Numerical software 2
Anisotropic mesh adaptation

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Lecture 1

Overview

Numerical solution of PDE

- we seek $u : \Omega \to \mathbb{R}$ such that $\mathscr{L} u = f$ in Ω
- we define mesh $\mathscr{T}_h$ of Ω and finite dimensional space V_h
- approximate solution $u_h \in V_h$

Main goal

Define (create) a mesh $\mathscr{T}_h$ such that

1. the computational error is under the given tolerance
2. the number of elements of $\mathscr{T}_h$ is as small as possible

Fundamental question

How to fulfil the main goal?

- we adapt the given mesh based on the computed solution and its error estimation

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Interpolation error

Main idea

- let u be the exact solution and $u_h \in V_h$ the approximate one
- let $\Pi_h : V \to V_h$ be a projection
- we approximate $u - u_h \approx u - \Pi_h u$
- $u - \Pi_h u =$ interpolation error

Formulation of an abstract problem

- Let $u : \Omega \to \mathbb{R}$ be a given function and $\omega > 0$
- let $\Pi_h : V \to V_h$ be a projection
- We seek $\mathscr{T}_h$ such that
  1. $\|u - \Pi_h\| \leq \omega$
  2. $\#\mathscr{T}_h$ be minimal

exact solution u is unknown, it will be later approximated by u_h

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Concrete formulation

- $V_h = \{v_h \in L^2(\Omega);\ v_h|_K \in P^1(K)\ \forall K \in \mathscr{T}_h\}$ – discontinuous piecewise linear
- $\|\cdot\| := \|\cdot\|_{L^\infty(\Omega)}$
- $\Pi_h : V \to V_h$ such that

1. $\Pi_h u(x_K) = u(x_K)$, x_K is the barycentre of $K \in \mathscr{T}_h$
2. $\nabla \Pi_h u(x_K) = \nabla u(x_K)$, x_K is the barycentre of $K \in \mathscr{T}_h$

$\Pi_h u$ is a discontinuous piecewise linear, the same value and gradient as u in the barycentres of all $K \in \mathscr{T}_h$

Interpolation error

$$u(x) = \underbrace{u(x_K) + \nabla u(x_K)(x - x_K)}_{\Pi_h u(x)} + \frac{1}{2}(x - x_K)^T \mathscr{H}(u(x'))(x - x_K)$$

$\mathscr{H}(u(\cdot))$ ... Hessian matrix

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Interpolation error estimate

$$u(x) - \Pi_h u(x) \approx \frac{1}{2}(x - x_K)^T \mathscr{H}(u(x_K))(x - x_K),$$

$$\mathscr{H}(u(x)) = \begin{pmatrix} \frac{\partial^2 u(x)}{\partial x_1^2} & \frac{\partial^2 u(x)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 u(x)}{\partial x_1 \partial x_2} & \frac{\partial^2 u(x)}{\partial x_2^2} \end{pmatrix}$$

we assume that $\mathscr{H}(u)$ is positively definite then (up to a higher order terms)

Interpolation error estimate

$$|u(x) - \Pi_h u(x)| \leq \frac{1}{2}(x - x_K)^T \mathscr{H}(u(x_K))(x - x_K),$$

Optimal triangle

Interpolation error estimate

$$|u(x) - \Pi_h u(x)| \leq \frac{1}{2}(x - x_K)^T \mathscr{H}(u(x_K))(x - x_K), \quad x \in K$$

Our goal

$$\|u(x) - \Pi_h u(x)\| \leq \omega \quad \Leftrightarrow \quad |u(x) - \Pi_h u(x)| \leq \omega \ \forall x \in K$$

Equivalent condition

$$\frac{1}{2}(x - x_K)^T \mathscr{H}(u(x_K))(x - x_K) \leq \omega \quad \forall x \in K \tag{1}$$

Geometrical interpretation

all $x \in \mathbb{R}^2$ satisfying (1) form an ellipse

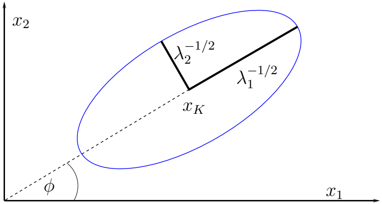
Ellipse

- let $\mathbb{M}$ be a symmetric, positively definite, then

$$\mathbb{M} = \mathbb{R}^T \mathbb{L} \mathbb{R},$$

where $\mathbb{R} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}, \qquad \mathbb{L} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$

$$(x - x_K)^T \mathbb{M} (x - x_K) \leq 1 \quad \Longleftrightarrow \quad x \in E$$



Fulfilling of requirements

let $\mathscr{H}(u(x_K))$ be the Hessian, E_K be the corresponding ellipse

Lemma

$\frac{1}{2}(x - x_K)^T \mathscr{H}(u(x_K))(x - x_K) \leq \omega \ \forall x \in K$ is valid $\Leftrightarrow$ $K \subset E_K$

Definition

K is optimal triangle $\Leftrightarrow K \subset E_K$ & area of K is maximal

Lemma

Let $\mathscr{H} = \mathscr{H}(u(x_K))$ be the Hessian, E_K the ellipse.
Then K is optimal triangle $\Leftrightarrow$

$$\|e_{K,i}\|_{\mathscr{H}} := \left(e_{K,i}^T \mathscr{H} e_{K,i}\right)^{1/2} = \sqrt{6\omega}, \quad i = 1, 2, 3,$$

where $e_{K,i}$, $i = 1, 2, 3$ are edges of K.

