

# FINITE ELEMENT METHODS: IMPLEMENTATIONS

Vít DOLEJŠÍ

Charles University Prague  
Faculty of Mathematics and Physics  
Department of Numerical Mathematics  
Sokolovská 83, 186 75 Prague  
Czech Republic

Praha

January 4, 2011



# Contents

<b>1</b>	<b>Implementation of FEM and DGM</b>	<b>5</b>
1.1	$P^1$ solution of the model problem . . . . .	5
1.1.1	Triangulation . . . . .	6
1.1.2	Finite element space . . . . .	6
1.1.3	Discretization . . . . .	8
1.1.4	Implementation . . . . .	8
1.1.5	Treatment of general boundary conditions . . . . .	15
1.2	General FE solution of the model problem . . . . .	17
1.2.1	Definition of reference elements . . . . .	17
1.2.2	Evaluation of integrals on reference element . . . . .	19
1.2.3	Grids and functional spaces . . . . .	21
1.2.4	Implementation . . . . .	23
1.3	Basis and shape functions . . . . .	24
1.3.1	The reference Lagrangian shape functions . . . . .	24
1.3.2	The reference Lobatto shape functions . . . . .	27
1.3.3	Global basis functions on $V_h$ . . . . .	30
1.4	Discontinuous finite elements . . . . .	33
1.4.1	DG discretization of the model problem . . . . .	33
1.4.2	$hp$ -discontinuous Galerkin method . . . . .	35
1.4.3	Definition of a DG basis . . . . .	35
1.4.4	Construction of the DG basis functions . . . . .	36
1.4.5	Evaluation of volume integrals . . . . .	38
1.4.6	Evaluation of face integrals . . . . .	38
1.4.7	Data structure . . . . .	41
1.5	Numerical quadratures . . . . .	41
1.5.1	Edge quadratures . . . . .	41
1.5.2	Quadrilaterals quadratures . . . . .	41
1.5.3	Triangles quadratures . . . . .	43
1.5.4	Data structure . . . . .	43
1.6	Basic visualization techniques . . . . .	43
1.6.1	Types of visualization . . . . .	45
1.6.2	Software for visualization . . . . .	46
1.6.3	Visualization of higher order polynomial function . . . . .	47



# Chapter 1

## Implementation of FEM and DGM

In this chapter we deal with the implementation of the finite element and the discontinuous Galerkin methods which were introduced in the previous chapters. However, the implementation of these methods is rather complicated problem which can differ based on the practical applications. Our aim is only to introduce some basic ideas. This chapter is relatively autonomous, all necessary definitions and notations are recalled.

In order to state these notes more readable, we start in Section 1.1 with the implementation of a simple problem, namely  $P^1$  continuous finite element solution of the Poisson problem with homogeneous Dirichlet boundary condition. We describe the implementation of this problem in details including the code subroutines which are written in the Fortran 90 syntax. We suppose that the non-Fortran readers will be able to understand this syntax too since the subroutines contain only standard 'if then' conditions and 'do' cycles. The subroutines are commented in the standard way, i.e., text behind the character '!' is ignored.

In Section 1.2 we describe a general implementation of the conforming finite element method. We employ the concept of the reference element which seems (in the authors' opinion) more suitable for the implementation. Moreover, we include a description of the necessary data structures. In Section 1.3 we present a construction of the basis functions based on two possible sets of shape functions, the Lagrangian and the Lobatto shape functions. Section 1.4 contains an implementation of the discontinuous Galerkin method with the emphasizes on the differences with the conforming methods. Since DG methods allow a simple treatment of  $hp$ -methods, we consider an approximation of different polynomial degrees on different elements. Finally, Sections 1.5 and 1.6 describes possible numerical quadratures and visualizations techniques, respectively.

### 1.1 $P^1$ solution of the model problem

In order to explain the basic aspects of the implementation of FEM and DGM, we start with the model problem represented by the Poisson problem. We recall the weak formulation and the finite element formulation of this elliptic problem.

Let  $\Omega$  be a bounded polygonal domain in  $\mathbb{R}^d$ ,  $d = 2, 3$ , with a boundary  $\partial\Omega$ . We

seek a function  $u : \Omega \rightarrow \mathbb{R}$  such that

$$-\Delta u(x) = g(x), \quad x \in \Omega, \quad (1.1)$$

$$u(x) = 0, \quad x \in \partial\Omega, \quad (1.2)$$

where  $g \in L^2(\Omega)$ .

Let  $V := H_0^1(\Omega)$  then the *weak formulation* of (1.1) reads

$$\text{find } u \in V : \quad (\nabla u, \nabla v) = (g, v) \quad \forall v \in V. \quad (1.3)$$

Let  $V_h$  be a finite dimensional subspace of  $V$  than the *finite element* approximation (1.3) can be written as

$$\text{find } u_h \in V_h : \quad (\nabla u_h, \nabla v_h) = (g_h, v_h) \quad \forall v_h \in V. \quad (1.4)$$

In the following, we describe in details the implementation of the method (1.4) with the aid of a continuous piecewise linear approximation constructed over triangular grid.

### 1.1.1 Triangulation

Let  $\mathcal{T}_h$  ( $h > 0$ ) be a partition of the closure  $\bar{\Omega}$  of the domain  $\Omega$  into a finite number of closed  $d$ -dimensional simplexes  $T$  with mutually disjoint interiors such that

$$\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} T. \quad (1.5)$$

In two-dimensional problems ( $d = 2$ ), the elements  $T \in \mathcal{T}_h$  are triangles and in three-dimensional problems ( $d = 3$ ) the elements  $T \in \mathcal{T}_h$  are tetrahedra. Figure 1.1 shows two examples of triangular grid constructed over a square domain.

Moreover, we use the following notation for vertices and edges (faces for  $d = 3$ ) of  $\mathcal{T}_h$ . We denote by  $\{\mathbf{v}_i\}_{i=1}^N$  the set of all vertices of  $T \in \mathcal{T}_h$  lying inside of  $\Omega$  and by  $\{\mathbf{v}_i\}_{i=N+1}^{N+N_b}$  the set of all vertices of  $T \in \mathcal{T}_h$  lying on  $\partial\Omega$ . Finally,  $\{e_i\}_{i=1}^E$  denotes the set of all edges (faces) of all  $T \in \mathcal{T}_h$  lying inside of  $\Omega$  and  $\{e_i\}_{i=E+1}^{E+E_b}$  the set of all edges (faces) of all  $T \in \mathcal{T}_h$  lying on  $\partial\Omega$ .

### 1.1.2 Finite element space

Let

$$V_h := \{v_h, v_h \in C(\bar{\Omega}) \cap H_0^1(\Omega), v_h|_T \in P^1(T) \forall T \in \mathcal{T}_h\} \quad (1.6)$$

denote the space of continuous piecewise linear functions on  $\mathcal{T}_h$  vanishing on  $\partial\Omega$ . Obviously, each  $v_h \in V_h$  is uniquely defined by its values in  $\{\mathbf{v}_i\}_{i=1}^N$ .

We denote by  $B := \{\varphi_i\}_{i=1}^N$  the set of basis functions of  $V_h$  defined by

$$\varphi_i \in V_h, \quad \varphi_i(\mathbf{v}_j) = \delta_{ij}, \quad i, j = 1, \dots, N, \quad (1.7)$$

where  $\delta_{ij}$  is the Kronecker symbol. Obviously, support of each  $\varphi_i$ ,  $i = 1, \dots, N$  is small and consists of all triangles  $T \in \mathcal{T}_h$  having  $\mathbf{v}_i$  as vertex, see Figure 1.2.

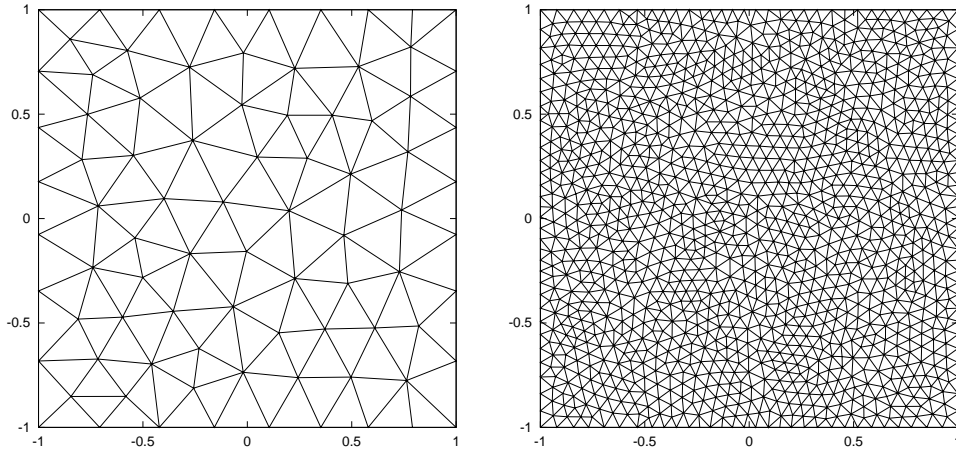


Figure 1.1: Examples of triangular meshes constructed over  $\Omega = (-1, 1) \times (-1, 1)$ : a coarser grid (left) and a finer one (right)

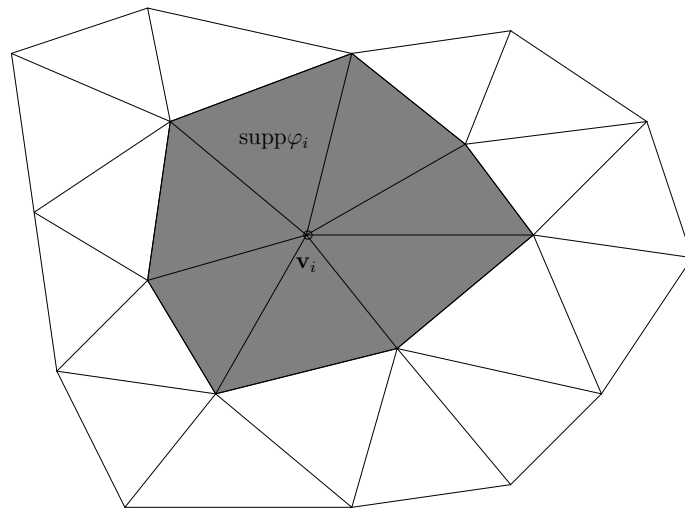


Figure 1.2: Support of the basis function  $\varphi_i$  corresponding to the node  $\mathbf{v}_i$

### 1.1.3 Discretization

The approximate solution  $u_h \in V_h$  can be expressed by

$$u_h(x) = \sum_{j=1}^N u_j \varphi_j(x), \quad u_j \in \mathbb{R}, \quad j = 1, \dots, N. \quad (1.8)$$

Moreover, since  $\varphi_i$ ,  $i = 1, \dots, N$  form the basis of  $V_h$ , it is sufficient that the relation (1.4) is valid for each  $\varphi_i$ ,  $i = 1, \dots, N$ . Then, using (1.8), the approximate problems (1.8) is equivalent to

$$\text{find } \{u_j\}_{j=1}^N \in \mathbb{R}^N \quad \sum_{j=1}^N u_j (\nabla \varphi_j, \nabla \varphi_i) = (g, \varphi_i) \quad \forall i = 1, \dots, N. \quad (1.9)$$

Let us define the *stiffness matrix*

$$\mathbb{S} = \{S_{ij}\}_{i,j=1}^N, \quad S_{ij} = (\nabla \varphi_j, \nabla \varphi_i), \quad (1.10)$$

the *solution vector*  $\mathbf{u} = \{u_i\}_i^N$  and the right-hand side

$$\mathbf{g} = \{g_i\}_{i=1}^N, \quad g_i = (g, \varphi_i), \quad i = 1, \dots, N. \quad (1.11)$$

Then, the problem (1.9) is equivalent to the linear algebraic problem

$$\mathbb{S}\mathbf{u} = \mathbf{g}. \quad (1.12)$$

Therefore, in order to implement the conforming piecewise linear FEM to (1.3), we have to

- evaluate  $\mathbb{S}$  and  $\mathbf{g}$  according to (1.10) – (1.11),
- solve the algebraic problem (1.12),
- reconstruct the approximate solution  $u_h \in V_h$  from  $\mathbf{u}$  using (1.8).

### 1.1.4 Implementation

#### Data structures

In order to present algorithms, we introduce the global data structures defining the triangulation  $\mathcal{T}_h$ . For simplicity, we restrict to  $d = 2$ .

Let  $\mathbf{N}$  denote the number of inner vertices of  $\mathcal{T}_h$  and  $\mathbf{Nb}$  denote the number of vertices of  $\mathcal{T}_h$  lying on  $\partial\Omega$ . Let the real arrays  $\mathbf{xp}(1:\mathbf{N}+\mathbf{Nb})$  and  $\mathbf{yp}(1:\mathbf{N}+\mathbf{Nb})$  be the  $x_1$ - and  $x_2$ -coordinates of the nodes of mesh  $\mathcal{T}_h$  indexed by  $i = 1, \dots, \mathbf{N} + \mathbf{Nb}$ , respectively. We assume that first  $\mathbf{N}$  components of  $\mathbf{xp}(\cdot)$  and  $\mathbf{yp}(\cdot)$  correspond to inner vertices and the last  $\mathbf{Nb}$  components of  $\mathbf{xp}(\cdot)$  and  $\mathbf{yp}(\cdot)$  to boundary vertices.

Furthermore, let the vertexes of triangle  $T_i \in \mathcal{T}_h$ ,  $i = 1, \dots, M$  be indexed by the integer (two-dimensional) array  $\mathbf{node}(i, 1:3)$ . Obviously, if some vertex of  $T_i$  lies on  $\partial\Omega$  then we have for the corresponding value  $\mathbf{node}(i, j) > \mathbf{N}$ .

For simplicity, we assume that  $\mathbf{xp}(\cdot)$ ,  $\mathbf{yp}(\cdot)$ ,  $\mathbf{node}(\cdot, \cdot)$  are global arrays so that we have access to them from any subroutine.



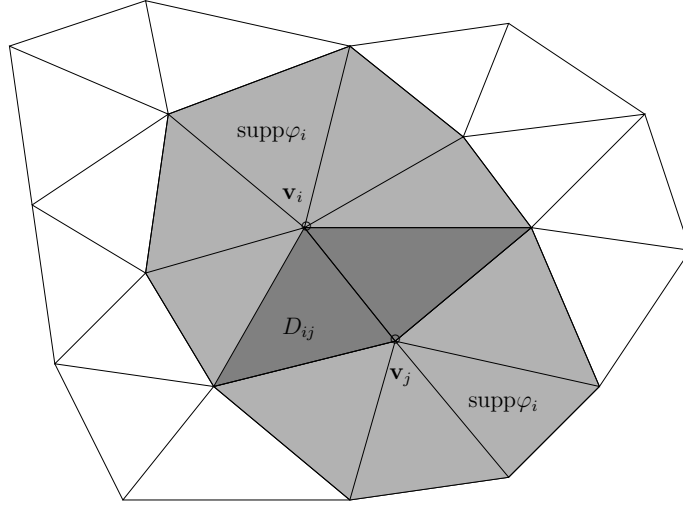


Figure 1.3: Volume  $D_{ij} = \text{supp}(\varphi_i) \cap \text{supp}(\varphi_j)$  for nodes  $\mathbf{v}_i$  and  $\mathbf{v}_j$  connected by an edge

### Setting of the stiff matrix – non-optimized algorithm

Here we present a non-optimized algorithm for the evaluation of the entries of the stiff matrix  $\mathbb{S}$ . According to (1.10), the entries of the stiff matrix satisfy

$$S_{ij} = \int_{\Omega} \nabla \varphi_j \cdot \nabla \varphi_i \, d\mathbf{x}. \quad (1.13)$$

Since the support of each  $\varphi_i$ ,  $i = 1, \dots, N$  is located around the corresponding node  $\mathbf{v}_i$ , we can distinguish three cases:

$$S_{ij} = \begin{cases} \int_{\text{supp}(\varphi_i)} |\nabla \varphi_i|^2 \, d\mathbf{x} & \text{if } i = j, \\ \int_{D_{ij}} \nabla \varphi_j \cdot \nabla \varphi_i \, d\mathbf{x} & \text{if } \mathbf{v}_i \text{ and } \mathbf{v}_j \text{ are connected by an edge,} \\ 0 & \text{otherwise.} \end{cases} \quad (1.14)$$

Here,  $D_{ij} = \text{supp}(\varphi_i) \cap \text{supp}(\varphi_j)$ , see Figure 1.3.

For the diagonal entries of the stiff matrix  $S_{ii}$ ,  $i = 1, \dots, N$ , we have

$$S_{ii} = \int_{\text{supp}(\varphi_i)} |\nabla \varphi_i|^2 \, d\mathbf{x} = \sum_{T \subset \text{supp}(\varphi_i)} \int_T |\nabla \varphi_i|^2 \, d\mathbf{x} = \sum_{T \subset \text{supp}(\varphi_i)} |T| |\nabla \varphi_i|_T|^2. \quad (1.15)$$

The last equality follows from the fact that  $\varphi_i$  is linear on each  $T \subset \text{supp}(\varphi_i)$  and thus  $\nabla \varphi_i$  is constant on  $T$ . Therefore, it is necessary to evaluate  $\nabla \varphi_i$  on triangles having  $\mathbf{v}_i$  as an vertex.

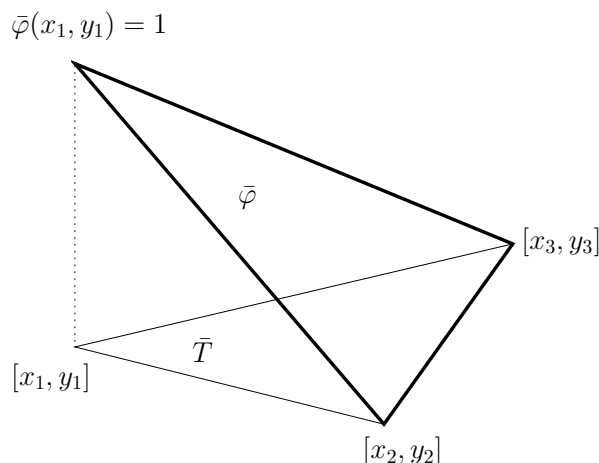


Figure 1.4: Triangle  $\bar{T}$  and the function  $\bar{\varphi}$

Let us consider a triangle  $\bar{T}$  with vertices  $[x_1, y_1]$ ,  $[x_2, y_2]$  and  $[x_3, y_3]$ . Let  $\bar{\varphi}(x, y)$  be the linear function on  $\bar{T}$  such that

$$\bar{\varphi}(x_1, y_1) = 1, \quad \bar{\varphi}(x_2, y_2) = 0, \quad \bar{\varphi}(x_3, y_3) = 0, \quad (1.16)$$

see Figure 1.4.

It is possible to derive the explicit relations for the gradient of  $\bar{\varphi}$ . Let us put

$$D = \det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}, \quad D_x = \det \begin{pmatrix} 1 & y_1 & 1 \\ 0 & y_2 & 1 \\ 0 & y_3 & 1 \end{pmatrix}, \quad D_y = \det \begin{pmatrix} x_1 & 1 & 1 \\ x_2 & 0 & 1 \\ x_3 & 0 & 1 \end{pmatrix}. \quad (1.17)$$

Then

$$\nabla \bar{\varphi} = \left( \frac{D_x}{D}, \frac{D_y}{D} \right) = \frac{(y_2 - y_3, x_3 - x_2)}{2|\bar{T}|}, \quad (1.18)$$

where  $|\bar{T}|$  is the area of the triangle  $\bar{T}$ .

