NMSA409, topic 6: Invertibility of ARMA series

Definition 6.1: Let $\{X_t, t \in \mathbb{Z}\}$ be a stationary ARMA(m, n) random sequence defined by

$$X_t + a_1 X_{t-1} + \cdots + a_m X_{t-m} = Y_t + b_1 Y_{t-1} + \cdots + b_n Y_{t-n}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise WN$(0, \sigma^2)$. If there exists a sequence of constants $\{d_j, j \in \mathbb{N}_0\}$ such that $\sum_{j=0}^{\infty} |d_j| < \infty$ and

$$Y_t = \sum_{j=0}^{\infty} d_j X_{t-j}, \quad t \in \mathbb{Z},$$

then $\{X_t, t \in \mathbb{Z}\}$ is called *invertible* (it has an AR(∞) representation).

Theorem 6.1: Let $\{X_t, t \in \mathbb{Z}\}$ be a stationary ARMA(m, n) random sequence. Let the polynomials $a(z) = 1 + a_1 z + \cdots + a_m z^m$ and $b(z) = 1 + b_1 z + \cdots + b_n z^n$ have no common roots and let the polynomial $b(z) = 1 + b_1 z + \cdots + b_n z^n$ have all the roots outside the unit circle. Then $\{X_t, t \in \mathbb{Z}\}$ is invertible and the coefficients d_j are given by

$$\sum_{j=0}^{\infty} d_j z^j = \frac{1 + a_1 z + \cdots + a_m z^m}{1 + b_1 z + \cdots + b_n z^n}, \quad |z| \leq 1.$$

Remark: We may obtain the coefficients d_j by solving the equations we get by plugging $Y_t = \sum_{j=0}^{\infty} d_j X_{t-j}$ into the defining formula of the ARMA sequence and comparing the coefficients on both sides.

Consider a causal and invertible ARMA(m, n) sequence. Invertibility implies (note that $d_0 = 1$)

$$X_{t+1} = -\sum_{j=1}^{\infty} d_j X_{t+1-j} + Y_{t+1}, \quad t \in \mathbb{Z}.$$

Causality implies that the random variable Y_{t+1} is independent of X_t, $X_{t-1} \ldots$ Thus the best linear prediction of X_{t+1} based on the whole history X_t, $X_{t-1}, \ldots$ is the prediction

$$\widehat{X}_{t+1} = -\sum_{j=1}^{\infty} d_j X_{t+1-j}.$$

The prediction error is

$$\mathbb{E}|X_{t+1} - \widehat{X}_{t+1}|^2 = \mathbb{E}|Y_{t+1}|^2 = \sigma^2.$$

Exercise 6.1: Consider the ARMA(1,1) model defined by

$$X_t + 0{,}7 X_{t-1} = Y_t + 0{,}3 Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise WN$(0, \sigma^2)$. Determine the coefficients of the AR(∞) representation. Find the prediction of X_{n+1}, X_{n+2} based on the history X_n, $X_{n-1}, \ldots$ Determine the prediction error.

Exercise 6.2: Consider the ARMA(2,1) model defined by

$$X_t - 0{,}1 X_{t-1} - 0{,}12 X_{t-2} = Y_t - 0{,}7 Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise WN$(0, \sigma^2)$. Express $\{X_t, t \in \mathbb{Z}\}$ as a causal linear process. Determine its autocovariance function and spectral density. Decide whether it is invertible. Assume that the whole history up to time n is known. Find the prediction of X_{n+1}, X_{n+2} based on X_n, $X_{n-1}, \ldots$

NMSA409, topic 7: Linear filters

Definition 7.1: Let $\{Y_t, t \in \mathbb{Z}\}$ be a centered weakly stationary sequence. Let $\{c_j, j \in \mathbb{Z}\}$ be a sequence of (complex-valued) numbers such that $\sum_{j=-\infty}^{\infty} |c_j| < \infty$.
We say that a random sequence $\{X_t, t \in \mathbb{Z}\}$ is obtained by filtration of the sequence $\{Y_t, t \in \mathbb{Z}\}$ if

$$X_t = \sum_{j=-\infty}^{\infty} c_j Y_{t-j}, \quad t \in \mathbb{Z}.$$

The sequence $\{c_j, j \in \mathbb{Z}\}$ is called *time-invariant linear filter.*
Provided that $c_j = 0$ for all $j < 0$, we say that the filter $\{c_j, j \in \mathbb{Z}\}$ is *causal.*

