcal{S}_{n+1} \subset \mathcal{S}_n$ for all $n \in \mathbb{N}$ and $\mathcal{S} = \cap_{n=1}^{\infty} \mathcal{S}_n$.
b) Show that $\mathcal{T}_{n+1} \subset \mathcal{T}_n$ for all $n \in \mathbb{N}$ and $\mathcal{T} = \cap_{n=1}^{\infty} \mathcal{T}_n$.
c) Check that $\mathcal{S}$, $\mathcal{I}$ and $\mathcal{T}$ are σ-algebras.
d) Prove that $\mathcal{I} \subset \mathcal{T}_n \subset \mathcal{S}_n$ for all $n \in \mathbb{N}$ and hence $\mathcal{I} \subset \mathcal{T} \subset \mathcal{S}$.
e) Show that the previous inclusions are strict, i.e. the sets are not equal. Give examples of invariant, symmetric and terminal sets.

Definition 1.10: We call the set $B \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ *finite-dimensional* if there are $n \in \mathbb{N}$ and $B_n \in \mathcal{B}(\mathbb{R}^n)$ such that $B = B_n \times \mathbb{R}^{\mathbb{N}}$.

Exercise 1.6: (Proposition 1.6) Denote by $\mathcal{A}$ the system of finite-dimensional sets from $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$. Prove that $\mathcal{A}$ is an algebra generating $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$, i.e. it holds that $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R}^{\mathbb{N}})$.

NMSA405: exercise 2 – random sequences

Definition 1.13: *Binary expansion* of the number $x \in (0, 1]$ is the sequence $x_1, x_2, \ldots$ of zeroes and ones such that it contains infinitely many ones and

$$x = \sum_{k=1}^{\infty} \frac{x_k}{2^k}.$$

Binary expansion of the number 0 is the sequence of zeroes.

Exercise 2.1: (Proposition 1.14) Prove that if X is a random variable with uniform distribution on the interval $[0, 1]$ and

$$X(\omega) = \sum_{k=1}^{\infty} \frac{X_k(\omega)}{2^k} \tag{1}$$

is its binary expansion then $X_1, X_2, \ldots$ is a sequence of independent random variables with Bernoulli distribution with parameter $1/2$.
Conversely, consider a sequence of independent random variables with Bernoulli distribution with parameter $1/2$ and define X using the equation (1). Prove that X has uniform distribution on the interval $[0, 1]$.

Exercise 2.2: Show that there is a random sequence $W_1, W_2, \ldots$ such that its increments W_1, $W_2 - W_1$, $W_3 - W_2, \ldots$ are independent random variables with standard normal distribution. Determine the distribution of the vector $(W_1, \ldots, W_n)$.

Definition 1.14: We call the random sequence $X = (X_1, X_2, \ldots)$

- *iid* if the random variables X_j, $j \in \mathbb{N}$, are independent and identically distributed,
- *n-symmetric* if the distributions of $(X_1, \ldots, X_n, X_{n+1}, \ldots)$ and $(X_{k_1}, \ldots, X_{k_n}, X_{n+1}, \ldots)$ coincide for each finite permutation $(k_1, \ldots, k_n)$ of order $n \in \mathbb{N}$,
- *symmetric* if it is n-symmetric for each $n \in \mathbb{N}$,
- *stationary* if the distributions of $(X_1, \ldots, X_n, X_{n+1}, \ldots)$ and $(X_{n+1}, X_{n+2}, \ldots)$ coincide for each $n \in \mathbb{N}$.

Exercise 2.3: Show that the following statements are equivalent:

a) random sequence $X = (X_1, X_2, \ldots)$ is stationary,
b) X and $s(X)$ have the same distribution,
c) random vectors $(X_1, \ldots, X_{n-1})$ and $(X_2, \ldots, X_n)$ have the same distribution for each $n \in \mathbb{N}$.

Exercise 2.4: Prove the following assertions.

a) Each iid sequence is symmetric.
b) Each symmetric sequence is stationary.
c) Each $(n + 1)$-symmetric sequence is n-symmetric for any $n \in \mathbb{N}$.
d) Let $X = (X_1, X_2, \ldots)$ be an iid random sequence and $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ Borel-measurable mapping such that $f \circ s = s \circ f$ (f and the shift commute). Prove that in such a case $f(X) = (Y_1, Y_2, \ldots)$ is stationary. Does this assertion hold if we instead assumed only stationarity of X?

Exercise 2.5: Give an example of

a) a symmetric sequence which is not iid,
b) a stationary sequence which is not symmetric,
c) n-symmetric sequence which is not $(n + 1)$-symmetric.