Hence, based on (1.17) – (1.18), we can define the following subroutine **GRADn**, which evaluate the gradient of the linear function on the triangle with vertices  $[x_1, y_1]$ ,  $[x_2, y_2]$  and  $[x_3, y_3]$  such that this function takes value 1 in  $[x_1, y_1]$  and vanish at  $[x_2, y_2]$  and  $[x_3, y_3]$ . We use the convention that  $x_1 = \mathbf{x1}$ ,  $y_1 = \mathbf{y1}$  etc.

```

subroutine GRADn (x1,y1, x2, y2, x3, y3, Dx, Dy)
  real, intent(in) :: x1,y1, x2, y2, x3, y3 ! input parameters
  real, intent(out) :: Dx, Dy ! output parameters
  real :: D ! local variable

  D = x1*(y2-y3) + x2*(y3-y1) + x3*(y1-y2)
  Dx = (y2 - y3) / D
  Dy = (x3 - x2) / D
end subroutine GRADn

```

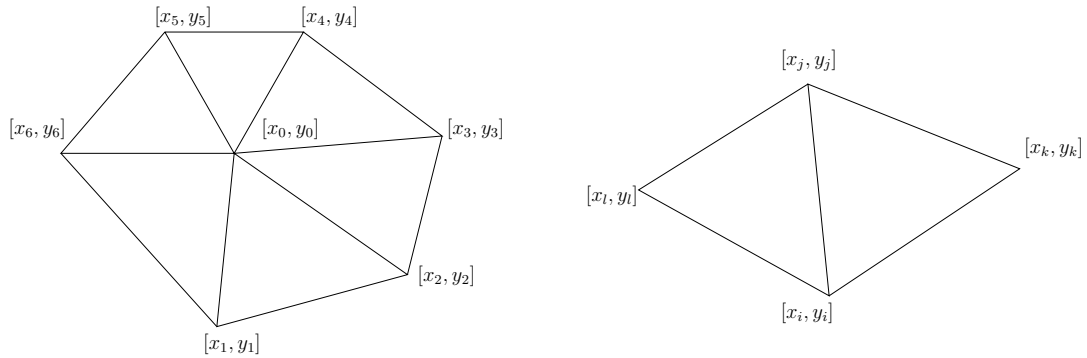


Figure 1.5: Notation for the evaluation of the diagonal entries (left) and off-diagonal entries (right) of the stiff matrix

Now, let us go back to the evaluation of  $S_{ii}$ . Let  $T_i^j$ ,  $j = 1, \dots, n$  be triangles having the node  $\mathbf{v}_i$  as an vertex. Obviously,  $\text{supp}(\varphi_i) = \cup_{j=1}^n T_i^j$ . We assume that  $\mathbf{v}_i$  has the coordinates  $[x_0, y_0]$ . Moreover, the vertices of  $T_i^j$ ,  $j = 1, \dots, n$  have coordinates  $[x_j, y_j]$ ,  $j = 1, \dots, n$  indexed counter-clock-wise, see Figure 1.5, left.

Finally, taking into account (1.15) we can evaluate  $S_{ii}$  by the following subroutine:

```

subroutine STIFF_DIAGn(n, x(0:n), y(0:n), sii )
  integer, intent(in) :: n           ! input parameters
  real, intent(in) :: x(0:n), y(0:n) ! input array
  real, intent(out) :: sii           ! output parameters
  real :: D, Dx, Dy                  ! local variables
  integer :: j, j1                    ! local variables

  sii = 0.
  do j=1,n
    j1 = mod(j+1, n) + 1
    call GRADn(x(0), y(0), x(j), y(j), x(j1), y(j1), Dx, Dy)
    D = x(0)*(y(j)-y(j1) ) + x(j)*(y(j1)-y(0) ) + x(j1)*(y(0)-y(j) )

    sii = sii + (Dx*Dx + Dy*Dy)*D/2 ! D/2 = area of triangle
  end do
end subroutine STIFF_DIAGn

```

Comparing the subroutines `GRADn` and `STIFF_DIADn`, we observe that the values `Dx` and `Dy` are divided by `D` in `GRADn`, then we square this value and multiply by `D` in `STIFF_DIADn`. It is more efficient to avoid the division and the following multiplication by the same value. Hence, we can modify the subroutines `GRADn` and `STIFF_DIADn` by their more efficient variants `GRAD` and `STIFF_DIAD` defined bellow.

```

subroutine GRAD (x1,y1, x2, y2, x3, y3, Dx, Dy)
  real, intent(in) :: x1,y1, x2, y2, x3, y3

```

```

real, intent(out) :: Dx, Dy

Dx = (y2 - y3)
Dy = (x3 - x2)
end subroutine GRAD

```

and

```

subroutine STIFF_DIAG( n, x, y, sii )
integer, intent(in) :: n
real, intent(in) :: x(0:n), y(0:n)
real, intent(out) :: sii
real :: D, Dx, Dy      ! local variables
integer :: j, j1
sii = 0.
do j=1,n
  j1 = mod(j+1, n) + 1
  call GRAD(x(0), y(0), x(j), y(j), x(j1), y(j1), Dx, Dy)

  D = x(0)*(y(j)-y(j1) ) + x(j)*(y(j1)-y(0) ) + x(j1)*(y(0)-y(j) )
  sii = sii + (Dx*Dx + Dy*Dy)/D/2  ! D/2 = area of triangle
end do
end subroutine STIFF_DIAG

```

Moreover, we have to evaluate the off diagonal terms of the stiff matrix, i.e.,  $S_{ij}$  where  $\mathbf{v}_i$  and  $\mathbf{v}_j$  are connected by an edge of  $\mathcal{T}_h$ . We denote the coordinates of vertices of triangles sharing the edge  $\mathbf{v}_i\mathbf{v}_j$  according to Figure 1.5, right. Then we define the subroutine for the evaluation of the off-diagonal term of the stiff matrix:

```

subroutine STIFF_OFFDIAG(xi, yi, xj, yj, xk, yk, xl, yl, sij )
real, intent(in) :: xi, yi, xj, yj, xk, yk, xl, yl
real, intent(out) :: sij
real :: D, Dx, Dy      ! local variables

sij = 0.
call GRAD(xi, yi, xk, yk, xj, yj, Dx, Dy)
D = xi*(yk-yj) + xk*(yj-yi) + xj*(yi-yk)
sij = sij + (Dx*Dx + Dy*Dy)/D/2  ! D/2 = area of triangle

call GRAD(xi, yi, xj, yj, xl, yl, Dx, Dy)
D = xi*(yj-yl) + xj*(yl-yi) + xl*(yi-yj)
sij = sij + (Dx*Dx + Dy*Dy)/D/2  ! D/2 = area of triangle
end subroutine STIFF_OFFDIAG

```

Finally, in order to evaluate all entries of the stiff matrix  $\mathbb{S}$  we should call either subroutine `STIFF_DIAG` or `STIFF_OFFDIAG` for all  $S_{ij} \neq 0$ . However, it requires a setting of the appropriate data structures, namely some link to the nodes connected

by an edge to a given node (see 1.5, left) and some link between elements sharing an face (see 1.5, right.) Since we are going to introduce better algorithm in the following section, we skip this one.

### Setting of the stiff matrix – optimized algorithm

We can notice that when we evaluate all non-vanishing terms of matrix  $\mathbb{S}$ , we call subroutine GRAD for each  $T \in \mathcal{T}_h$  several times. It is more efficient to go over all  $T \in \mathcal{T}_h$  only once time. This means that we compute the gradients of all possible linear functions on  $T$  vanishing in two vertices of  $T$  and equal to 1 in the rest node. Then we evaluate the corresponding entries of the stiff matrix.

Here we introduce more efficient algorithm for the evaluation of  $\mathbb{S}$ . It evaluates all entries of  $\mathbb{S}$  which correspond to inner vertices of  $\mathcal{T}_h$  given by (1.10). First, we define a subroutine, which evaluate the gradient of three possible restrictions of the test functions on the given triangle.

```

subroutine GRADIENTS( x, y, Dphi )
  real, intent(in) :: x(1:3), y(1:3)
  real, intent(out) :: Dphi(1:3, 1:2)
  ! Dphi(i,j) = derivative of i-th function with respect x_j

  Dphi(1,1) = y(2) - y(3)
  Dphi(1,2) = x(3) - x(2)

  Dphi(2,1) = y(3) - y(1)
  Dphi(2,2) = x(1) - x(3)

  Dphi(3,1) = y(1) - y(2)
  Dphi(3,2) = x(2) - x(1)
end subroutine GRADIENTS

```

Then we have the following subroutine for the evaluation of all entries of  $\mathbb{S}$ . Let us recall that `txp(:)`, `yp(:)`, `node(:, :)` are global arrays storing the coordinates of vertices and the indexes of vertices of triangles, see begin of Section 1.1.4. Hence, we have access to them from the subroutines without their specification as input parameters.

```

subroutine STIFF(M, N, S )
  integer, intent(in) :: M  ! = number of triangles of triangulation
  integer, intent(in) :: N  ! = number of vertices of triangulation
  real, dimension(1:N, 1:N), intent(out) :: S  ! = stiff matrix
  real :: Dphi(1:3, 1:2) , D  ! local variables
  integer :: i,j,ki, kj

  S(1:N, 1:N) = 0

```

```

do k=1,M
  x(1:3) = xp(node(k, 1:3) )
  y(1:3) = yp(node(k, 1:3) )
  D = x(1)*(y(2)-y(3)) + x(2)*(y(3)-y(1)) + x(3)*(y(1)-y(2))
  call GRADIENT( x(1:3), y(1:3), Dphi(1:3, 1:2) )

  do ki=1,3
    i = node(k,ki)
    if( i <= N) then          !! inner vertex?
      do kj=1,3
        j = node(k,kj)
        if( i <= N) then      !! inner vertex?
          S(i,j) = S(i,j) + (Dphi(ki,1)* Dphi(kj,1) &
            + Dphi(ki,2)* Dphi(kj,2) )/D/2
        endif
      enddo
    endif
  enddo
enddo
enddo
end subroutine STIFF

```

**Remark 1** *The subroutine STIFF evaluate only entries of the stiff matrix corresponding to the inner vertices of  $\mathcal{T}_h$ . It is assured by two if then conditions in subroutine STIFF. If more general boundary condition than (1.2) is given also the entries corresponding to the boundary vertices of  $\mathcal{T}_h$  has to be evaluated. In this case,  $N$  denotes the number of all vertices including the boundary ones, see Section 1.1.5. Moreover, we can simply remove both if then conditions in subroutine STIFF.*

### Setting of the right-hand side

Here we present an algorithm which evaluate the entries of the vector  $\mathbf{g}$  given by (1.11). Let  $i = 1, \dots, N$ , we have

$$g_i = (g, \varphi_i) = \sum_{T \subset \text{supp}(\varphi_i)} \int_T g \varphi_i \, d\mathbf{x}, \quad (1.19)$$

where  $\text{supp}(\varphi_i)$  is shown in Figure 1.2. Generally, the integrals in (1.19) are evaluated with the aid of a numerical quadratures, which are discussed in Section 1.5. Since we considered in this section the piecewise linear approximation, we will consider only the following simple quadrature rule

$$\int_T g(\mathbf{x}) \varphi(\mathbf{x}) \, d\mathbf{x} \approx \frac{1}{3} |T| \sum_{l=1}^3 \varphi(\mathbf{x}_l) g(\mathbf{x}_l), \quad (1.20)$$

where  $\mathbf{x}_l$ ,  $l = 1, 2, 3$  are the vertices of an triangle  $T \in \mathcal{T}_h$ . This quadrature rule integrate exactly the linear function, hence it has the first order of accuracy.

The application of (1.19) for the evaluation of (1.20) leads to a very simple relation since  $\varphi_i(x_0, y_0) = 1$  and  $\varphi_i(x_l, y_l) = 1$ ,  $l = 1, \dots, n$ , see Figure 1.5, left. Then we have

$$g_i \approx \frac{1}{3} \text{meas}(\text{supp}(\varphi_i)) g(\mathbf{v}_i), \quad i = 1, \dots, N \quad (1.21)$$

where  $\mathbf{v}_i$ ,  $i = 1, \dots, N$  are inner vertices of  $\mathcal{T}_h$ .

Finally, let us note that the numerical quadrature (1.20) is not sufficiently accurate generally.

### Solution of the linear algebraic system (1.12)

Once we have evaluated the entries of matrix  $\mathbb{S}$  and vector  $\mathbf{g}$ , we have to solve the linear algebraic system (1.12). In the case of the Laplace problem (1.1) the situation is very simple. First we introduce the following term

**Definition 1** *Let  $\mathcal{T}_h$  be a mesh of  $\Omega$ . Let  $T$  and  $T'$  be a pair of triangles from  $\mathcal{T}_h$  sharing an edge  $\mathbf{v}_i\mathbf{v}_j$ . Let  $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k$  and  $\mathbf{v}_j, \mathbf{v}_i, \mathbf{v}_l$  be the vertices of  $T$  and  $T'$ , respectively. We denote by  $\text{SumAngles}(T, T')$  the sum of angles of  $T$  and  $T'$  adjacent to vertices  $\mathbf{v}_k$  and  $\mathbf{v}_l$ , respectively. We say that  $\mathcal{T}_h$  is of the Delaunay type if  $\text{SumAngles}(T, T') \leq \pi$  for each pair of triangles  $T, T'$  from  $\mathcal{T}_h$  sharing an edge.*

It is possible to show that if  $\mathcal{T}_h$  is of Delaunay type then the corresponding stiff matrix  $\mathbb{S}$  is symmetric and diagonally dominant. In this case in fact arbitrary direct or iterative solver can be used with success for the solution of (1.12).

### Reconstruction of the approximate solution

Once we have found the vector  $\mathbf{u}$  which is the solution of (1.12), we will reconstruct the piecewise linear function  $u_h \in V_h$  from  $\mathbf{u}$  using (1.8). However, in practice we need not know an explicit relation for  $u_h$ . Usually, we require a visualization of the solution and/or an evaluation of some characteristics. The basic visualization techniques are mentioned in Section 1.6.

#### 1.1.5 Treatment of general boundary conditions

In the previous sections, the homogeneous Dirichlet boundary condition was considered. However, in practice, more general boundary conditions have to be taken into account.

Let  $\Omega$  be a bounded polygonal domain in  $\mathbb{R}^d$ ,  $d = 2, 3$ , with a boundary  $\partial\Omega$  which consists of two disjoint parts  $\partial\Omega_D$  and  $\partial\Omega_N$ . We seek find a function  $u : \Omega \rightarrow \mathbb{R}$  such that

$$-\Delta u(x) = g(x), \quad x \in \Omega, \quad (1.22)$$

$$u(x) = u_D(x), \quad x \in \partial\Omega_D, \quad (1.23)$$

$$\nabla u(x) \cdot \mathbf{n} = g_N(x), \quad x \in \partial\Omega_D, \quad (1.24)$$

where  $g \in L^2(\Omega)$ ,  $g_N \in L^2(\partial\Omega)$  and  $u_D$  is a trace of some  $u^* \in H^1(\Omega)$ .

Let

$$V := \{v \in H^1(\Omega), v|_{\partial\Omega_D} = 0 \text{ in the sense of traces} \} \quad (1.25)$$

then the *weak formulation* of (1.22) – (1.24) reads

$$\begin{aligned} \text{find } u \in H^1(\Omega) : \quad & u - u^* \in V, \\ & (\nabla u, \nabla v) = (g, v) + (g_N, v)_{L^2(\partial\Omega_N)} \quad \forall v \in V. \end{aligned} \quad (1.26)$$

Let  $\mathcal{T}_h$  be a mesh reflecting the  $\partial\Omega$  consists of two disjoint parts  $\partial\Omega_D$  and  $\partial\Omega_N$ . Let  $X_h$  be a finite dimensional subspace of  $H^1(\Omega)$  constructed over mesh  $\mathcal{T}_h$  and  $V_h \subset X_h$  its subspace such that  $v_h|_{\partial\Omega_D} = 0 \quad \forall v_h \in V_h$ . The *finite element* approximation (1.26) can be written as

$$\begin{aligned} \text{find } u_h \in X_h : \quad & (u_h - u_D, v_h)_{L^2(\partial\Omega_D)} = 0 \quad \forall v_h \in X_h, \\ & (\nabla u_h, \nabla v_h) = (g, v_h) + (g_N, v_h)_{L^2(\partial\Omega_N)} \quad \forall v_h \in V_h. \end{aligned} \quad (1.27)$$

The numerical implementation of (1.27) is more complicated than the implementation of (1.4). In this place we skip the evaluation of the boundary integral on the right hand side of (1.27) and refer to Section 1.4.6. We focus on the assuring of the Dirichlet boundary condition on  $\partial\Omega_D$ .

We use the following notation for vertices of  $\mathcal{T}_h$  which differs from the notation introduced in the previous sections. We denote by  $\{\mathbf{v}_i\}_{i=1}^N$  a set of all vertices of  $T \in \mathcal{T}_h$  lying inside of  $\Omega$  or on  $\partial\Omega_N$  and by  $\{\mathbf{v}_i\}_{i=N+1}^{N+N_b}$  a set of all vertices of  $T \in \mathcal{T}_h$  lying on  $\partial\Omega_D$ .

Similarly as in the previous, we restrict to the piecewise linear approximations. Therefore, each function from  $X_h$  is uniquely given by its values in  $\{\mathbf{v}_i\}_{i=1}^{N+N_b}$  and each function from  $V_h$  is uniquely given by its values in  $\{\mathbf{v}_i\}_{i=1}^N$ . We denote by  $\varphi_i \in X_h$  the basis functions corresponding to  $\mathbf{v}_i$ ,  $i = 1, \dots, N + N_b$  given by

$$\varphi_i \in X_h, \quad \varphi_i(\mathbf{v}_j) = \delta_{ij}, \quad i, j = 1, \dots, N + N_b. \quad (1.28)$$

First, we introduce a piecewise linear approximation of  $u_D$  from the boundary condition. We define a function  $u_{D,h} \in X_h \setminus V_h$  by

$$(u_{D,h} - u_D, v_h)_{L^2(\partial\Omega_D)} = 0 \quad \forall v_h \in X_h \setminus V_h. \quad (1.29)$$

Such  $u_{D,h}$  can be simply constructed since  $X_h$  is a finite dimensional spaces and then (1.29) represents a system of linear algebraic equations. Namely, we have

$$u_{D,h}(\mathbf{x}) = \sum_{j=N+1}^{N_b} u_{D,j} \varphi_j(\mathbf{x}), \quad (1.30)$$

therefore, we have from (1.29) – (1.30) the following system of the linear algebraic equations

$$\sum_{j=N+1}^{N_b} u_{D,j} (\varphi_j, \varphi_i)_{L^2(\partial\Omega_D)} = (u_D, \varphi_i)_{L^2(\partial\Omega_D)}, \quad i = N + 1, \dots, N_b. \quad (1.31)$$



The evaluation of terms  $(\varphi_j, \varphi_i)_{L^2(\partial\Omega_D)}$ ,  $i, j = N+1, \dots, N_b$  is simple since these terms are non-vanishing only if  $\mathbf{v}_i$  and  $\mathbf{v}_j$  are connected by an edge on  $\partial\Omega_D$ . Moreover, the evaluation of  $(u_D, \varphi_i)_{L^2(\partial\Omega_D)}$ ,  $i = N+1, \dots, N_b$  reduces to an integration over two edges on  $\partial\Omega_D$  having  $\mathbf{v}_i$  as an end-point.