Theorem 7.1: Let $\{Y_t, t \in \mathbb{Z}\}$ be a centered weakly stationary sequence with the autocovariance function R_Y and the spectral density f_Y and let $\{c_k, k \in \mathbb{Z}\}$ be a linear filter such that $\sum_{k=-\infty}^{\infty} |c_k| < \infty$. Then $\{X_t, t \in \mathbb{Z}\}$, where $X_t = \sum_{k=-\infty}^{\infty} c_k Y_{t-k}$, is a centered weakly stationary sequence with the autocovariance function

$$R_X(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_j \overline{c_k} R_Y(t - j + k), \quad t \in \mathbb{Z},$$

and spectral density

$$f_X(\lambda) = |\Psi(\lambda)|^2 f_Y(\lambda), \quad \lambda \in [-\pi, \pi],$$

where

$$\Psi(\lambda) = \sum_{k=-\infty}^{\infty} c_k e^{-ik\lambda}, \quad \lambda \in [-\pi, \pi],$$

is called *the transfer function of the filter.*

Exercise 7.1: Let $\{Z_t, t \in \mathbb{Z}\}$ be a white noise $\mathrm{WN}(0, 1)$ and let $\{X_t, t \in \mathbb{Z}\}$ be a causal linear process defined by

$$X_t - 0.99X_{t-3} = Z_t, \quad t \in \mathbb{Z}.$$

Let $\{Y_t, t \in \mathbb{Z}\}$ be the process obtained by the filtration $Y_t = \frac{1}{3}(X_{t-1} + X_t + X_{t+1})$. Determine the transfer function of the filter and compute the spectral density of $\{Y_t\}$.

Exercise 7.2: Let $\{Y_t, t \in \mathbb{Z}\}$ be a white noise $\mathrm{WN}(0, \sigma^2)$. Let it be transformed by a linear filter into $\{X_t, t \in \mathbb{Z}\}$ so that

$$X_t - 2X_{t-1} = Y_t, \quad t \in \mathbb{Z},$$

holds. Determine the coefficients of the linear filter, the transfer function of the filter and compute the autocovariance function and the spectral density of $\{X_t, t \in \mathbb{Z}\}$.

NMSA409, topic 8: Ergodicity of stochastic processes

Definition 8.1: We say that a stationary sequence $\{X_t, t \in \mathbb{Z}\}$ with mean μ is *mean square ergodic* or it follows the law of large numbers in $L_2(\Omega, \mathcal{A}, P)$ if, as $n \to \infty$,

$$\overline{X}_n = \frac{1}{n}\sum_{t=1}^{n} X_t \to \mu \quad \text{in mean square (in } L_2). \tag{1}$$

If $\{X_t, t \in \mathbb{Z}\}$ is a mean square ergodic sequence then

$$\frac{1}{n}\sum_{t=1}^{n} X_t \xrightarrow{P} \mu,$$

i.e., $\{X_t, t \in \mathbb{Z}\}$ satisfies the weak law of large numbers for stationary sequences.

Definition 8.2: A stationary mean square continuous process $\{X_t, t \in \mathbb{R}\}$ with mean μ is *mean square ergodic* if, as $\tau \to \infty$,

$$\overline{X}_\tau = \frac{1}{\tau}\int_0^\tau X_t \, \mathrm{d}t \to \mu \quad \text{in the mean square (in } L_2). \tag{2}$$

Remark: The convergences above imply that the empirical average (1) or the integral (2) are weakly consistent estimates of the mean value μ of the random sequence or the process $\{X_t\}$, respectively.

Theorem 8.1: A stationary random sequence $\{X_t, t \in \mathbb{Z}\}$ with mean μ and autocovariance function R is mean square ergodic if and only if

$$\frac{1}{n}\sum_{t=1}^{n} R(t) \to 0 \quad \text{as } n \to \infty.$$

If the sequence is real-valued and it also satisfies $\sum_{t=-\infty}^{\infty} |R(t)| < \infty$ then $n\mathrm{var}(\overline{X}_n) \to \sum_{k=-\infty}^{\infty} R(k)$.

Theorem 8.2: A stationary mean square continuous process $\{X_t, t \in \mathbb{R}\}$ is mean square ergodic if and only if its autocovariance function satisfies the condition

$$\frac{1}{\tau}\int_0^\tau R(t)\, \mathrm{d}t \to 0 \quad \text{as } \tau \to \infty.$$

If the process is real-valued and it also satisfies $\int_{\infty}^{\infty} |R(t)|\, \mathrm{d}t < \infty$ then $\tau\mathrm{var}(\overline{X}_\tau) \to \int_{\infty}^{\infty} R(t)dt$.

Exercise 8.1: Are the AR models from exercises 5.3–5.5 mean square ergodic? And what about the ARMA(2,1) model from exercise 5.7?

Exercise 8.2: Is the mean square continuous process with spectral density $f(\lambda) = |\lambda| I(|\lambda| \leq 1), \lambda \in \mathbb{R}$, mean square ergodic?