Edge-based optimality

- Let K and K' share edge e, $\mathscr{H}$ and $\mathscr{H}'$ be the Hessians
- then $\|e\|_{\mathscr{H}} = \sqrt{6\omega} = \|e\|_{\mathscr{H}'}$ can not be valid!!
- it is impossible to consider $\mathscr{T}_h$ as a set of "optimal triangles"

Edge-based mesh

- Let $\mathscr{F}_h$ denotes the set of edges e of the mesh $\mathscr{T}_h$
- Let $\mathscr{H}_e$ be the Hessian evaluated at edge $e \in \mathscr{F}_h$.
- Def: mesh $\mathscr{T}_h$ is edge-optimal $\Leftrightarrow$ $\|e\|_{\mathscr{H}_e} = \sqrt{6\omega} \quad \forall e \in \mathscr{F}_h$

Edge optimal mesh exists only in special situation

Definition

Mesh $\mathscr{T}_h$ is optimal $\iff$ $Q_{\mathscr{T}_h} = \min_{\mathscr{T}_{h'}} Q_{\mathscr{T}_{h'}}$

where $Q_{\mathscr{T}_{h'}} = \frac{1}{\#\mathscr{F}_{h'}} \sum_{e\in\mathscr{F}_{h'}} (\|e\|_{\mathscr{H}_e} - \sqrt{6\omega})^2$

$Q_{\mathscr{T}_h} \geq 0$ ... parameter of "quality"

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Mesh optimization process

Quality of mesh

- $Q_{\mathscr{T}_h} = \frac{1}{\#\mathscr{F}_h} \sum_{e \in \mathscr{F}_h} (\|e\|_{\mathscr{H}_e} - \sqrt{6\omega})^2 \geq 0$
- smaller $Q_{\mathscr{T}_h}$ means "better" mesh.

Idea of mesh optimization

modify locally mesh in such a way that $Q_{\mathscr{T}_h}$ is decreasing

Mesh optimization algorithm

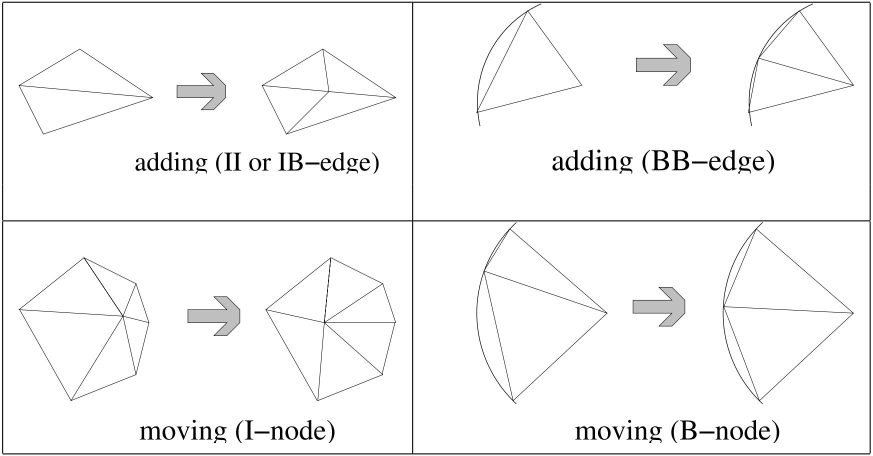
- several local operations (adding a node, removing an edge, moving a node, etc.)
- tested and performed if $Q_{\mathscr{T}_h}$ is decreasing

Riemann metric

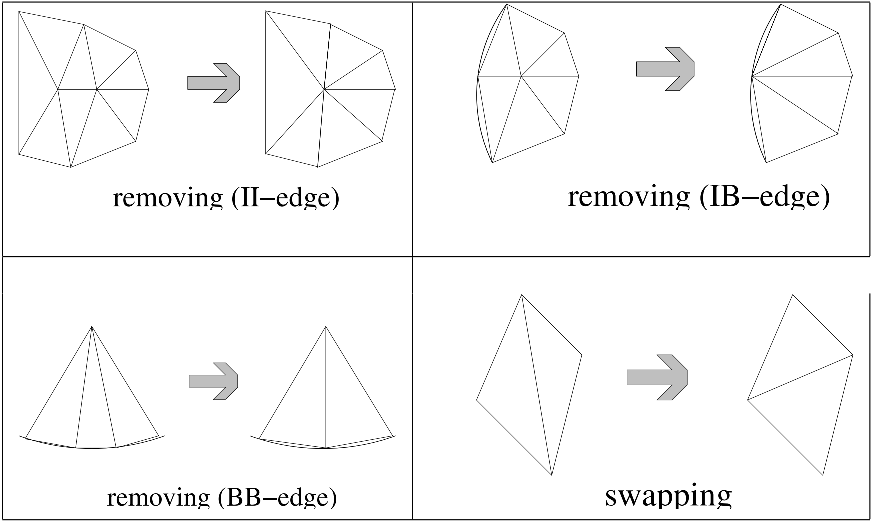
Optimal mesh

$\Leftrightarrow$ mesh is uniform in the Riemann metric generated by $\mathscr{H}$