The solution of the (1.27) can be written in the form

$$u_h(\mathbf{x}) = \sum_{j=1}^N u_j \varphi_j(\mathbf{x}) + \sum_{j=N+1}^{N_b} u_{D,j} \varphi_j(\mathbf{x}) = \sum_{j=1}^N u_j \varphi_j(\mathbf{x}) + u_{D,h}, \quad (1.32)$$

where the coefficients  $u_{D,j}$ ,  $j = N+1, \dots, N_b$  are known and the coefficients  $u_j$ ,  $j = 1, \dots, N$  are unknown. Inserting (1.32) into (1.27) and putting  $v_h := \varphi_i$ , we have

$$\sum_{j=1}^N u_j (\nabla \varphi_j, \nabla \varphi_i) = (g, v_h) + (g_N, v_h)_{L^2(\partial\Omega_N)} - (\nabla u_{D,h}, \nabla \varphi_i), \quad i = 1, \dots, N,$$

which is very close to the problem (1.10), where the right-hand-side contains additionally terms arising from the Dirichlet and the Neumann boundary conditions. Moreover, symbol  $N$  has little different meanings in (1.10) and in (1.33), in the former case denotes the number of interior vertices, in the latter case the number of inner and ‘‘Dirichlet’’ nodes. However, all subroutines presented in Section 1.1.4 can be employed with this minor changes.

## 1.2 General FE solution of the model problem

In Section 1.1 we described the implementation of the conforming piecewise linear finite element approximation of problem (1.1) – (1.2). Here we consider a higher degree of polynomial approximation. The implementation is more complicated since the gradient of test function is not constant over  $T \in \mathcal{T}_h$ .

It is more comfortable employ the concept of the reference element. This approach allows us to deal with more general elements than triangles, e.g., quadrilaterals, curvilinear elements, etc.

### 1.2.1 Definition of reference elements

In this lecture notes, we consider

- the *reference simplex* (triangle for  $d = 2$  and tetrahedron for  $d = 3$ )

$$\widehat{T}_t := \left\{ \hat{\mathbf{x}} = (x_1, \dots, x_d), x_i \geq 0, i = 1, \dots, d, \sum_{i=1}^d x_i \leq 1 \right\} \quad (2.33)$$

- the *reference hexagon* (square for  $d = 2$  and cube for  $d = 3$ )

$$\widehat{T}_q := \{ \hat{\mathbf{x}} = (x_1, \dots, x_d), 0 \leq x_i \leq 1, i = 1, \dots, d, \}. \quad (2.34)$$

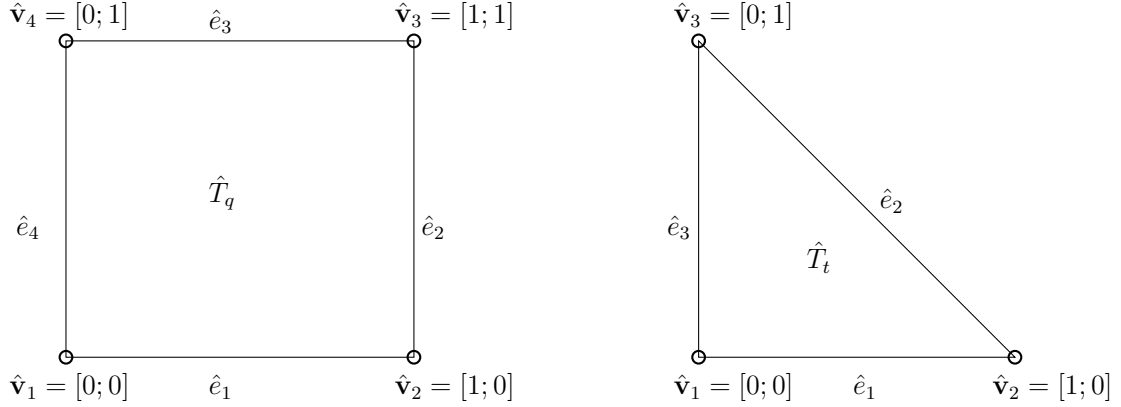


Figure 1.6: Two dimensional reference elements: quadrilateral  $\widehat{T}_q$  (left) and triangle  $\widehat{T}_t$  (right) with the vertices  $\widehat{\mathbf{v}}_i$  the vertices and the edges  $\widehat{e}_i$

For simplicity,  $\widehat{T}$  denotes either the reference simplex or the reference hexagon based on the context. We call  $\widehat{T}$  the *reference element*.

Moreover, we denote by  $\widehat{\mathbf{v}}_1, \dots, \widehat{\mathbf{v}}_{d+1}$  the vertices and by  $\widehat{e}_1, \dots, \widehat{e}_{d+1}$  the edges of  $\widehat{T}_t$ , similarly by  $\widehat{\mathbf{v}}_1, \dots, \widehat{\mathbf{v}}_{2d}$  the vertices and by  $\widehat{e}_1, \dots, \widehat{e}_{2d}$  the edges of  $\widehat{T}_q$ . For  $d = 2$ , we assume that the numbering is oriented counterclockwise, see Figure 1.6, which shows the two-dimensional reference triangle and the reference quadrilateral.

We assume that for each element  $T \in \mathcal{T}_h$  there exist a mapping

$$F_T = F_T(\widehat{\mathbf{x}}) = (F_{T,1}(\widehat{\mathbf{x}}), \dots, F_{T,d}(\widehat{\mathbf{x}})) : \widehat{T} \rightarrow \mathbb{R}^d \quad \text{such that} \quad F_T(\widehat{T}) = T. \quad (2.35)$$

If  $F_T$  is an affine (linear) mapping then  $T$  is either a triangle or a parallelogram based on the type of  $\widehat{T}$ . Generally, the construction of the mappings  $F_T$  is simple for several types of elements:

- *Triangles*: Let  $T$  be a triangle defined by its vertices  $\mathbf{x}_1$ ,  $\mathbf{x}_2$  and  $\mathbf{x}_3$ . Then the linear mapping

$$F_T(\widehat{\mathbf{x}}) = F_T(\widehat{x}_1, \widehat{x}_2) = \mathbf{x}_1 + \widehat{x}_1(\mathbf{x}_2 - \mathbf{x}_1) + \widehat{x}_2(\mathbf{x}_3 - \mathbf{x}_1) \quad (2.36)$$

maps  $\widehat{T}_t$  on  $T$  and moreover,  $F_T(0, 0) = \mathbf{x}_1$ ,  $F_T(1, 0) = \mathbf{x}_2$  and  $F_T(0, 1) = \mathbf{x}_3$ .

- *Tetrahedron*: Let  $T$  be a tetrahedron defined by its vertices  $\mathbf{x}_1$ ,  $\mathbf{x}_2$ ,  $\mathbf{x}_3$  and  $\mathbf{x}_4$ . Then the linear mapping

$$F_T(\widehat{\mathbf{x}}) = F_T(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) = \mathbf{x}_1 + \widehat{x}_1(\mathbf{x}_2 - \mathbf{x}_1) + \widehat{x}_2(\mathbf{x}_3 - \mathbf{x}_1) + \widehat{x}_3(\mathbf{x}_4 - \mathbf{x}_1) \quad (2.37)$$

maps  $\widehat{T}_q$  on  $T$  and moreover,  $F_T(0, 0, 0) = \mathbf{x}_1$ ,  $F_T(1, 0, 0) = \mathbf{x}_2$ ,  $F_T(0, 1, 0) = \mathbf{x}_3$  and  $F_T(0, 0, 1) = \mathbf{x}_4$ .

- *Quadrilaterals*: Let  $T$  be a quadrilateral defined by its vertices  $\mathbf{x}_1$ ,  $\mathbf{x}_2$ ,  $\mathbf{x}_3$  and  $\mathbf{x}_4$ . Then the mapping

$$F_T(\widehat{\mathbf{x}}) = F_T(\widehat{x}_1, \widehat{x}_2) = \mathbf{x}_1 + \widehat{x}_1(\mathbf{x}_2 - \mathbf{x}_1) + \widehat{x}_2(\mathbf{x}_3 - \mathbf{x}_1) + \widehat{x}_1\widehat{x}_2(\mathbf{x}_4 - \mathbf{x}_2 - \mathbf{x}_3 + \mathbf{x}_1) \quad (2.38)$$

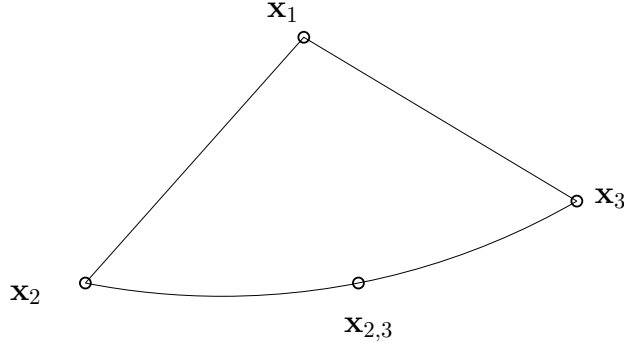


Figure 1.7: One-edge  $P^2$ -curvilinear triangles

maps  $\widehat{T}_q$  on  $T$  and moreover,  $F_T(0,0) = \mathbf{x}_1$ ,  $F_T(1,0) = \mathbf{x}_2$ ,  $F_T(0,1) = \mathbf{x}_3$  and  $F_T(1,1) = \mathbf{x}_4$ . Let us note that if  $T$  is a parallelogram then  $\mathbf{x}_4 - \mathbf{x}_2 - \mathbf{x}_3 + \mathbf{x}_1 = \mathbf{0}$  and (2.38) is identical with (2.36).

- *One-edge  $P^2$ -curvilinear triangles:* This types of elements are often used for an approximation of non-polygonal boundaries. Let  $T$  be a one-edge  $P^2$ -curvilinear triangle, which is defined by its vertices  $\mathbf{x}_1$ ,  $\mathbf{x}_2$ ,  $\mathbf{x}_3$  and the node  $\mathbf{x}_{2,3}$  lying “not far” from  $(\mathbf{x}_2 + \mathbf{x}_3)/2$ . Element  $T$  is bounded by two straight lines  $\mathbf{x}_1\mathbf{x}_2$  and  $\mathbf{x}_1\mathbf{x}_3$  and one quadratic curve passing  $\mathbf{x}_2, \mathbf{x}_{2,3}, \mathbf{x}_3$ , see Figure 1.7. Then the quadratic mapping

$$F_T(\hat{x}_1, \hat{x}_2) = \mathbf{x}_1 + \hat{x}_1(\mathbf{x}_2 - \mathbf{x}_1) + \hat{x}_2(\mathbf{x}_3 - \mathbf{x}_1) + 4\hat{x}_1\hat{x}_2(\mathbf{x}_{2,3} - (\mathbf{x}_2 + \mathbf{x}_3)/2) \quad (2.39)$$

maps  $\widehat{T}_q$  on  $T$ . Moreover,  $F_T(0,0) = \mathbf{x}_1$ ,  $F_T(1,0) = \mathbf{x}_2$ ,  $F_T(0,1) = \mathbf{x}_3$  and  $F_T(1/2, 1/2) = \mathbf{x}_{2,3}$ . Let us note that if  $\mathbf{x}_{2,3} = (\mathbf{x}_2 + \mathbf{x}_3)/2$  then  $T$  is a triangle and (2.39) is identical with (2.36).

It is possible to define mappings  $F_K$  for more general elements. Let us note that in the following sections we require that  $F_K$  is invertible and continuously differentiable on  $\widehat{T}$ . The existence of  $F_K^{-1}$  follows from the assumption that the *Jacobi matrix*

$$J_{F_T}(\hat{\mathbf{x}}) := \frac{D}{D\hat{\mathbf{x}}} F_T(\hat{\mathbf{x}}) \quad (2.40)$$

does not change a sign on  $\widehat{T}$ . Very often, only polynomial mappings are considered from practical reasons.

### 1.2.2 Evaluation of integrals on reference element

The concept of the reference elements is based in the Fubini theorem, which we present in the following form.

**Theorem 1** (Fubini theorem) *Let  $T$  be the image of  $F_T(\widehat{T})$  where  $\widehat{T}$  is the reference element and  $F_T$  is the continuously differentiable mapping. Let  $f(\mathbf{x}) : T \rightarrow \mathbb{R}$  be an integrable function defined on  $T$ . We define the function  $\hat{f} : \widehat{T} \rightarrow \mathbb{R}$  by  $\hat{f}(\hat{\mathbf{x}}) = f(\mathbf{x})$  where  $\mathbf{x} = F_T(\hat{\mathbf{x}})$ . Then*

$$\int_T f(\mathbf{x}) \, d\mathbf{x} = \int_{\widehat{T}} \hat{f}(\hat{\mathbf{x}}) |\det J_{F_T}(\hat{\mathbf{x}})| \, d\hat{\mathbf{x}}, \quad (2.41)$$

where  $J_{F_T}$  is the Jacobi matrix of  $F_T$ .

Let us note that if  $F_T$  is a linear mapping (i.e.,  $T$  is an simplex or a parallelogram) then  $J_F$  is a constant matrix and  $|\det J_{F_T}| = 2|T|$ , where  $|T|$  denotes the  $d$ -dimensional Lebesgue measure of  $T$ .

Theorem 1 gives us a tool for the evaluation of the various type of integrals in the implementation of finite element and discontinuous Galerkin methods. Instead of integration over  $T$  we integrate over  $\widehat{T}$  which is generally more easy. This is demonstrated in the following paragraphs.

Let  $T \in \mathcal{T}_h$  and  $\varphi$  be a function defined on  $T$ . Our aim is to compute

$$\int_T \frac{\partial \varphi(\mathbf{x})}{\partial x_i} \, d\mathbf{x}. \quad (2.42)$$

Let  $\hat{\varphi} : \widehat{T} \rightarrow \mathbb{R}$  be defined as usual by  $\hat{\varphi}(\hat{\mathbf{x}}) = \varphi(\mathbf{x})$  where  $\mathbf{x} = F_T(\hat{\mathbf{x}})$ ,

$$F_T = (F_{T,1}, \dots, F_{T,d})$$

and  $T = F_T(\widehat{T})$ . Moreover, we have  $\hat{\mathbf{x}} = F_T^{-1}(\mathbf{x})$  where

$$F_T^{-1} = (F_{T,1}^{-1}, \dots, F_{T,d}^{-1})$$

is the inverse mapping of  $F_T$ . With the aid of Theorem 1 and the chain rule, we have

$$\begin{aligned} \int_T \frac{\partial}{\partial x_i} \varphi(\mathbf{x}) \, d\mathbf{x} &= \int_T \frac{\partial}{\partial x_i} \hat{\varphi}(\hat{\mathbf{x}}) |\det J_{F_T}(\hat{\mathbf{x}})| \, d\hat{\mathbf{x}} \\ &= \int_T \sum_{k=1}^d \frac{\partial \hat{\varphi}(\hat{\mathbf{x}})}{\partial \hat{x}_k} \frac{\partial \hat{x}_k}{\partial x_i} |\det J_{F_T}(\hat{\mathbf{x}})| \, d\hat{\mathbf{x}} = \int_T \sum_{k=1}^d \frac{\partial \hat{\varphi}(\hat{\mathbf{x}})}{\partial \hat{x}_k} \frac{\partial F_{T,k}^{-1}}{\partial x_i} |\det J_{F_T}(\hat{\mathbf{x}})| \, d\hat{\mathbf{x}}. \end{aligned} \quad (2.43)$$

Therefore, we have to evaluate the partial derivative of  $F_T^{-1}$ . It is possible to use the following determination. Let  $i, k = 1, \dots, d$ , then

$$\begin{aligned} x_k &= F_{T,k}(F_T^{-1}(\mathbf{x})), & \Big| \frac{\partial}{\partial x_i} \\ \frac{\partial}{\partial x_i} x_k &= \frac{\partial}{\partial x_i} F_{T,k}(F_T^{-1}(\mathbf{x})), \\ \delta_{ik} &= \sum_{l=1}^d \frac{\partial F_{T,k}}{\partial \hat{x}_l} \frac{\partial F_{T,l}^{-1}}{\partial x_i}, \\ \delta_{ik} &= \sum_{l=1}^d (J_{F_T})_{kl} (J_{F_T^{-1}})_{li}, \end{aligned} \quad (2.44)$$

where  $(J_{F_T})_{kl}$ ,  $k, l = 1, \dots, d$  and  $(J_{F_T^{-1}})_{li}$ ,  $l, i = 1, \dots, d$  denote entries of the Jacobi matrices  $J_{F_T}$  and  $J_{F_T^{-1}}$ , respectively. The relation (2.44) implies

$$\mathbb{I} = J_{F_T} J_{F_T^{-1}} \implies J_{F_T^{-1}} = (J_{F_T})^{-1}. \quad (2.45)$$

Therefore, instead of evaluation of the Jacobi matrix of  $F_T^{-1}$  it is sufficient to evaluate  $(J_{F_T})^{-1}$  = the inversion of  $J_{F_T}$ . This is simple since  $J_{F_T}$  is (usually)  $2 \times 2$  or  $3 \times 3$  matrix.

Therefore, from (2.43) and (2.45) we have

$$\begin{aligned} \int_T \frac{\partial}{\partial x_i} \varphi(\mathbf{x}) \, d\mathbf{x} &= \int_T \sum_{k=1}^d \frac{\partial \hat{\varphi}(\hat{\mathbf{x}})}{\partial \hat{x}_k} (J_{F_T}(\hat{\mathbf{x}})^{-1})_{ki} |\det J_{F_T}(\hat{\mathbf{x}})| \, d\hat{\mathbf{x}} \\ &= \int_T \sum_{k=1}^d \frac{\partial \hat{\varphi}(\hat{\mathbf{x}})}{\partial \hat{x}_k} (J_{F_T}(\hat{\mathbf{x}})^{-T})_{ik} |\det J_{F_T}(\hat{\mathbf{x}})| \, d\hat{\mathbf{x}} \\ &= \int_T \left( J_{F_T}(\hat{\mathbf{x}})^{-T} \widehat{\nabla} \hat{\varphi}(\hat{\mathbf{x}}) \right)_i |\det J_{F_T}(\hat{\mathbf{x}})| \, d\hat{\mathbf{x}}, \end{aligned} \quad (2.46)$$

where  $J_{F_T}^{-T}$  is the transposed matrix to  $J_{F_T}^{-1}$ ,  $(J_{F_T}^{-T} \widehat{\nabla} \hat{\varphi})_i$  denotes the  $i$ -th component of vector  $J_{F_T}^{-T} \widehat{\nabla} \hat{\varphi}$  arising from the matrix-vector product of  $J_{F_T}^{-T}$  by  $\widehat{\nabla} \hat{\varphi}$  and  $\widehat{\nabla}$  is the reference gradient operator given by

$$\widehat{\nabla} := \left( \frac{\partial}{\partial \hat{x}_1}, \dots, \frac{\partial}{\partial \hat{x}_d} \right). \quad (2.47)$$

In the same manner it is possible to derive, e.g.,

$$\int_T \frac{\partial \varphi_a(\mathbf{x})}{\partial x_i} \frac{\partial \varphi_b(\mathbf{x})}{\partial x_j} \, d\mathbf{x} = \int_T \left( J_{F_T}(\hat{\mathbf{x}})^{-T} \widehat{\nabla} \hat{\varphi}_a(\hat{\mathbf{x}}) \right)_i \left( J_{F_T}(\hat{\mathbf{x}})^{-T} \widehat{\nabla} \hat{\varphi}_b(\hat{\mathbf{x}}) \right)_j |\det J_{F_T}(\hat{\mathbf{x}})| \, d\hat{\mathbf{x}} \quad (2.48)$$

or more simply

$$\int_T \frac{\partial \varphi_a}{\partial x_i} \frac{\partial \varphi_b}{\partial x_j} \, d\mathbf{x} = \int_T \left( J_{F_T}^{-T} \widehat{\nabla} \hat{\varphi}_a \right)_i \left( J_{F_T}^{-T} \widehat{\nabla} \hat{\varphi}_b \right)_j |\det J_{F_T}| \, d\hat{\mathbf{x}} \quad (2.49)$$

which is type of integral appearing in the evaluation of the stiff matrix.

### 1.2.3 Grids and functional spaces

We again consider the model Poisson problem (1.1) – (1.2), the weak formulation (1.3) and the finite element discretization (1.4). In contrast to Section 1.1, we consider more general partitions of  $\Omega$  and finite element spaces. Let a grid  $\mathcal{T}_h$  be a set of non-overlapping elements  $T \in \mathcal{T}_h$  such that

$$\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} T. \quad (2.50)$$

We consider two following type of grids.

### Triangular grid

Each element  $T \in \mathcal{T}_h$  is an images of the reference triangle (for  $d = 2$ ) or the reference tetrahedron (for  $d = 3$ ) given by a polynomial mapping  $F_T : \widehat{T}_t \rightarrow \mathbb{R}$ ,  $T \in \mathcal{T}_h$ . If  $F_T$  is a linear mapping then  $T$  is a triangle (for  $d = 2$ ) or a tetrahedron (for  $d = 3$ ). If  $F_T$  has the form (2.39) then  $T$  is an one-edge  $P^2$ -curvilinear triangle. The approximate solution is sought in the functional space

$$\begin{aligned} V_h^{t,p} &:= \{v_h, v_h \in C(\overline{\Omega}) \cap H_0^1(\Omega), v_h|_{T \circ F_T} \in P^{t,p}(\widehat{T}_t) \forall T \in \mathcal{T}_h\}, \quad (2.51) \\ P^{t,p}(\omega) &:= \{\hat{z} : \omega \rightarrow \mathbb{R}, \hat{z}(\hat{x}_1, \dots, \hat{x}_d) = \sum_{\substack{i_1+\dots+i_d \leq p \\ i_1, \dots, i_d=0}} a_{i_1, \dots, i_d} \hat{x}_1^{i_1} \dots \hat{x}_d^{i_d}, a_{i_1, \dots, i_d} \in \mathbb{R}\}, \end{aligned}$$

where  $\omega \subset \mathbb{R}^d$  is an arbitrary domain. Let us note that the notation  $v_h|_{T \circ F_T} \in P^{t,p}(\widehat{T}_t)$  means that there exists a function  $\hat{v}_h \in P^{t,p}(\widehat{T}_t)$  such that

$$v_h(\mathbf{x}) = v_h(F_T(\hat{\mathbf{x}})) = \hat{v}_h(\hat{\mathbf{x}}) \quad \forall \hat{\mathbf{x}} \in \widehat{T}_t. \quad (2.52)$$

Obviously, if  $F_T$  is a linear mapping then  $v_h|_T \in P^{t,p}(T)$ . Otherwise, (e.g.,  $T$  is a curvilinear triangle),  $v_h|_T \notin P^{t,p}(T)$ .

The spaces  $P^{t,p}(\widehat{T}_t)$ ,  $p = 1, 2, 3$  can be expressed (for  $d = 2$ ) as

$$\begin{aligned} P^{t,1}(\widehat{T}_t) &= \text{span}\{1, \hat{x}_1, \hat{x}_2\}, \quad (2.53) \\ P^{t,2}(\widehat{T}_t) &= \text{span}\{1, \hat{x}_1, \hat{x}_2, \hat{x}_1^2, \hat{x}_1\hat{x}_2, \hat{x}_2^2\}, \\ P^{t,3}(\widehat{T}_t) &= \text{span}\{1, \hat{x}_1, \hat{x}_2, \hat{x}_1^2, \hat{x}_1\hat{x}_2, \hat{x}_2^2, \hat{x}_1^3, \hat{x}_1^2\hat{x}_2, \hat{x}_1\hat{x}_2^2, \hat{x}_2^3\}, \end{aligned}$$

where  $\text{span}\{S\}$  is the linear hull of the set  $\{S\}$ . Obviously,  $\hat{N}^{t,p} := \dim P^{t,p}(\widehat{T}_t) = (p+1)(p+2)/2$  for  $d = 2$  and  $\hat{N}^{t,p} := \dim P^{t,p}(\widehat{T}_t) = (p+1)(p+2)(p+3)/6$  for  $d = 3$ .

### Quadrilateral grids

Each element  $T \in \mathcal{T}_h$  is an images of the reference square (for  $d = 2$ ) or the reference cube (for  $d = 3$ ) given by a polynomial mapping  $F_T : \widehat{T}_q \rightarrow \mathbb{R}$ ,  $T \in \mathcal{T}_h$ . If  $F_T$  is a linear mapping then  $T$  is a parallelogram. The approximate solution is sought in the functional space

$$\begin{aligned} V_h^{q,p} &:= \{v_h, v_h \in C(\overline{\Omega}) \cap H_0^1(\Omega), v_h|_{T \circ F_T} \in P^{q,p}(\widehat{T}_q) \forall T \in \mathcal{T}_h\}, \quad (2.54) \\ P^{q,p}(\omega) &:= \{\hat{z} : \omega \rightarrow \mathbb{R}, \hat{z}(x_1, \dots, x_d) = \sum_{i_1, \dots, i_d=0}^p a_{i_1, \dots, i_d} x_1^{i_1} \dots x_d^{i_d}, a_{i_1, \dots, i_d} \in \mathbb{R}\}, \end{aligned}$$

where  $\omega \subset \mathbb{R}^d$  is an arbitrary domain.

The spaces  $P^{q,p}(\widehat{T}_q)$ ,  $p = 1, 2, 3$  can be expressed (for  $d = 2$ ) as

$$\begin{aligned} P^{q,1}(\widehat{T}_q) &= \text{span}\{1, \hat{x}_1, \hat{x}_2, \hat{x}_1\hat{x}_2\}, \quad (2.55) \\ P^{q,2}(\widehat{T}_q) &= \text{span}\{1, \hat{x}_1, \hat{x}_2, \hat{x}_1\hat{x}_2, \hat{x}_1^2, \hat{x}_1^2\hat{x}_2, \hat{x}_1\hat{x}_2^2, \hat{x}_1\hat{x}_2^2, \hat{x}_2^2\}, \\ P^{q,3}(\widehat{T}_q) &= \text{span}\{1, \hat{x}_1, \hat{x}_2, \hat{x}_1\hat{x}_2, \hat{x}_1^2, \hat{x}_1^2\hat{x}_2, \hat{x}_1\hat{x}_2^2, \hat{x}_1\hat{x}_2^2, \hat{x}_2^2, \\ &\quad \hat{x}_1^3, \hat{x}_1^3\hat{x}_2, \hat{x}_1^2\hat{x}_2^2, \hat{x}_1^3\hat{x}_2^2, \hat{x}_1^2\hat{x}_2^3, \hat{x}_1\hat{x}_2^3, \hat{x}_2^3\}. \end{aligned}$$

Obviously,  $\hat{N}^{q,p} := \dim P^{q,p}(\hat{T}_q) = (p+1)^d$  for  $d = 2, 3$ .

Since the most of the following considerations are valid for triangular as well as quadrilateral grids we call  $\mathcal{T}_h$  a triangulation in both cases. Moreover, the corresponding reference element is denoted by  $\hat{T}$ ,  $P^p(\hat{T})$  the corresponding space of polynomial functions on  $\hat{T}$  with the dimension  $\hat{N}$  and the corresponding finite element space is denoted by  $V_h$ .

## 1.2.4 Implementation

### Basis of $V_h$

Let  $N$  denote the dimension of the space  $V_h$  and  $B_h := \{\varphi_i(\mathbf{x})\}_{i=1}^N$  be its basis. The construction of  $B_h$  will be discussed in Section 1.3. Here we only note that we construct the set *reference shape functions*  $\hat{B} := \{\hat{\varphi}_j, \hat{\varphi}_j : \hat{T} \rightarrow \mathbb{R}\}_{j=1}^{\hat{N}}$  such that

- $\hat{\varphi}_j(\hat{\mathbf{x}}) \in P^p(\hat{T})$ ,  $j = 1, \dots, \hat{N}$ ,  $\hat{N}$  is the dimension of  $P^p(\hat{T})$ .
- $\hat{\varphi}_j(\hat{\mathbf{x}})$ ,  $j = 1, \dots, \hat{N}$  are linearly independent,
- $\text{span}(\hat{B}) = P^p(\hat{T})$ .

We call  $\hat{B}$  the *reference basis*.

Moreover, in virtue of (2.51) or (2.54), the basis functions  $\varphi_i \in B_h$  satisfy: for each  $T \in \mathcal{T}_h$ ,  $T \subset \text{supp}(\varphi_i)$  there exists a function  $\hat{\varphi}_{i,T} \in \hat{B}$  such that

$$\varphi_i(\mathbf{x})|_T = \varphi_i(F_T(\hat{\mathbf{x}}))|_T = \hat{\varphi}_{i,T}(\hat{\mathbf{x}}) \quad \forall \hat{\mathbf{x}} \in \hat{T}, \quad i = 1, \dots, N. \quad (2.56)$$

More details are given in Section 1.3.

### A use of the concept of the reference elements

Our aim is to evaluate the integrals of type

$$\int_T \frac{\partial \varphi_i}{\partial x_k} \frac{\partial \varphi_j}{\partial x_l} \, d\mathbf{x}, \quad i, j = 1, \dots, N, \quad k, l = 1, \dots, d, \quad T \in \mathcal{T}_h, \quad (2.57)$$

which appears, e.g., in the definition of the stiff matrix.

A use of the approach from Section 1.1 requires an evaluation (and storing) of  $\nabla \varphi_i|_T$  for all  $i = 1, \dots, N$  and all  $T \in \mathcal{T}_h$ . Let us note, that  $\nabla \varphi_i|_T = 0$  for a lot of  $T \in \mathcal{T}_h$  since supports of test functions contain only a few elements from  $\mathcal{T}_h$ .

It is more efficient to employ the concept of the reference elements, namely relation (2.49), where the integration is carried out on the reference element  $\hat{T}$ .

Therefore, using (2.49) and (2.57), we can evaluate (2.57) with the aid of

$$\int_T \frac{\partial \varphi_i}{\partial x_k} \frac{\partial \varphi_j}{\partial x_l} \, d\mathbf{x} = \int_{\hat{T}} \left( J_{F_T}^{-T} \hat{\nabla} \hat{\varphi}_{i,T} \right)_k \left( J_{F_T}^{-T} \hat{\nabla} \hat{\varphi}_{j,T} \right)_l |\det J_{F_T}| \, d\hat{\mathbf{x}}. \quad (2.58)$$

## Data structures

In order to evaluate the right-hand side of (2.58) for all  $i, j = 1, \dots, N$ ,  $k, l = 1, \dots, d$ ,  $T \in \mathcal{T}_h$  it is enough to evaluate and store the following data.

- (S1) for each  $T \in \mathcal{T}_h$ , the determinant of the Jacobi matrix  $\det J_{F_T}$  and the transposed matrix of the inversion of the Jacobi matrix  $J_{F_T}^{-T}$ ,
- (S2) for each  $\hat{\varphi}_i$ ,  $i = 1, \dots, \hat{N}$ , the gradient  $\widehat{\nabla} \hat{\varphi}_i$  on  $\widehat{T}$ ,
- (S3) for each pair  $(\varphi_i, T)$ ,  $T \in \mathcal{T}_h$ ,  $T \subset \text{supp}(\varphi_i)$ ,  $i = 1, \dots, N$ , an index  $j \in \{1, \dots, \hat{N}\}$  such that  $\varphi_i(F_T(\hat{\mathbf{x}}))|_T = \hat{\varphi}_j(\hat{\mathbf{x}})$ .

Let us note that (S1) is given by the geometry of the mesh, (S2) by the reference shape functions and (S3) by a combination of both. However, in (S3) we have to store only one integer index for any acceptable pair  $(\varphi_i, T)$ .

## 1.3 Basis and shape functions

In this section we describe two possible way (among others) of the construction of basis of spaces  $V_h^{t,p}$  and  $V_h^{q,p}$  given by (2.54) and (2.52), respectively. In Section 1.3.1 we describe the construction of Lagrangian basis, which is easy for the determination and also for the implementation. However, in case when we use, e.g., different degrees of polynomial approximations at different elements (*hp*-methods), the efficiency of this approach is low. Therefore, in Section 1.3.2 we present the construction of the Lobatto basis, which is more efficient generally.

As we have already mentioned in Section 1.2.4, first we define the *reference shape functions*  $\hat{B} := \{\hat{\varphi}_j\}_{j=1}^{\hat{N}}$  defined on the reference element  $\widehat{T}$ . Then the basis of  $V_h$  is generated from  $\hat{B}$ . We present the construction of the shape functions for  $d = 2$ . For  $d = 3$ , it is possible use the similar (but technically more complicated) approach.

### 1.3.1 The reference Lagrangian shape functions

The reference Lagrangian shape functions are defined with the aid of a set of the reference Lagrangian nodes  $\hat{\mathbf{z}}_i = (\hat{z}_{i,1}, \hat{z}_{i,2})$ ,  $i = 1, \dots, \hat{N}$  within the reference element. Then we simply put  $\hat{\varphi}_i(\hat{z}_k) = \delta_{ij}$ ,  $i, j = 1, \dots, \hat{N}$ . In the following we introduce the reference Lagrangian nodes and the reference Lagrangian shape function for the reference triangle and the reference square separately.

#### The reference Lagrangian shape functions on the square

**Definition 2** (Lagrangian nodes) *Let  $p \geq 1$ , then the Lagrangian nodes on the reference square  $\widehat{T}_q$  corresponding to the polynomial degree  $p$  is the set of nodes*

$$\{\hat{\mathbf{z}}_k^{q,p}\}_{k=1}^{\hat{N}^{q,p}}, \quad \hat{\mathbf{z}}_k^{q,p} = (\hat{z}_{k,1}^{q,p}, \hat{z}_{k,2}^{q,p}) \quad (3.59)$$



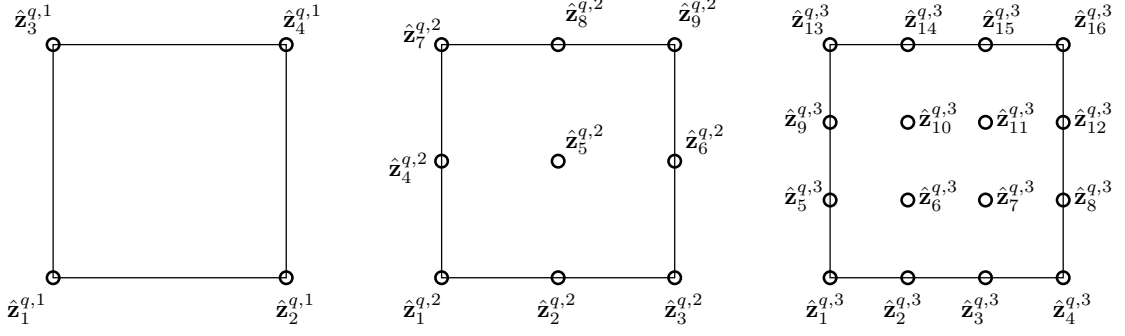


Figure 1.8: The reference Lagrangian nodes on the reference square  $\widehat{T}_q$ ,  $p = 1$  (left),  $p = 2$  (center) and  $p = 3$  (right)

where

$$\widehat{z}_{j(p+1)+i+1,1}^{q,p} = i/p, \quad \widehat{z}_{j(p+1)+i+1,2}^{q,p} = j/p, \quad i, j = 0, \dots, p. \quad (3.60)$$

Obviously,  $\widehat{N}^{q,p} = (p+1)^2$ .

Figure 1.8 shows the Lagrangian nodes for  $p = 1, 2, 3$ . We define the *reference Lagrangian shape functions*  $\{\widehat{\varphi}_l^{q,p}\}_{l=1}^{\widehat{N}^{q,p}}$  for  $p \geq 1$  on the reference square  $\widehat{T}_q$  by the relation

$$\widehat{\varphi}_l^{q,p} \in P^p(\widehat{T}_q), \quad \widehat{\varphi}_l^{q,p}(\widehat{\mathbf{z}}_k^{q,p}) = \delta_{lk}, \quad l, k = 1, \dots, \widehat{N}^{q,p}. \quad (3.61)$$

It is possible to derive the following explicit relations for the *Lagrangian shape functions on the reference square*  $\widehat{T}_q$

$$\begin{aligned}
p = 1 \quad & \widehat{\varphi}_1^{q,1} = (1 - \lambda_1)(1 - \lambda_2), \quad \widehat{\varphi}_2^{q,1} = \lambda_1(1 - \lambda_2), \\
& \widehat{\varphi}_3^{q,1} = \lambda_2(1 - \lambda_1), \quad \widehat{\varphi}_4^{q,1} = \lambda_1\lambda_2 \\
p = 2 \quad & \widehat{\varphi}_1^{q,2} = 4(1 - \lambda_1)\left(\frac{1}{2} - \lambda_1\right)(1 - \lambda_2)\left(\frac{1}{2} - \lambda_2\right), \\
& \widehat{\varphi}_2^{q,2} = 8\lambda_1(1 - \lambda_1)(1 - \lambda_2)\left(\frac{1}{2} - \lambda_2\right), \\
& \widehat{\varphi}_3^{q,2} = -4\lambda_1\left(\frac{1}{2} - \lambda_1\right)(1 - \lambda_2)\left(\frac{1}{2} - \lambda_2\right), \\
& \widehat{\varphi}_4^{q,2} = 8(1 - \lambda_1)\left(\frac{1}{2} - \lambda_1\right)\lambda_2(1 - \lambda_2), \\
& \widehat{\varphi}_5^{q,2} = 16\lambda_1(1 - \lambda_1)\lambda_2(1 - \lambda_2), \\
& \widehat{\varphi}_6^{q,2} = -8\lambda_1\left(\frac{1}{2} - \lambda_1\right)\lambda_2(1 - \lambda_2), \\
& \widehat{\varphi}_7^{q,2} = -4(1 - \lambda_1)\left(\frac{1}{2} - \lambda_1\right)\lambda_2\left(\frac{1}{2} - \lambda_2\right), \\
& \widehat{\varphi}_8^{q,2} = -8\lambda_1(1 - \lambda_1)\lambda_2\left(\frac{1}{2} - \lambda_2\right), \\
& \widehat{\varphi}_9^{q,2} = 4\lambda_1\left(\frac{1}{2} - \lambda_1\right)\lambda_2\left(\frac{1}{2} - \lambda_2\right),
\end{aligned} \quad (3.62)$$

where  $\lambda_1 = \widehat{x}_1$  and  $\lambda_2 = \widehat{x}_2$  are the barycentric coordinates on  $\widehat{T}_q$ .