Local operations: adding and moving



Local operations: removing and swapping

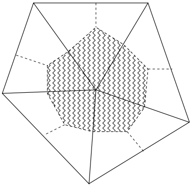


Towards practical realization

Mesh optimization algorithm

For given $u \in V$ and $\omega > 0$, we can construct optimal mesh $\mathscr{T}_h$

- u has to be approximated by u_h
- in practice, we need to approximate $\mathscr{H}(u)$ on each edge only
- we approximate $\mathscr{H}(u)$ at vertices P_k of mesh
- let D_k be a polygon around P_k



$$\frac{\partial^2 u}{\partial x_i \partial x_j}(P_k) \approx \frac{1}{|D_k|} \int_{D_k} \frac{\partial^2 u}{\partial x_i \partial x_j} \,\mathrm{d}x = \frac{1}{|D_k|} \int_{\partial D_k} \frac{\partial u}{\partial x_i} n_j \,\mathrm{d}S$$
$$\approx \frac{1}{|D_k|} \int_{\partial D_k} \frac{\partial u_h}{\partial x_i} n_j \,\mathrm{d}S$$

Towards practical realization

Approximation of $\mathscr{H}$

- previous approximation gives $\mathscr{H}(u(P_k)) \approx \mathbb{H}(u_h(P_k))$
- $\mathbb{H}$ is symmetric, not positively definite
- we put $\bar{\mathbb{H}} := |\mathbb{H}|$ using the eigenvalue decomposition

Regularization

- if u_h is linear then $\bar{\mathbb{H}} = 0 \ldots$ problem
- if u_h is discontinuous then $\bar{\mathbb{H}}$ can blow up
- in order to overcome this problem, we set

$$\mathbb{M}(P_k) = c \left[\mathbb{I} + \frac{\varepsilon_1}{\varepsilon_1/p + \|\bar{\mathbb{H}}(P_k)\|} \bar{\mathbb{H}}(P_k) \right] \tag{2}$$

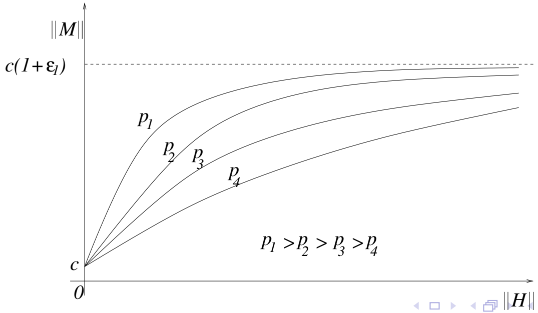
- we replace $\bar{\mathbb{H}}$ by $\mathbb{M}$
- c, ε_1, p suitably chosen constants

Setting of matrices $\mathbb{M}$ (metric)

$$\mathbb{M}(P_k) = c\left[\mathbb{I} + \frac{\varepsilon_1}{\varepsilon_1/p + \|\bar{\mathbb{H}}(P_k)\|}\bar{\mathbb{H}}(P_k)\right]$$

Setting of constant

- if $\bar{\mathbb{H}} = 0$ then $\mathbb{M} = c\mathbb{I}$, c density of the coarsest mesh $\sim$numel
- if $\|\bar{\mathbb{H}}\| \to \infty$ then $\|\mathbb{M}\| \to c(1+\varepsilon_1) \approx c\varepsilon_1$, $\quad \varepsilon_1 = \left(\frac{\ell_{\max}}{\ell_{\min}}\right)^2$
- p the "speed" of transition (from coarse to fine parts)



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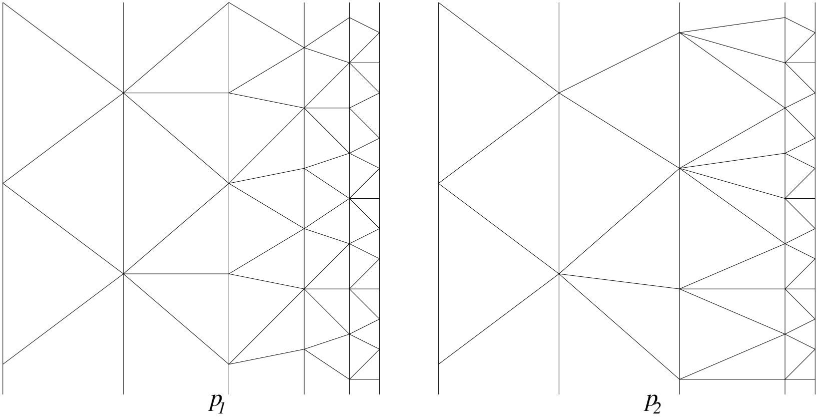

$$p_1 \quad > \quad p_2$$

Multilevel computation