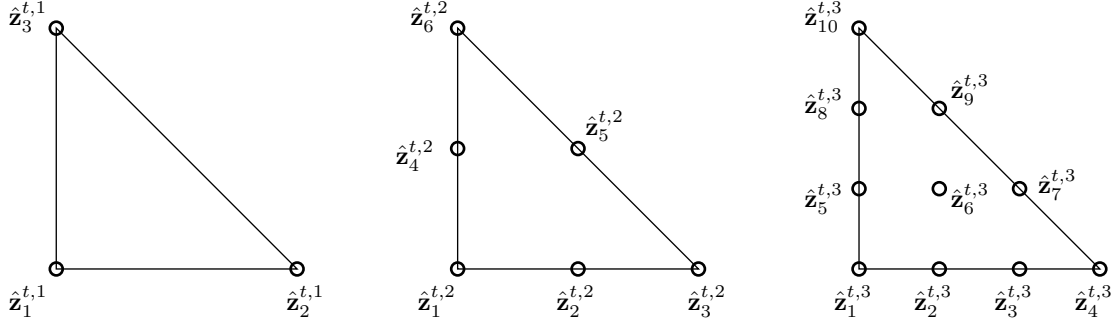


Figure 1.9: Lagrangian nodes on the reference triangle  $\widehat{T}_t$ ,  $p = 1$  (left),  $p = 2$  (center) and  $p = 3$  (right)

### The reference Lagrangian shape functions on the triangle

**Definition 3** (*Lagrangian nodes*) Let  $p \geq 1$ , then the reference Lagrangian nodes on the reference triangle  $\widehat{T}_t$  corresponding to the polynomial degree  $p$  is the set of nodes

$$\{\widehat{\mathbf{z}}_k^{t,p}\}_{k=1}^{\widehat{N}^{t,p}}, \widehat{\mathbf{z}}_k^{t,p} = (\widehat{z}_{k,1}^{t,p}, \widehat{z}_{k,2}^{t,p}) \quad (3.63)$$

generated by the following algorithm

$$\begin{aligned} &k := 0 \\ &\text{for } i = 0, \dots, p \\ &\quad \text{for } j = 0, \dots, p - i \\ &\quad\quad k := k + 1 \\ &\quad\quad\quad \widehat{z}_{k,1}^{t,p} = j/p, \\ &\quad\quad\quad \widehat{z}_{k,2}^{t,p} = i/p. \end{aligned} \quad (3.64)$$

Obviously,  $\widehat{N}^{t,p} = (p + 1)(p + 2)/2$ .

Figure 1.9 shows the Lagrangian nodes for  $p = 1, 2, 3$ . We define the *reference Lagrangian shape functions*  $\{\widehat{\varphi}_l^{t,p}\}_{l=1}^{\widehat{N}^{t,p}}$  by the relation

$$\widehat{\varphi}_l^{t,p} \in P^{t,p}(\widehat{K}_t), \widehat{\varphi}_l^{t,p}(\widehat{\mathbf{z}}_k^{t,p}) = \delta_{lk}, \quad l, k = 1, \dots, \widehat{N}^{t,p}. \quad (3.65)$$

It is possible to derive the following explicit relation for the *Lagrangian shape functions* on the reference triangle  $\widehat{T}_t$

$$p = 1 \quad \widehat{\varphi}_1^{t,1} = \lambda_1, \widehat{\varphi}_2^{t,1} = \lambda_2, \widehat{\varphi}_3^{t,1} = \lambda_3, \quad (3.66)$$

$$\begin{aligned} p = 2 \quad \widehat{\varphi}_2^{t,2} &= \lambda_1(2\lambda_1 - 1), \widehat{\varphi}_3^{t,2} = 4\lambda_1\lambda_2, \widehat{\varphi}_4^{t,2} = \lambda_2(2\lambda_2 - 1), \\ \widehat{\varphi}_5^{t,2} &= 4\lambda_2\lambda_3, \widehat{\varphi}_6^{t,2} = \lambda_3(2\lambda_3 - 1), \widehat{\varphi}_7^{t,2} = 4\lambda_3\lambda_1, \end{aligned}$$

$$\begin{aligned}
p = 3 \quad & \hat{\varphi}_1^{t,3} = \lambda_1(3\lambda_1 - 1)(3\lambda_1 - 2)/2, \quad \hat{\varphi}_2^{t,3} = 9\lambda_1\lambda_2(3\lambda_1 - 1)/2, \\
& \hat{\varphi}_3^{t,3} = 9\lambda_1\lambda_2(3\lambda_2 - 1)/2, \quad \hat{\varphi}_4^{t,3} = \lambda_2(3\lambda_2 - 1)(3\lambda_2 - 2)/2, \\
& \hat{\varphi}_5^{t,3} = 9\lambda_3\lambda_1(3\lambda_1 - 2)/2, \quad \hat{\varphi}_6^{t,3} = 27\lambda_1\lambda_2\lambda_3, \\
& \hat{\varphi}_7^{t,3} = 9\lambda_2\lambda_3(3\lambda_2 - 1)/2, \quad \hat{\varphi}_8^{t,3} = 9\lambda_3\lambda_1(3\lambda_2 - 1)/2, \\
& \hat{\varphi}_9^{t,3} = 9\lambda_2\lambda_3(3\lambda_2 - 2)/2, \quad \hat{\varphi}_{10}^{t,3} = \lambda_3(3\lambda_3 - 1)(3\lambda_3 - 2), \\
\\
p = 4 \quad & \hat{\varphi}_1^{t,4} = 6\lambda_1(4\lambda_1 - 1)(4\lambda_1 - 2)(4\lambda_1 - 3), \\
& \hat{\varphi}_2^{t,4} = 8\lambda_1\lambda_2(4\lambda_1 - 1)(4\lambda_1 - 2)/3, \\
& \hat{\varphi}_3^{t,4} = 4\lambda_1\lambda_2(4\lambda_1 - 1)(4\lambda_2 - 1), \\
& \hat{\varphi}_4^{t,4} = 8\lambda_1\lambda_2(4\lambda_2 - 1)(4\lambda_2 - 2)/3, \\
& \hat{\varphi}_5^{t,4} = 6\lambda_2(4\lambda_2 - 1)(4\lambda_2 - 2)(4\lambda_2 - 3), \\
& \hat{\varphi}_6^{t,4} = 8\lambda_3\lambda_1(4\lambda_1 - 1)(4\lambda_3 - 2)/3, \\
& \hat{\varphi}_7^{t,4} = 32\lambda_1\lambda_2\lambda_3(4\lambda_1 - 1), \\
& \hat{\varphi}_8^{t,4} = 32\lambda_1\lambda_2\lambda_3(4\lambda_2 - 1), \\
& \hat{\varphi}_9^{t,4} = 8\lambda_2\lambda_3(4\lambda_2 - 1)(4\lambda_2 - 2)/3, \\
& \hat{\varphi}_{10}^{t,4} = 4\lambda_3\lambda_1(4\lambda_3 - 1)(4\lambda_1 - 1), \\
& \hat{\varphi}_{11}^{t,4} = 32\lambda_1\lambda_2\lambda_3(4\lambda_3 - 1), \\
& \hat{\varphi}_{12}^{t,4} = 4\lambda_2\lambda_3(4\lambda_2 - 1)(4\lambda_3 - 1), \\
& \hat{\varphi}_{13}^{t,4} = 8\lambda_3\lambda_1(4\lambda_3 - 1)(4\lambda_3 - 2)/3, \\
& \hat{\varphi}_{14}^{t,4} = 8\lambda_2\lambda_3(4\lambda_3 - 1)(4\lambda_3 - 2)/3, \\
& \hat{\varphi}_{15}^{t,4} = 6\lambda_3(4\lambda_3 - 1)(4\lambda_3 - 2)(4\lambda_3 - 3).
\end{aligned}$$

Here  $\lambda_1 = \hat{x}_1$ ,  $\lambda_2 = \hat{x}_2$  and  $\lambda_3 = 1 - \hat{x}_1 - \hat{x}_2$  are the barycentric coordinates on  $\widehat{T}_t$ .

### 1.3.2 The reference Lobatto shape functions

The disadvantage of the Lagrangian shape functions is that they are not hierarchical, which means that

$$\{\hat{\varphi}_k^{q,p}\}_{k=1}^{\hat{N}^{t,p}} \not\subset \{\hat{\varphi}_k^{q,p+1}\}_{k=1}^{\hat{N}^{t,p+1}} \quad \text{and} \quad \{\hat{\varphi}_k^{t,p}\}_{k=1}^{\hat{N}^{t,p}} \not\subset \{\hat{\varphi}_k^{t,p+1}\}_{k=1}^{\hat{N}^{t,p+1}}, \quad p = 1, 2, \dots \quad (3.67)$$

This cause no troubles in case when our aim is to implement FEM for one given degree of polynomial approximation only. In case, when we need to implement several degrees of polynomial approximation (e.g.,  $hp$ -FEM), then the use of the Lagrangian shape functions requires an evaluation and a storing data structure (S1) for a lot of test functions.

Therefore, it is more efficient to use *Lobatto shape functions*, which are hierarchical, since they satisfy the conditions

$$\{\hat{\varphi}_k^{q,p}\}_{k=1}^{\hat{N}^{t,p}} \subset \{\hat{\varphi}_k^{q,p+1}\}_{k=1}^{\hat{N}^{t,p+1}} \quad \text{and} \quad \{\hat{\varphi}_k^{t,p}\}_{k=1}^{\hat{N}^{t,p}} \subset \{\hat{\varphi}_k^{t,p+1}\}_{k=1}^{\hat{N}^{t,p+1}}, \quad p = 1, 2, \dots, \quad (3.68)$$

for triangles and quadrilaterals, respectively.

Now, we recall two well-known sets of functions.

**Definition 4** (*Legendre polynomials*) Legendre polynomials  $L_n(x)$ ,  $n = 0, 1, \dots$  are eigenvectors of the Legendre operator,

$$-\frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} L_n(x) \right] = n(n+1)L_n, \quad x \in (-1, 1). \quad (3.69)$$

Legendre polynomials can be evaluated using the recursive formula

$$\begin{aligned} L_0(x) &= 1, \\ L_1(x) &= x, \\ L_k(x) &= \frac{2k-1}{k} x L_{k-1}(x) - \frac{k-1}{k} L_{k-2}(x), \quad k = 2, 3, \dots \end{aligned}$$

Legendre polynomials form an orthogonal basis of the space  $L^2(-1, 1)$ , namely

$$\int_{-1}^1 L_k(x) L_m(x) dx = \begin{cases} \frac{2}{2k+1} & \text{for } k = m, \\ 0 & \text{otherwise.} \end{cases} \quad (3.70)$$

**Definition 5** (*Lobatto functions on  $(-1, 1)$* ) Let us define functions

$$\begin{aligned} \tilde{\ell}_0(x) &= \frac{1-x}{2}, \\ \tilde{\ell}_1(x) &= \frac{x+1}{2}, \\ \tilde{\ell}_k(x) &= \frac{1}{\|L_{k-1}\|_2} \int_{-1}^x L_{k-1}(\xi) d\xi, \quad k = 2, 3, \dots \end{aligned} \quad (3.71)$$

From (3.70) we have  $\|L_{k-1}\|_2 = \sqrt{2/(2k-1)}$ . Obviously  $\tilde{\ell}_k(-1) = 0$  and  $\tilde{\ell}_k(1) = 0$  since  $L_{k-1}$ ,  $k \geq 2$  are orthogonal to  $L_0 = 1$ , i.e.,

$$\int_{-1}^1 L_k(x) dx = \int_{-1}^1 L_k(x) L_0(x) dx = 0, \quad k = 2, 3, \dots \quad (3.72)$$

Therefore, the Lobatto functions  $\tilde{\ell}_k$ ,  $k = 0, \dots, p$  form a basis of the space  $P_p(-1, 1)$ .

For the purposes of our reference elements, we have to transform the Lobatto functions from interval  $(-1, 1)$  onto  $(0, 1)$ .

**Definition 6** (*Lobatto functions on  $(0, 1)$* ) Let us define functions

$$\ell_k(x) = \tilde{\ell}_k \left( \frac{x+1}{2} \right), \quad k = 0, 1, \dots, \quad (3.73)$$

where  $\tilde{\ell}_k$ ,  $k = 0, 1, \dots$ , are given by (5).

Obviously  $\ell_k(0) = 0$ ,  $\ell_k(1) = 0$  and the Lobatto functions  $\ell_k$ ,  $k = 0, \dots, p$  form a basis of the space  $P_p(0, 1)$ .

For the definition of the shape function on triangles, it is convenient to decompose the higher-order Lobatto functions  $\ell_k$ ,  $k \geq 2$  from Definition 6 into products of the form

$$\ell_k(x) = \ell_0(x) \ell_1(x) \phi_{k-2}(x), \quad k = 2, 3, \dots, \quad (3.74)$$

where  $\phi_{k-2} \in P^{k-2}([0, 1])$ ,  $k = 2, 3, \dots$  are the *kernel functions*.

Let us recall that we denote by  $\{\hat{\mathbf{v}}_i\}$  the vertices and by  $\{\hat{e}_i\}$  the edges of the reference element  $\hat{T}$ .

Node type	Pol. degree	# of shape functions	# of objects
Vertex	$p \geq 1$	1	4
Edge	$p \geq 2$	$p - 1$	4
Bubble	$p \geq 3$	$(p - 1)(p - 2)$	1

Table 1.1: Lobatto shape functions on  $\widehat{T}_q$

### The reference Lobatto shape functions on the square

The Lobatto shape functions on the quadrilateral  $\widehat{T}_q$  are constructed as tensor products of the one dimensional Lobatto functions. They are grouped into 3 subsets according to which node they belong.

The *Lobatto vertex functions*  $\hat{\varphi}_{\hat{\mathbf{v}}_l}^{q,p}$ ,  $l = 1, \dots, 4$  assigned to vertices  $\hat{\mathbf{v}}_l$ ,  $l = 1, \dots, 4$  are equal to one at  $\hat{\mathbf{v}}_l$  and vanishes at all remaining vertices. They are chosen bilinear, defined via products of Lobatto functions as follows

$$\begin{aligned}
\hat{\varphi}_{\hat{\mathbf{v}}_1}^{q,p}(\hat{x}_1, \hat{x}_2) &= \ell_0(\hat{x}_1)\ell_0(\hat{x}_2), \\
\hat{\varphi}_{\hat{\mathbf{v}}_2}^{q,p}(\hat{x}_1, \hat{x}_2) &= \ell_1(\hat{x}_1)\ell_0(\hat{x}_2), \\
\hat{\varphi}_{\hat{\mathbf{v}}_3}^{q,p}(\hat{x}_1, \hat{x}_2) &= \ell_1(\hat{x}_1)\ell_1(\hat{x}_2), \\
\hat{\varphi}_{\hat{\mathbf{v}}_4}^{q,p}(\hat{x}_1, \hat{x}_2) &= \ell_0(\hat{x}_1)\ell_1(\hat{x}_2),
\end{aligned} \tag{3.75}$$

for  $(\hat{x}_1, \hat{x}_2) \in \widehat{T}_q$ .

For  $p \geq 2$ , the *Lobatto edge functions*  $\hat{\varphi}_{\hat{e}_j}^{q,p}$ ,  $k = 2, \dots, p$ ,  $j = 1, \dots, 4$  are associated with the corresponding edge  $\hat{e}_j$ . Their trace on  $\hat{e}_j$  coincides with the Lobatto functions  $\ell_k$  and their trace vanishes on all remaining edges:

$$\begin{aligned}
\hat{\varphi}_{\hat{e}_{1,k}}^{q,p}(\hat{x}_1, \hat{x}_2) &= \ell_k(\hat{x}_1)\ell_0(\hat{x}_2), \quad 2 \leq k \leq p, \\
\hat{\varphi}_{\hat{e}_{2,k}}^{q,p}(\hat{x}_1, \hat{x}_2) &= \ell_1(\hat{x}_1)\ell_k(\hat{x}_2), \quad 2 \leq k \leq p, \\
\hat{\varphi}_{\hat{e}_{3,k}}^{q,p}(\hat{x}_1, \hat{x}_2) &= \ell_k(\hat{x}_1)\ell_1(\hat{x}_2), \quad 2 \leq k \leq p, \\
\hat{\varphi}_{\hat{e}_{4,k}}^{q,p}(\hat{x}_1, \hat{x}_2) &= \ell_0(\hat{x}_1)\ell_k(\hat{x}_2), \quad 2 \leq k \leq p,
\end{aligned} \tag{3.76}$$

for  $(\hat{x}_1, \hat{x}_2) \in \widehat{T}_q$ . Therefore,  $p - 2$  shape functions correspond to each edge of  $\widehat{T}_q$ .

For  $p \geq 3$ , the shape functions are completed by *Lobatto bubble functions*  $\hat{\varphi}_{b,n_1,n_2}^{q,p}$ , which vanish everywhere on the boundary of  $\widehat{T}'_q$

$$\hat{\varphi}_{b,n_1,n_2}^{q,p}(\hat{x}_1, \hat{x}_2) = \ell_{n_1}(\hat{x}_1)\ell_{n_2}(\hat{x}_2), \quad 2 \leq n_1 \leq p, \quad 2 \leq n_2 \leq p, \tag{3.77}$$

for  $(\hat{x}_1, \hat{x}_2) \in \widehat{T}_q$ . The total number of the Lobatto shape functions on  $\widehat{T}_q$  is summarized in Table 1.1.

### The reference Lobatto shape functions on the triangle

Lobatto basis of the space  $V_h$  consist again of vertex, edge and bubble shape functions defined on  $\widehat{T}_t$ .

Node type	Pol. degree	# of shape functions	# of objects
Vertex	$p \geq 1$	1	3
Edge	$p \geq 2$	$p - 1$	3
Bubble	$p \geq 3$	$(p - 1)(p - 2)/2$	1

Table 1.2: Lobatto shape functions on  $\widehat{T}_t$

*Vertex functions*  $\hat{\varphi}_{\hat{\mathbf{v}}_1}^{t,p}$ ,  $\hat{\varphi}_{\hat{\mathbf{v}}_2}^{t,p}$ ,  $\hat{\varphi}_{\hat{\mathbf{v}}_3}^{t,p}$ , are assigned to vertices  $\hat{\mathbf{v}}_1$ ,  $\hat{\mathbf{v}}_2$ ,  $\hat{\mathbf{v}}_3$ . Each function is equal to one at corresponding vertex and vanishes at remaining two vertices. Vertex functions are linear, defined by following formulas

$$\hat{\varphi}_{\hat{\mathbf{v}}_k}^{t,p}(\hat{x}_1, \hat{x}_2) = \lambda_k, \quad k = 1, 2, 3, \quad (3.78)$$

where  $\lambda_1 = \hat{x}_1$ ,  $\lambda_2 = \hat{x}_2$  and  $\lambda_3 = 1 - \hat{x}_1 - \hat{x}_2$  are the barycentric coordinates on  $\widehat{T}_t$ .

For  $p \geq 2$ , the *Edge functions*  $\hat{\varphi}_{\hat{e}_j,k}^{t,p}$ ,  $k = 2, \dots, p$ ,  $j = 1, 2, 3$  coincide with one-dimensional Lobatto functions on corresponding edge and vanish on all remaining edges. They can be written using the kernel functions  $\phi_{k-2}$ , defined by (3.74), in the form

$$\begin{aligned} \hat{\varphi}_{\hat{e}_1,k}^{t,p}(\hat{x}_1, \hat{x}_2) &= \lambda_1 \lambda_2 \phi_{k-2}(\lambda_2 - \lambda_1), \quad 2 \leq k \leq p, \\ \hat{\varphi}_{\hat{e}_2,k}^{t,p}(\hat{x}_1, \hat{x}_2) &= \lambda_2 \lambda_3 \phi_{k-2}(\lambda_3 - \lambda_2), \quad 2 \leq k \leq p, \\ \hat{\varphi}_{\hat{e}_3,k}^{t,p}(\hat{x}_1, \hat{x}_2) &= \lambda_3 \lambda_1 \phi_{k-2}(\lambda_1 - \lambda_3), \quad 2 \leq k \leq p. \end{aligned} \quad (3.79)$$

For  $p \geq 3$ , the *bubble shape functions*  $\hat{\varphi}_{b,n_1,n_2}^{t,p}$  complete the basis on  $\widehat{T}_t$ . These functions vanish on the whole element boundary. They will be defined using affine coordinates and kernel functions as follows

$$\hat{\varphi}_{b,n_1,n_2}^{t,p} = \lambda_1 \lambda_2 \lambda_3 \phi_{n_1-1}(\lambda_3 - \lambda_2) \phi_{n_2-1}(\lambda_2 - \lambda_1), \quad 1 \leq n_1, 1 \leq n_2, n_1 + n_2 \leq p - 1, \quad (3.80)$$

where  $\phi_{n_1-1}$ ,  $\phi_{n_2-1}$  are the kernel functions defined by (3.74). Total number of the Lobatto shape functions on triangular reference element is summarized in Table 1.2.

**Remark 2** *On both reference domains  $\widehat{T}_q$ ,  $\widehat{T}_t$  shape functions coincide on edges with Lobatto functions  $\ell_k$ ,  $k = 0, 1, \dots$ . Moreover, the bubble shape functions does not affect a possible compatibility between quadrilateral and triangular meshes This allows us to combine triangular and quadrilateral elements in one hybrid mesh, see [1].*

### 1.3.3 Global basis functions on $V_h$

In Sections 1.3.1 – 1.3.2, we introduced two possible sets of the shape functions defined on the reference element  $\widehat{T}$ .

Here we describe a construction of a global basis of the finite-dimensional space  $V_h$ , which is carried out by gluing together reference shape functions (transformed to the physical element by mapping  $F_T$ ). It is done in such a way, that resulting basis functions satisfy conformity requirements of the space  $V_h \subset H^1(\Omega)$ .

**Remark 3** In order to fulfill the conformity requirement of global continuity of basis functions, the orientation of physical mesh edges has to be taken into account. For two-dimensional problems, it is possible to index edges of the reference element as well as physical elements counterclockwise. Then the orientation of the reference element edge  $\hat{e} = \hat{\mathbf{v}}_i \hat{\mathbf{v}}_j \subset \partial \hat{T}$  coincides with the orientation of the edge  $e = \mathbf{v}_i \mathbf{v}_j \subset \partial T$  such that  $\mathbf{v}_i = F_T(\hat{\mathbf{v}}_i)$  and  $\mathbf{v}_j = F_T(\hat{\mathbf{v}}_j)$ . In other case, when the orientations differ, all reference edge functions have to be transformed by a suitable mapping, which inverts the parameterization of the edge  $\hat{e}$ .

In order to avoid a complication with a notation, we assume that  $\mathcal{T}_h$  is a triangular grid. All statements are valid also for quadrilateral grid, only the superscript  $t$  has to be replaced by the superscript  $q$  and the number of vertices and edges of  $T \in \mathcal{T}_h$  from 3 to 4.

### Global Lagrangian shape functions

Let  $p \geq 1$  be a given degree of polynomial approximation and  $\hat{\mathbf{z}}_i^{t,p}$ ,  $i = 1, \dots, \hat{N}$  be the reference Lagrangian nodes defined by (3.64). For each  $T \in \mathcal{T}_h$ , we define the nodes

$$\mathbf{z}_{T,i}^{t,p} = F_T(\hat{\mathbf{z}}_i^{t,p}), \quad i = 1, \dots, \hat{N}, \quad (3.81)$$

where  $F_T$  maps  $\hat{T}_t$  onto  $T$ . Let us note that some of these nodes coincides, namely  $\mathbf{z}_{T,i}^{t,p} = \mathbf{z}_{T',i'}^{t,p}$  for  $T \neq T'$  sharing a vertex or an edge.

Moreover, we define the set of *Lagrangian nodes*

$$\mathcal{L}^{t,p} := \left\{ \mathbf{z} \in \Omega \text{ such that } \exists (T, i) \in \mathcal{T}_h \times \{1, \dots, \hat{N}\} \text{ satisfying } \mathbf{z} = \mathbf{z}^{t,p} \right\}. \quad (3.82)$$

It means that  $\mathcal{L}^{t,p}$  contains all  $\mathbf{z}_{T,i}^{t,p}$ ,  $i = 1, \dots, \hat{N}$ ,  $T \in \mathcal{T}_h$  lying inside of  $\Omega$  (we exclude nodes lying on  $\partial\Omega$ ) and each node appears in  $\mathcal{L}^{t,p}$  only one times. We index the set  $\mathcal{L}^{t,p}$  by  $\mathcal{L}^{t,p} = \{\mathbf{z}_j^{t,p}, j = 1, \dots, N_L\}$ .

The *global Lagrangian basis functions*  $\varphi_i^{t,p}$ ,  $i = 1, \dots, N_L$  is defined by

$$\varphi_i^{t,p}(\mathbf{x}) \in V_h, \quad \varphi_i^{t,p}(\mathbf{z}_j^{t,p}) = \delta_{ij}, \quad i, j = 1, \dots, N_L. \quad (3.83)$$

Let us note that each global Lagrangian basis function  $\varphi_i^{t,p}$ ,  $i = 1, \dots, N_L$  can be, due to (3.83), classified as

- *vertex basis function* if the corresponding Lagrangian node is an vertex of  $\mathcal{T}_h$ , its support is formed by a patch of elements  $T \in \mathcal{T}_h$  having this vertex common,
- *edge basis function* if the corresponding Lagrangian node lies on an edge of  $\mathcal{T}_h$ , its support is formed by a patch of two elements  $T \in \mathcal{T}_h$  sharing this edge,
- *bubble basis function* if the corresponding Lagrangian node lies in the interior of a  $T \in \mathcal{T}_h$ , which is also its support.

The global Lagrangian basis function are defined in the following way.

**Definition 7** (*Vertex basis function*) Let  $\mathbf{z}_i^{t,p}$  is a Lagrangian node which is identical with the vertex  $\mathbf{v}_l$  of  $\mathcal{T}_h$ . Then the corresponding vertex Lagrangian basis function  $\varphi_i^{t,p}$  is a continuous function defined on  $\Omega$  which equals to one at  $\mathbf{v}_l$  and vanishes outside of the patch  $S_l$  formed by all elements  $T \in \mathcal{T}_h$  sharing the vertex  $\mathbf{v}_l$ . For each element  $T_j \in S_l$ ,

$$\varphi_i^{t,p}(\mathbf{x})|_{T_j} = \hat{\varphi}_k^{t,p}(\hat{\mathbf{x}}), \quad \mathbf{x} = F_T(\hat{\mathbf{x}}), \quad (3.84)$$

where  $\hat{\varphi}_k^{t,p}$  is the reference Lagrangian shape function such that  $\hat{\mathbf{z}}_k = F_T^{-1}(\mathbf{z}_i^{t,p})$ .

**Definition 8** (*Edge basis function*) Let  $\mathbf{z}_i^{t,p}$  is a Lagrangian node which lies on an edge  $e_l$  of  $\mathcal{T}_h$ . Then the corresponding edge Lagrangian basis function  $\varphi_i^{t,p}$  is a continuous function defined on  $\Omega$  which vanishes outside of the patch  $S_l$  formed by all elements sharing the edge  $e_l$ . For each element  $T_j \in S_l$ ,

$$\varphi_i^{t,p}(\mathbf{x})|_{T_j} = \hat{\varphi}_k^{t,p}(\hat{\mathbf{x}}), \quad \mathbf{x} = F_T(\hat{\mathbf{x}}), \quad (3.85)$$

where  $\hat{\varphi}_k^{t,p}$  is the reference Lagrangian shape function such that  $\hat{\mathbf{z}}_k = F_T^{-1}(\mathbf{z}_i^{t,p})$ .

**Definition 9** (*Bubble basis function*) Let  $\mathbf{z}_i^{t,p}$  is a Lagrangian node which lies in the interior of an element  $T \in \mathcal{T}_h$ . Then the corresponding bubble Lagrangian basis function  $\varphi_i^{t,p}$  which vanishes outside of  $T$ ,

$$\varphi_i^{t,p}(\mathbf{x})|_T = \hat{\varphi}_k^{t,p}(\hat{\mathbf{x}}), \quad \mathbf{x} = F_T(\hat{\mathbf{x}}), \quad (3.86)$$

where  $\hat{\varphi}_k^{t,p}$  is the reference Lagrangian shape function such that  $\hat{\mathbf{z}}_k = F_T^{-1}(\mathbf{z}_i^{t,p})$ .

## Global Lobatto shape functions

Let  $p \geq 1$  be a given degree of polynomial approximation and The global basis  $B$  of  $V_h$  consists of three type of basis functions:

- *Vertex basis functions*  $\varphi_{\mathbf{v}_l}^{t,p}$ ,  $l = 1, \dots, N$ , which are associated to each inner vertex  $\mathbf{v}_l$ ,  $l = 1, \dots, N$  of  $\mathcal{T}_h$ . The support of  $\varphi_{\mathbf{v}_l}^{t,p}$  is formed by a patch of elements  $T \in \mathcal{T}_h$  having  $\mathbf{v}_l$  as an vertex.
- *Edge basis functions*  $\varphi_{e_l,k}^{t,p}$ ,  $2 \leq k \leq p$ ,  $l = 1, \dots, E$ , which are associated to each inner edge  $e_l$ ,  $l = 1, \dots, E$  of  $\mathcal{T}_h$ . The support of  $\varphi_{e_l,k}^{t,p}$ ,  $2 \leq k \leq p$  is formed by the patch of two elements from  $T \in \mathcal{T}_h$  sharing the face  $e_l$ .
- *Bubble basis functions*  $\varphi_{T,n_1,n_2}^{t,p}$ ,  $1 \leq n_1$ ,  $1 \leq n_2$ ,  $n_1 + n_2 \leq p - 1$ ,  $T \in \mathcal{T}_h$ , which are associated to each element  $T \in \mathcal{T}_h$ . Their support consists just from the one element  $T$ .

The global Lobatto basis function are defined in the following way.

**Definition 10** (*Vertex basis function*) The vertex basis function  $\varphi_{\mathbf{v}_l}^{t,p}$  associated with a mesh vertex  $\mathbf{v}_l$ ,  $l = 1, \dots, N$  is a continuous function defined on  $\Omega$  which equals to



one at  $\mathbf{v}_l$  and vanishes outside of the patch  $S_l$  formed by all elements  $T \in \mathcal{T}_h$  sharing the vertex  $\mathbf{v}_l$ . For each element  $T_i \in S_l$ ,

$$\varphi_{\mathbf{v}_l}^{t,p}(\mathbf{x})|_{T_i} = \hat{\varphi}_{\hat{\mathbf{v}}_k}^{t,p}(\hat{\mathbf{x}}), \quad \mathbf{x} = F_T(\hat{\mathbf{x}}), \quad (3.87)$$

where  $\hat{\varphi}_{\hat{\mathbf{v}}_k}^{t,p}$  is the reference vertex shape function associated to the vertex  $\hat{\mathbf{v}}_k$  of  $\hat{T}_t$  such that  $\hat{\mathbf{v}}_k = F_T^{-1}(\mathbf{v}_l)$ .

**Definition 11** (*Edge basis function*) The edge function  $\varphi_{e_l,k}^{t,p}$  associated with a mesh edge  $e_l$  and  $k = 2, \dots, p$ , is a continuous function defined on  $\Omega$  which vanishes outside of the patch  $S_l$  formed by all elements sharing the edge  $e_l$ . For each element  $T \in S_l$ ,

$$\varphi_{e_l,k}^{t,p}(\mathbf{x})|_{T_i} = \hat{\varphi}_{\hat{e}_j,k}^{t,p}(\hat{\mathbf{x}}), \quad \mathbf{x} = F_T(\hat{\mathbf{x}}), \quad (3.88)$$

where  $\hat{\varphi}_{\hat{e}_j,k}^{t,p}$  is the reference edge shape function associated to the edge  $\hat{e}_j$  of  $\hat{T}_t$  such that  $\hat{e}_j = F_T^{-1}(e_l)$ .

**Definition 12** (*Bubble basis function*) The bubble function  $\varphi_{T,n_1,n_2}^{t,p}$  associated with a  $T \in \mathcal{T}_h$  and  $1 \leq n_1, 1 \leq n_2, n_1 + n_2 \leq p - 1, k = 2, \dots, p$ , is a continuous function defined on  $\Omega$  which vanishes outside of  $T$ ,

$$\varphi_{T,n_1,n_2}^{t,p}(\mathbf{x})|_T = \hat{\varphi}_{b,n_1,n_2}^{t,p}(\hat{\mathbf{x}}), \quad \mathbf{x} = F_T(\hat{\mathbf{x}}), \quad (3.89)$$

where  $\hat{\varphi}_{b,n_1,n_2}^{t,p}$  is the reference bubble shape function associated to  $\hat{T}_t$ .

**Remark 4** We already mentioned that a use of hierarchical finite elements is advantageous in case where different degree of polynomial approximation are used on different elements. However, in case of conforming finite element methods, the basis functions have to be continuous. Therefore, it is necessary to construct a transition from one degree of polynomial approximation to the another one, see [1].

## 1.4 Discontinuous finite elements

In the previous sections of this chapter, we discussed implementations of conforming finite elements methods, where the finite element spaces consists of continuous functions. In this section we focus on an implementation of methods based on discontinuous approximation, namely the *discontinuous Galerkin (DG) method*.

Although the DG formulation is more complicated than for the conforming FEM since additional terms appear there, its implementation is usually more simple since the test functions are chosen separately for each  $T \in \mathcal{T}_h$ .

### 1.4.1 DG discretization of the model problem

In the following we recall the DG discretization of the model problem (1.1) – (1.2). We consider here the IIPG variant of the DG method since it has the simplest formulation.

Let  $\mathcal{T}_h$  ( $h > 0$ ) be a partition of the closure  $\bar{\Omega}$  of the domain  $\Omega$  into a finite number of closed  $d$ -dimensional elements, where we admit a combination of triangular and quadrilateral elements including hanging nodes. For simplicity, we write again,

$$\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} T. \quad (4.90)$$

Moreover,  $\mathcal{F}_h$  denotes the set of all edges (faces) of  $\mathcal{T}_h$ .

We assume that for each element  $T \in \mathcal{T}_h$  there exist a mapping

$$F_T = F_T(\hat{\mathbf{x}}) = (F_{T,1}(\hat{\mathbf{x}}), \dots, F_{T,d}(\hat{\mathbf{x}})) : \hat{T} \rightarrow \mathbb{R}^d \quad \text{such that} \quad F_T(\hat{T}) = T, \quad (4.91)$$

where  $\hat{T}$  is either the reference triangle  $\hat{T}_t$  or the reference quadrilateral  $\hat{T}_q$  given by (2.33) or (2.34), respectively.

We define the space of *discontinuous piecewise polynomial functions*  $S_{hp}$  by

$$S_{hp} = \{v_h, v_h \in L^2(\Omega), v_h|_T \circ F_T \in P^p(\hat{T}) \forall T \in \mathcal{T}_h\}, \quad (4.92)$$

where  $\hat{T}$  denotes either  $\hat{T}_t$  or  $\hat{T}_q$  depending on the type of  $T$ . Furthermore,  $P^p$  is the space of polynomials of degree  $\leq p$  on  $\hat{T}$  defined by (2.51) and/or (2.54).

Since DG methods use a discontinuous approximation, it is possible to define the space  $P^p(\hat{T})$  in (4.92) by several manners, namely

- $P^{t,p}(\hat{T}_t)$  for triangles and  $P^{t,p}(\hat{T}_q)$  for quadrilaterals,
- $P^{t,q}(\hat{T}_t)$  for triangles and  $P^{t,q}(\hat{T}_q)$  for quadrilaterals,
- $P^{t,p}(\hat{T}_t)$  for triangles and  $P^{t,q}(\hat{T}_q)$  for quadrilaterals,

In this text we consider the first variants, the others can be obtain with a small modification.

Moreover,  $\mathbf{n}_e$  denotes a unit normal vector to  $e \in \mathcal{F}_h$ , its orientation is arbitrary but fixed. Furthermore, the symbols  $\langle v_h \rangle_e$  and  $[[v_h]]_e$  denote the mean value and the jump in direction of  $\mathbf{n}_e$  of  $v_h \in S_{hp}$  on inner face  $e \in \mathcal{F}_h$  ( $e \subset \Omega$ ). For  $e \subset \partial\Omega$ , we put  $[[v_h]]_e = \langle v_h \rangle_e = v_h|_e$ . Finally, in case that  $\mathbf{n}_e$  or  $[[\cdot]]_e$  or  $\langle \cdot \rangle_e$  are arguments of  $\int_e \dots d\sigma$ ,  $e \in \mathcal{F}_h$ , we omit very often the subscript  $e$  and write simply  $\mathbf{n}$  or  $[[\cdot]]$  or  $\langle \cdot \rangle$ , respectively.

Now, we are ready to introduce the DG approximation of the model problem.

**Definition 13** We say that  $u_h \in S_{hp}$  is the approximate DG solution of (1.1) – (1.2) if

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \int_T \nabla u_h \cdot \nabla v_h \, dx &- \sum_{e \in \mathcal{F}_h} \int_e \langle \nabla u_h \rangle \cdot \mathbf{n} [[v_h]] \, d\sigma \\ &+ \sum_{e \in \mathcal{F}_h} \int_e \rho [[u_h]] [[v_h]] \, d\sigma = \int_{\Omega} g v_h \, dx \quad \forall v_h \in S_{hp}, \end{aligned} \quad (4.93)$$

where  $\rho$  denotes the penalty parameter.

### 1.4.2 $hp$ -discontinuous Galerkin method

Let us assume that the elements of  $\mathcal{T}_h$  are indexed by

$$\mathcal{T}_h = \{T_\mu, \mu = 1, \dots, M\}. \quad (4.94)$$

DGM allows the use of different polynomial degrees over different elements without a necessity to construct a transition from one degree of polynomial approximation to the another one. In this section, we will consider this general case since its treatment is relatively simple. We assign a positive integer  $p_\mu$  (local polynomial degree) to each  $T_\mu \in \mathcal{T}_h$ . In case when the one uses the same degree of polynomial approximation in the whole  $\Omega$ , we have  $p_\mu = p \forall T_\mu \in \mathcal{T}_h$ . We define the vector

$$\mathbf{p} = \{p_\mu, T_\mu \in \mathcal{T}_h\}. \quad (4.95)$$

Over the triangulation  $\mathcal{T}_h$ , we define the finite dimensional space of discontinuous piecewise polynomial functions associated with the vector  $\mathbf{p}$  by

$$S_{hp} = \{v; v \in L^2(\Omega), v|_\mu \circ F_K \in P_{p_\mu}(\widehat{T}) \forall \mu = 1, \dots, M\}, \quad (4.96)$$

where  $P^{p_\mu}(\widehat{T})$  denotes the space of all polynomials on  $\widehat{T}$  of degree  $\leq p_\mu$ ,  $K \in \mathcal{T}_h$ .

Now, we can simply formulate the  $hp$ -approximation of the model problem. We replace

**Definition 14** *We say that  $u_h \in S_{hp}$  is the  $hp$ -approximate DG solution of (1.1) – (1.2) if*

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \int_T \nabla u_h \cdot \nabla v_h \, dx - \sum_{e \in \mathcal{F}_h} \int_e \langle \nabla u_h \rangle \cdot \mathbf{n} [v_h] \, d\sigma \\ + \sum_{e \in \mathcal{F}_h} \int_e \rho [u_h] [v_h] \, d\sigma = \int_\Omega g v_h \, dx \quad \forall v_h \in S_{hp}, \end{aligned} \quad (4.97)$$

where  $\rho$  denotes the penalty parameter.

In comparison with (1.4) the face integrals have to be implemented. On the other hand, their implementation of DGM is easier since a discontinuous approximation is used therefore it is possible constructed basis functions with a support consisting from one  $T \in \mathcal{T}_h$ .

In the following we describe the construction of basis functions of  $S_{hp}$ . First, we introduce a (global) basis of the space  $S_{hp}$  as a composition of local shape constructed for each  $T \in \mathcal{T}_h$  separately. Then, we introduce the construction of local shape functions. Finally, we explain the evaluation of the face integrals.

### 1.4.3 Definition of a DG basis

Since  $S_{hp}$  is a space of discontinuous piecewise polynomial functions, it is possible to consider a set of linearly independent polynomial functions on  $T_\mu$  for each  $T_\mu \in \mathcal{T}_h$

$$\begin{aligned} B_\mu = \left\{ \varphi_{\mu,j}; \varphi_{\mu,j} \in S_{hp}, \text{supp}(\varphi_{\mu,j}) \subseteq T_\mu, \right. \\ \left. \varphi_{\mu,j} \text{ are linearly independent for } j = 1, \dots, \text{dof}_\mu \right\}, \end{aligned} \quad (4.98)$$

$p_\mu$	1	2	3	4	5
$d = 2$	3	6	10	15	21
$d = 3$	4	10	20	35	56

Table 1.3: Values of  $\text{dof}_\mu$  for  $p_\mu = 1, \dots, 5$  and  $d = 2, 3$

where

$$\text{dof}_\mu = \begin{cases} \frac{1}{2} \sum_{j=1}^2 (p_\mu + j) & \text{for } d = 2 \\ \frac{1}{6} \sum_{j=1}^3 (p_\mu + j) & \text{for } d = 3 \end{cases}, \quad \mu = 1, \dots, M, \quad (4.99)$$

denotes the number of *local degrees of freedom* for each element  $T_\mu \in \mathcal{T}_h$ . The values of  $\text{dof}_\mu$  are shown in Table 1.3 for  $p_\mu = 1, \dots, 5$  and  $d = 2, 3$ . We call  $B_\mu$  the *local basis* on  $T_\mu$ . For the construction of the basis  $B_\mu$ ,  $\mu = 1, \dots, M$ , see Section 1.4.4.

A composition of the local bases  $B_\mu$ ,  $\mu = 1, \dots, M$  defines a basis of  $S_{hp}$ , i.e.,

$$B = \{\varphi_j; \varphi_j \in S_{hp}, j = 1, \dots, \text{dof}\}. \quad (4.100)$$

By  $\text{dof}$ , we denote the dimension of  $S_{hp}$  (=number of elements of the basis  $B$ ) which is equal to

$$\text{dof} = \sum_{\mu=1}^M \text{dof}_\mu. \quad (4.101)$$

where  $\text{dof}_\mu$  is given by (4.99).

Therefore, a function  $u_h \in S_{hp}$  can be written in the form

$$u_h(x) = \sum_{\mu=1}^M \sum_{j=1}^{\text{dof}_\mu} u_{\mu,j} \varphi_{\mu,j}(x), \quad x \in \Omega, u_{\mu,j} \in \mathbb{R}, j = 1, \dots, \text{dof}_\mu, \mu = 1, \dots, M \quad (4.102)$$

where  $u_{\mu,j} \in \mathbb{R}$ ,  $j = 1, \dots, \text{dof}_\mu$ ,  $\mu = 1, \dots, M$ . Moreover, for  $u_h \in S_{hp}$ , we define the vector of its basis coefficients by

$$\mathbf{u} = \{u_{\mu,j}\}_{j=1, \dots, \text{dof}_\mu}^{\mu=1, \dots, M} \in \mathbb{R}^{\text{dof}}, \quad k = 0, 1, \dots, r. \quad (4.103)$$

Therefore, using (4.102) – (4.103) we have an isomorphism

$$u_h \in S_{hp} \quad \longleftrightarrow \quad \mathbf{u} \in \mathbb{R}^{\text{dof}}. \quad (4.104)$$

#### 1.4.4 Construction of the DG basis functions

In order to achieve a reasonable efficiency, hierarchical basis functions should be employed. Let

$$\bar{p} = \max_{\mu=1, \dots, M} p_\mu$$

and let  $P^{\bar{p}}(\widehat{T})$  denotes the space of all polynomials on  $\widehat{T}$  of degree  $\leq p_{\bar{p}}$ . Let  $\widehat{S}_{\bar{p}}$  be a hierarchical basis of  $P^{\bar{p}}(\widehat{T})$  which will be specified later. We call  $\widehat{S}_{\bar{p}}$  the *reference basis* and its elements the *reference shape function*.

Furthermore, let  $F_\mu := F_{T_\mu}$ ,  $\mu = 1, \dots, M$ , be the mapping introduced by (4.91) such that  $F_\mu(\widehat{T}) = T_\mu$ . We put

$$B_\mu := \{\varphi_{\mu,j}; \varphi_{\mu,j}(\mathbf{x}) = \varphi_{\mu,j}(F_\mu(\widehat{\mathbf{x}})) = \widehat{\varphi}_j(\widehat{\mathbf{x}}), \widehat{\mathbf{x}} \in \widehat{T}, j = 1, \dots, \text{dof}_\mu\}, \quad (4.105)$$

which defines a basis  $B_\mu$  introduced in (4.98) for each element  $T_\mu \in \mathcal{T}_h$  separately. Finally, (4.100) defines the *global basis* of  $S_{hp}$ .

### Reference shape functions

It is possible to use an arbitrary set of (hierarchical or non-hierarchical) shape functions for the definition of  $\widehat{S}_{\bar{p}}$ . An example are the Lagrangian and Lobatto shape functions introduced in Sections 1.3.1 and 1.3.2, respectively. However, since  $S_{hp}$  is the space of discontinuous functions, it is not necessary to construct vertex, edge and bubble functions in order to fulfil the continuity of the basis functions.

Therefore, it is possible to use the *Taylor shape functions* given by

$$\left\{ \widehat{\varphi}_{i_1, \dots, i_d}(\widehat{x}_1, \dots, \widehat{x}_d) = \prod_{i=1}^d (\widehat{x}_i - \widehat{x}_i^c)^{i_i}; \quad i_1, \dots, i_d \geq 0, \sum_{j=1}^d i_j \leq \bar{p} \right\}, \quad (4.106)$$

where  $(\widehat{x}_1^c, \dots, \widehat{x}_d^c)$  is the barycentre of  $\widehat{T}$ . (It is possible to use also  $\widehat{x}_1^c = \dots = \widehat{x}_d^c = 0$ ). The advantage of the Taylor shape functions is a simpler implementation since we need not to distinguish among the vertex, edge and bubble functions, one subroutine is sufficient.

However, a direct use the Taylor shape functions is not too much practical since these functions are not normalized which can cause problems in the solution of the corresponding linear algebraic systems. Moreover, in many application, namely in evolution problems, it is advantageous to employ an  $L^2$ -orthonormal basis because the corresponding linear systems have favorable computational properties. Let us note that a suitable preconditioning can reduce many drawback following from a possible use of a non-normalized shape functions.

Therefore, we employ the local character of the shape functions and construct a basis of  $\widehat{S}_{\bar{p}}$  which is *orthonormal* with respect to the  $L^2$ -*scalar product*. The simplest way, how to obtain the  $L^2$ -orthonormal basis is the performance of the Gram-Schmidt orthogonalization process applied to the functions from (4.106). Then we obtain the orthonormal set of the reference shape functions

$$\widehat{S}_{\bar{p}} := \{\widehat{\varphi}_j, j = 1, \dots, \text{dof}_{\bar{p}}\}, \quad (4.107)$$

where  $\text{dof}_{\bar{p}}$  is given by (4.99) with  $p_{\bar{p}} := \bar{p}$ .

It is well known fact that the Gram-Schmidt orthogonalization is an unstable algorithm. However, in practice, this approach works since  $\text{dof}_{\bar{p}}$  is usually not too large number (between 10 and 200) and moreover, the instability of the Gram-Schmidt process cause a small violation of the  $L^2$ -orthogonality of the reference shape functions which do not affect good computational properties of the corresponding linear algebraic systems.

The Gram-Schmidt orthogonalization on the reference element can be carried out symbolically by some suitable software (e.g., Maple) or numerically. Our numerical experiments based on the numerical realization of the Gram-Schmidt orthogonalization work with success for  $\bar{p} = 10 \Rightarrow \text{dof}_{\bar{\mu}} = 45$  (for  $d = 2$ ).

Let us finally note, the global basis of  $S_{h\mathbf{p}}$  obtained from (4.100), (4.105) and (4.107) is orthonormal if the mappings  $F_{\mu} := F_{T_{\mu}}$ ,  $\mu = 1, \dots, M$  are linear. Otherwise, some violations of the  $L^2$ -orthogonality are presented. However, the mappings  $F_{\mu}$ ,  $\mu = 1, \dots, M$  are close to linear mappings, hence these violations are small and (again) do not affect good computational properties of the corresponding linear algebraic systems.

### 1.4.5 Evaluation of volume integrals

The evaluation of the volume integrals in (4.97) is very simple. Similarly as in the conforming FEM, we have to compute the terms

$$\sum_{T \in \mathcal{T}_h} \int_T \nabla \varphi_{\mu,i} \cdot \nabla \varphi_{\nu,j} \, dx, \quad i = 1, \dots, \text{dof}_{\mu}, j = 1, \dots, \text{dof}_{\nu}, \mu, \nu = 1, \dots, M, \quad (4.108)$$

where  $\varphi_{\mu,i}, \varphi_{\nu,j} \in B$  ( $=$  basis of  $S_{h\mathbf{p}}$ ). Since each  $\varphi_{\mu,i}$  vanish outside of  $T_{\mu}$ , we have

$$\sum_{T \in \mathcal{T}_h} \int_T \nabla \varphi_{\mu,i} \cdot \nabla \varphi_{\nu,j} \, dx = \delta_{\mu,\nu} \int_{T_{\mu}} \nabla \varphi_{\mu,i} \cdot \nabla \varphi_{\mu,j} \, dx, \quad (4.109)$$

for all  $i = 1, \dots, \text{dof}_{\mu}$ ,  $j = 1, \dots, \text{dof}_{\nu}$ ,  $\mu, \nu = 1, \dots, M$ . Therefore, the stiff matrix  $\mathbb{S}$  is block diagonal with blocks  $\mathbb{S}_{\mu}$ ,  $\mu = 1, \dots, M$  given by

$$\mathbb{S}_{\mu} = \{S_{\mu,i,j}\}_{i,j=1}^{\text{dof}_{\mu}}, \quad S_{\mu,i,j} = \int_{T_{\mu}} \nabla \varphi_{\mu,i} \cdot \nabla \varphi_{\mu,j} \, dx, \quad i, j = 1, \dots, \text{dof}_{\mu}. \quad (4.110)$$

These integral can be simply evaluated with the aid of (2.58), namely

$$\int_{T_{\mu}} \nabla \varphi_{\mu,i} \cdot \nabla \varphi_{\mu,j} \, dx = \int_{\hat{T}} \sum_{l=1}^d \left( J_{F_{T_{\mu}}}^{-T} \widehat{\nabla} \hat{\varphi}_i \right)_l \left( J_{F_{T_{\mu}}}^{-T} \widehat{\nabla} \hat{\varphi}_j \right)_l |\det J_{F_{T_{\mu}}}| \, d\hat{\mathbf{x}}, \quad (4.111)$$

where  $\hat{\varphi}_i, \hat{\varphi}_j$  are the reference shape functions from  $\hat{S}_{\bar{p}}$  given by (4.107). In comparison with the conforming FEM, situation is more simple, since we know that  $\varphi_{\mu,i}$  corresponds to  $\hat{\varphi}_i$ ,  $i = 1, \dots, \text{dof}_{\mu}$ , compare with (2.56).

### 1.4.6 Evaluation of face integrals

In this section we deal with the evaluation of the face integrals in (4.97). Let us note that the necessity to evaluate face integrals appears also in the conforming finite element method, e.g., for the implementation of general boundary conditions, see Section 1.1.5.

Let  $e \in \mathcal{F}_h$  be an edge of  $T \in \mathcal{T}_h$ . We call  $e$  the physical edge. Our aim is to evaluate the integrals

$$\int_e f(\mathbf{x}) \, d\sigma, \quad \int_e \mathbf{f}(\mathbf{x}) \cdot \mathbf{n} \, d\sigma, \quad (4.112)$$

where  $\mathbf{n}$  is the normal vector to  $e$  and  $f : e \rightarrow \mathbb{R}$ ,  $\mathbf{f} : e \rightarrow \mathbb{R}^2$  are given functions. Let us recall the definition of the face integral. Let  $\psi = (\psi_1, \psi_2) : [0, 1] \rightarrow e$  be a parameterization of the edge  $e$ . Then

$$\int_e f(\mathbf{x}) \, d\sigma = \int_0^1 f(\psi(t)) \sqrt{(\psi_1(t)')^2 + (\psi_2(t)')^2} \, dt, \quad (4.113)$$

where  $\psi_i(t)'$ ,  $i = 1, 2$  denotes the derivative of  $\psi_i(t)$  with respect to  $t$ .

In order to evaluate the integrals (4.112), we use the approach based on the reference element. Let  $\hat{e}$  be an edge of the reference element  $\hat{T}$  such that  $T = F_T(\hat{T})$  and  $e = F_T(\hat{e})$ . We call  $\hat{e}$  the reference edge. Let

$$\hat{\mathbf{x}}_{\hat{e}}(t) = (\hat{x}_{\hat{e},1}(t), \hat{x}_{\hat{e},2}(t)) : [0, 1] \rightarrow \hat{e} \quad (4.114)$$

be a parameterization of the reference edge  $\hat{e}$  keeping the counterclockwise orientation of the element. Namely, using the notation from Figure 1.6, we have

$$\begin{aligned} \text{if } \hat{T} = \hat{T}_t \text{ then} \quad & \hat{\mathbf{x}}_{\hat{e}_1}(t) := (t, 0), \quad t \in (0, 1), \\ & \hat{\mathbf{x}}_{\hat{e}_2}(t) := (1 - t, t), \quad t \in (0, 1), \\ & \hat{\mathbf{x}}_{\hat{e}_3}(t) := (0, 1 - t), \quad t \in (0, 1), \end{aligned} \quad (4.115)$$

$$\begin{aligned} \text{if } \hat{T} = \hat{T}_q \text{ then} \quad & \hat{\mathbf{x}}_{\hat{e}_1}(t) := (t, 0), \quad t \in (0, 1), \\ & \hat{\mathbf{x}}_{\hat{e}_2}(t) := (1, t), \quad t \in (0, 1), \\ & \hat{\mathbf{x}}_{\hat{e}_3}(t) := (1 - t, 1), \quad t \in (0, 1), \\ & \hat{\mathbf{x}}_{\hat{e}_4}(t) := (0, 1 - t), \quad t \in (0, 1). \end{aligned}$$

Moreover, we define the infinitesimal increase of  $\hat{\mathbf{x}}$  by

$$d\hat{\mathbf{x}}_{\hat{e}}(t) := \frac{d}{dt} \hat{\mathbf{x}}_{\hat{e}}(t) \in \mathbb{R}^d, \quad (4.116)$$

namely

$$\begin{aligned} \text{if } \hat{T} = \hat{T}_t \text{ then} \quad & d\hat{\mathbf{x}}_{\hat{e}_1} := (1, 0), \quad t \in (0, 1), \\ & d\hat{\mathbf{x}}_{\hat{e}_2} := (-1, 1), \quad t \in (0, 1), \\ & d\hat{\mathbf{x}}_{\hat{e}_3} := (0, -1), \quad t \in (0, 1), \end{aligned} \quad (4.117)$$

$$\begin{aligned} \text{if } \hat{T} = \hat{T}_q \text{ then} \quad & d\hat{\mathbf{x}}_{\hat{e}_1} := (1, 0), \quad t \in (0, 1), \\ & d\hat{\mathbf{x}}_{\hat{e}_2} := (0, 1), \quad t \in (0, 1), \\ & d\hat{\mathbf{x}}_{\hat{e}_3} := (-1, 0), \quad t \in (0, 1), \\ & d\hat{\mathbf{x}}_{\hat{e}_4} := (0, -1), \quad t \in (0, 1). \end{aligned}$$

Therefore, the physical edge  $e$  is parameterized by

$$\begin{aligned} e : \quad \mathbf{x} &= F_T(\hat{\mathbf{x}}_{\hat{e}}(t)) = (F_{T,1}(\hat{\mathbf{x}}_{\hat{e}}(t)), F_{T,2}(\hat{\mathbf{x}}_{\hat{e}}(t))) \\ &= (F_{T,1}(\hat{x}_{\hat{e},1}(t), \hat{x}_{\hat{e},2}(t)), F_{T,2}(\hat{x}_{\hat{e},1}(t), \hat{x}_{\hat{e},2}(t))), \quad t \in [0, 1]. \end{aligned} \quad (4.118)$$

The first integral from (4.112) is given by

$$\begin{aligned}
\int_e f(\mathbf{x}) \, d\sigma &= \int_0^1 f(F_T(\hat{\mathbf{x}}_\varepsilon(t))) \left( \sum_{i=1}^2 \frac{d}{dt} F_{T,i}(\hat{\mathbf{x}}_\varepsilon(t)) \right)^{1/2} dt & (4.119) \\
&= \int_0^1 f(F_T(\hat{\mathbf{x}}_\varepsilon(t))) \left( \sum_{i,j=1}^2 \frac{\partial F_{T,i}(\hat{\mathbf{x}}_\varepsilon(t))}{\partial \hat{x}_j} \frac{d}{dt} \hat{x}_{e,j}(t) \right)^{1/2} dt \\
&= \int_0^1 f(F_T(\hat{\mathbf{x}}_\varepsilon(t))) |J_{F_T}(\hat{\mathbf{x}}_\varepsilon(t))| \, d\hat{\mathbf{x}}_\varepsilon \, dt,
\end{aligned}$$

where  $J_{F_T}$  is the Jacobian matrix of the mapping  $F_T$  multiplied by the vector  $d\hat{\mathbf{x}}_e$  (= the infinitesimal increase) and  $|\cdot|$  is the Euclid norm of the vector. Let us note that if  $F_T$  is a linear mapping then  $e$  is a straight edge and  $|J_{F_T}(\hat{\mathbf{x}}_\varepsilon(t))| \, d\hat{\mathbf{x}}_\varepsilon(t)$  is equal to its length.

Now, we evaluate the second integral from (4.112). For simplicity, we restrict only to the case  $d = 2$ . Let  $\mathbf{r}_e$  be the tangential vector to  $e$  (if  $e$  is a straight line then  $\mathbf{r}_e = e$ ). Using (4.118) and (4.119), we evaluate  $\mathbf{r}_e$  at  $\mathbf{x}(t) = F_T(\hat{\mathbf{x}}_\varepsilon(t))$ ,  $t \in [0, 1]$  by

$$\begin{aligned}
\mathbf{r}_e(\mathbf{x}(t)) &= (r_1(\mathbf{x}(t)), r_2(\mathbf{x}(t))) & (4.120) \\
&:= \frac{d}{dt} F_T(\hat{\mathbf{x}}_\varepsilon(t)) = (J_{F_{T,1}}(\hat{\mathbf{x}}_\varepsilon(t)) \, d\hat{\mathbf{x}}_\varepsilon, J_{F_{T,2}}(\hat{\mathbf{x}}_\varepsilon(t)) \, d\hat{\mathbf{x}}_\varepsilon).
\end{aligned}$$

Now, be a rotation we obtain the normal vector  $\mathbf{n}_e$  oriented outside of  $T$ , namely

$$\begin{aligned}
\mathbf{n}_e(\mathbf{x}(t)) &= (n_{e,1}(\mathbf{x}(t)), n_{e,2}(\mathbf{x}(t))) : & (4.121) \\
n_{e,1}(\mathbf{x}(t)) &= r_{e,2}(\mathbf{x}(t)), \quad n_{e,2}(\mathbf{x}(t)) = -r_{e,1}(\mathbf{x}(t)),
\end{aligned}$$

where  $(t_{e,1}, t_{e,2})$  is the tangential vector to  $e$  given by (4.120). Here it is important that the counterclockwise orientation of the elements is considered. Therefore, from (4.120) and (4.121), we have

$$\mathbf{n}_e(\mathbf{x}(t)) = (J_{F_{T,2}}(\hat{\mathbf{x}}_\varepsilon(t)) \, d\hat{\mathbf{x}}_\varepsilon, -J_{F_{T,1}}(\hat{\mathbf{x}}_\varepsilon(t)) \, d\hat{\mathbf{x}}_\varepsilon). \quad (4.122)$$

Let us note that  $\mathbf{n}_e(\mathbf{x}(t))$  is not normalized, it is necessary to divide it by  $|\mathbf{n}_e(\mathbf{x}(t))| = |J_{F_T}(\hat{\mathbf{x}}_\varepsilon(t)) \, d\hat{\mathbf{x}}_\varepsilon|$ . Finally, similarly as in (4.119), we obtain

$$\begin{aligned}
\int_e \mathbf{f}(\mathbf{x}) \cdot \mathbf{n} \, d\sigma &= \int_0^1 \mathbf{f}(F_T(\hat{\mathbf{x}}_\varepsilon(t))) \cdot \frac{\mathbf{n}_e(\mathbf{x}(t))}{|\mathbf{n}_e(\mathbf{x}(t))|} |J_{F_T}(\hat{\mathbf{x}}_\varepsilon(t)) \, d\hat{\mathbf{x}}_\varepsilon| \, dt, & (4.123) \\
&= \int_0^1 \mathbf{f}(F_T(\hat{\mathbf{x}}_\varepsilon(t))) \cdot \mathbf{n}_e(\mathbf{x}(t)) \, dt,
\end{aligned}$$

where  $\mathbf{n}_e(\mathbf{x}(t))$  is given by (4.122). Let us note that if  $F_T$  is a linear mapping then  $e$  is a straight edge and  $|\mathbf{n}_e(\mathbf{x}(t))|$  is equal to its length.



### 1.4.7 Data structure

In order to evaluate the integrals (4.111), (4.119) and (4.123), it is enough to evaluate and store the following data.

(S1) for each  $T \in \mathcal{T}_h$ , the determinant of the Jacobi matrix  $\det J_{F_T}$  and the transposed matrix of the inversion of the Jacobi matrix  $J_{F_T}^T$ ,

(S2) the shape functions  $\hat{\varphi}_i$ ,  $i = 1, \dots, \hat{N}$  with the gradients  $\hat{\nabla} \hat{\varphi}_i$ ,  $i = 1, \dots, \hat{N}$  on  $\hat{T}$ .

Comparison with the data structures of FEM implementation, we see that (S1) and (S2) are in fact the same but (S3) is missing. This is caused by the fact that we have the direct connection between the basis functions and the corresponding shape functions, see relation (4.105).

## 1.5 Numerical quadratures

In the previous sections we assumed that we are able to evaluate all integrals (over the reference element or the reference edge) exactly. It is true for the model problem which is linear and therefore all integrands are polynomial functions. However, when a problem is nonlinear, it is necessary to use an approximation, namely suitable numerical quadratures. Even in case, when the problem is linear, the implementation with numerical quadratures is more simple.

There exists many numerical quadrature rules with many advantages and disadvantages. In the following we exhibit some basis quadrature rules for edge integrals and volume integrals over the reference square and over the reference triangle, which exhibit a reasonable compromise between accuracy and the implementation complexity.

### 1.5.1 Edge quadratures

We consider the *Gauss quadrature rules* given by

$$\int_0^1 f(x) dx \approx G_k(f) := \sum_{i=1}^k w_i f(x_i), \quad (5.124)$$

where  $w_i$ ,  $i = 1, \dots, k$  and  $x_i$ ,  $i = 1, \dots, k$  are the Gauss weights and the Gauss nodes, respectively. Their values can be found in many textbooks, e.g., in [3]. Table 1.4 these values for  $G_k$ ,  $k = 1, \dots, 12$ .

### 1.5.2 Quadrilaterals quadratures

In order to integrate over the reference square  $\hat{T}_q$ , we employ the *bi-Gauss quadrature rules* given by

$$\int_{\hat{T}_q} f(\mathbf{x}) d\mathbf{x} \approx G_k^{\text{bi}}(f) := \sum_{i=1}^k \sum_{j=1}^k w_i w_j f(\mathbf{x}_{ij}), \quad (5.125)$$

$G_k$	$j$	$w_j$	$x_j$	$j$	$w_j$	$x_j$
$G_1$	1	1.000000000000000	0.500000000000000			
$G_2$	1	0.500000000000000	0.21132486540519	2	0.500000000000000	0.78867513459481
$G_3$	1	0.277777777777778	0.11270166537926	2	0.444444444444444	0.500000000000000
$G_3$	3	0.277777777777778	0.88729833462074			
$G_4$	1	0.17392742256873	0.06943184420297	2	0.32607257743127	0.33000947820757
$G_4$	3	0.32607257743127	0.66999052179243	4	0.17392742256873	0.93056815579703
$G_5$	1	0.11846344252809	0.04691007703067	2	0.23931433524968	0.23076534494716
$G_5$	3	0.284444444444444	0.500000000000000	4	0.23931433524968	0.76923465505284
$G_5$	5	0.11846344252809	0.95308992296933			
$G_6$	1	0.08566224618959	0.03376524289842	2	0.18038078652407	0.16939530676687
$G_6$	3	0.23395696728635	0.38069040695840	4	0.23395696728635	0.61930959304160
$G_6$	5	0.18038078652407	0.83060469323313	6	0.08566224618959	0.96623475710158
$G_7$	1	0.06474248308443	0.02544604382862	2	0.13985269574464	0.12923440720030
$G_7$	3	0.19091502525256	0.29707742431130	4	0.20897959183673	0.500000000000000
$G_7$	5	0.19091502525256	0.70292257568870	6	0.13985269574464	0.87076559279970
$G_7$	7	0.06474248308443	0.97455395617138			
$G_8$	1	0.05061426814519	0.01985507175123	2	0.11119051722669	0.10166676129319
$G_8$	3	0.15685332293894	0.23723379504184	4	0.18134189168918	0.40828267875218
$G_8$	5	0.18134189168918	0.59171732124782	6	0.15685332293894	0.76276620495816
$G_8$	7	0.11119051722669	0.89833323870681	8	0.05061426814519	0.98014492824877
$G_9$	1	0.04063719418079	0.01591988024619	2	0.09032408034743	0.08198444633668
$G_9$	3	0.13030534820147	0.19331428364970	4	0.15617353852000	0.33787328829810
$G_9$	5	0.16511967750063	0.500000000000000	6	0.15617353852000	0.66212671170190
$G_9$	7	0.13030534820147	0.80668571635030	8	0.09032408034743	0.91801555366332
$G_9$	9	0.04063719418079	0.98408011975381			
$G_{10}$	1	0.03333567215434	0.01304673574141	2	0.07472567457529	0.06746831665551
$G_{10}$	3	0.10954318125799	0.16029521585049	4	0.13463335965500	0.28330230293538
$G_{10}$	5	0.14776211235738	0.42556283050918	6	0.14776211235738	0.57443716949082
$G_{10}$	7	0.13463335965500	0.71669769706462	8	0.10954318125799	0.83970478414951
$G_{10}$	9	0.07472567457529	0.93253168334449	10	0.03333567215434	0.98695326425859
$G_{11}$	1	0.02783428355809	0.01088567092697	2	0.06279018473245	0.05646870011595
$G_{11}$	3	0.09314510546387	0.13492399721298	4	0.11659688229600	0.24045193539659
$G_{11}$	5	0.13140227225512	0.36522842202383	6	0.13646254338895	0.500000000000000
$G_{11}$	7	0.13140227225512	0.63477157797617	8	0.11659688229600	0.75954806460341
$G_{11}$	9	0.09314510546387	0.86507600278702	10	0.06279018473245	0.94353129988405
$G_{11}$	11	0.02783428355809	0.98911432907303			
$G_{12}$	1	0.02358766819326	0.00921968287664	2	0.05346966299766	0.04794137181476
$G_{12}$	3	0.08003916427167	0.11504866290285	4	0.10158371336153	0.20634102285669
$G_{12}$	5	0.11674626826918	0.31608425050091	6	0.12457352290670	0.43738329574427
$G_{12}$	7	0.12457352290670	0.56261670425573	8	0.11674626826918	0.68391574949909
$G_{12}$	9	0.10158371336153	0.79365897714331	10	0.08003916427167	0.88495133709715
$G_{12}$	11	0.05346966299766	0.95205862818524	12	0.02358766819326	0.99078031712336

Table 1.4: Gauss quadratures  $G_k$ ,  $k = 1, \dots, 12$  with weights  $w_i$  and nodes  $x_i$

where  $w_i$ ,  $i = 1, \dots, k$  are the Gauss weights introduced in (5.124) and

$$\mathbf{x}_{ij} = (x_i, x_j), \quad i, j = 1, \dots, k, \quad (5.126)$$

where  $x_i$ ,  $i = 1, \dots, k$  are the Gauss nodes introduced in (5.124). Obviously, the bi-Gauss quadrature rules are derived by the integration with respect to the first variable and then with respect to the second one. Similarly, it is possible to derive the quadrature rule for the reference cube for  $d = 3$ .

### 1.5.3 Triangles quadratures

In order to integrate over the reference triangle  $\hat{T}_t$ , we employ the *Dunavant quadrature rules* given by

$$\int_{\hat{T}_t} f(\mathbf{x}) \, d\mathbf{x} \approx D_k(f) := \sum_{i=1}^{m_k} w_i f(\mathbf{x}_i), \quad (5.127)$$

where  $m_k$  is the number of integration nodes,  $w_i$ ,  $i = 1, \dots, k$  are the Dunavant weights and  $\mathbf{x}_i = (\lambda_{i,1}, \lambda_{i,2}, \lambda_{i,3})$ ,  $i = 1, \dots, m_k$ , are the Dunavant integration nodes. Their values can be found, e.g., in [2], Table 1.5 shows these values for  $D_k$ ,  $k = 1, \dots, 6$ .

### 1.5.4 Data structure

If the integrals appearing in FEM and/or DGM are evaluated with the aid of numerical quadratures, we should revise the data structures (S1), (S2), (S3) introduced in Sections 1.2.4 and 1.4.7. Therefore, instead of (S1) and (S2), it is enough to evaluate and store the following data.

- (D1) for each  $T \in \mathcal{T}_h$ , the determinant of the Jacobi matrix  $\det J_{F_T}$  and the transposed matrix of the inversion of the Jacobi matrix  $J_{F_T}^T$  in the *used edge and volume quadrature nodes*,
- (D2) the shape functions  $\hat{\varphi}_i$ ,  $i = 1, \dots, \hat{N}$  with the gradients  $\hat{\nabla}\hat{\varphi}_i$ ,  $i = 1, \dots, \hat{N}$  on  $\hat{T}$  in the *used edge and volume quadrature nodes*.

However, there is a natural question which quadrature rules should be employed. For linear or closely linear problem, it is sufficient to use the quadrature rules such that integrate exactly polynomials of the degrees which arise in the volume and edge integrals. For strongly nonlinear problems, more accurate quadrature rule should be used, their degrees can be found empirically.

## 1.6 Basic visualization techniques

The result of a finite element computation is a (continuous or discontinuous) piecewise polynomial function  $u_h \in V_h$ . Usually, we are not interested of its analytical expression but we need its visualization. In this Section we discuss these aspects.

$D_k$	$j$	$w_j$	$\lambda_{j,1}$	$\lambda_{j,2}$	$\lambda_{j,3}$
$D_1$	1	1.00000000000000	0.33333333333333	0.33333333333333	0.33333333333333
$D_2$	1	0.33333333333333	0.66666666666667	0.16666666666667	0.16666666666667
$D_2$	2	0.33333333333333	0.16666666666667	0.16666666666667	0.66666666666667
$D_2$	3	0.33333333333333	0.16666666666667	0.66666666666667	0.16666666666667
$D_3$	1	-0.56250000000000	0.33333333333333	0.33333333333333	0.33333333333333
$D_3$	2	0.52083333333333	0.60000000000000	0.20000000000000	0.20000000000000
$D_3$	3	0.52083333333333	0.20000000000000	0.20000000000000	0.60000000000000
$D_3$	4	0.52083333333333	0.20000000000000	0.60000000000000	0.20000000000000
$D_4$	1	0.22338158967801	0.10810301816807	0.44594849091597	0.44594849091597
$D_4$	2	0.22338158967801	0.44594849091597	0.44594849091597	0.10810301816807
$D_4$	3	0.22338158967801	0.44594849091597	0.10810301816807	0.44594849091597
$D_4$	4	0.10995174365532	0.81684757298046	0.09157621350977	0.09157621350977
$D_4$	5	0.10995174365532	0.09157621350977	0.09157621350977	0.81684757298046
$D_4$	6	0.10995174365532	0.09157621350977	0.81684757298046	0.09157621350977
$D_5$	1	0.22500000000000	0.33333333333333	0.33333333333333	0.33333333333333
$D_5$	2	0.13239415278851	0.05971587178977	0.47014206410511	0.47014206410511
$D_5$	3	0.13239415278851	0.47014206410511	0.47014206410511	0.05971587178977
$D_5$	4	0.13239415278851	0.47014206410511	0.05971587178977	0.47014206410511
$D_5$	5	0.12593918054483	0.79742698535309	0.10128650732346	0.10128650732346
$D_5$	6	0.12593918054483	0.10128650732346	0.10128650732346	0.79742698535309
$D_5$	7	0.12593918054483	0.10128650732346	0.79742698535309	0.10128650732346
$D_6$	1	0.11678627572638	0.50142650965818	0.24928674517091	0.24928674517091
$D_6$	2	0.11678627572638	0.24928674517091	0.24928674517091	0.50142650965818
$D_6$	3	0.11678627572638	0.24928674517091	0.50142650965818	0.24928674517091
$D_6$	4	0.05084490637021	0.87382197101700	0.06308901449150	0.06308901449150
$D_6$	5	0.05084490637021	0.06308901449150	0.06308901449150	0.87382197101700
$D_6$	6	0.05084490637021	0.06308901449150	0.87382197101700	0.06308901449150
$D_6$	7	0.08285107561837	0.05314504984482	0.31035245103378	0.63650249912140
$D_6$	8	0.08285107561837	0.31035245103378	0.63650249912140	0.05314504984482
$D_6$	9	0.08285107561837	0.63650249912140	0.05314504984482	0.31035245103378
$D_6$	10	0.08285107561837	0.31035245103378	0.05314504984482	0.63650249912140
$D_6$	11	0.08285107561837	0.63650249912140	0.31035245103378	0.05314504984482
$D_6$	12	0.08285107561837	0.05314504984482	0.63650249912140	0.31035245103378
$D_7$	1	-0.14957004446768	0.33333333333333	0.33333333333333	0.33333333333333
$D_7$	2	0.17561525743321	0.47930806784192	0.26034596607904	0.26034596607904
$D_7$	3	0.17561525743321	0.26034596607904	0.26034596607904	0.47930806784192
$D_7$	4	0.17561525743321	0.26034596607904	0.47930806784192	0.26034596607904
$D_7$	5	0.05334723560884	0.86973979419557	0.06513010290222	0.06513010290222
$D_7$	6	0.05334723560884	0.06513010290222	0.06513010290222	0.86973979419557
$D_7$	7	0.05334723560884	0.06513010290222	0.86973979419557	0.06513010290222
$D_7$	8	0.07711376089026	0.04869031542532	0.31286549600487	0.63844418856981
$D_7$	9	0.07711376089026	0.31286549600487	0.63844418856981	0.04869031542532
$D_7$	10	0.07711376089026	0.63844418856981	0.04869031542532	0.31286549600487
$D_7$	11	0.07711376089026	0.31286549600487	0.04869031542532	0.63844418856981
$D_7$	12	0.07711376089026	0.63844418856981	0.31286549600487	0.04869031542532
$D_7$	13	0.07711376089026	0.04869031542532	0.63844418856981	0.31286549600487

Table 1.5: Dunavant quadratures  $D_k$ ,  $k = 1, \dots, 6$  with weights  $w_i$  and nodes  $x_i$

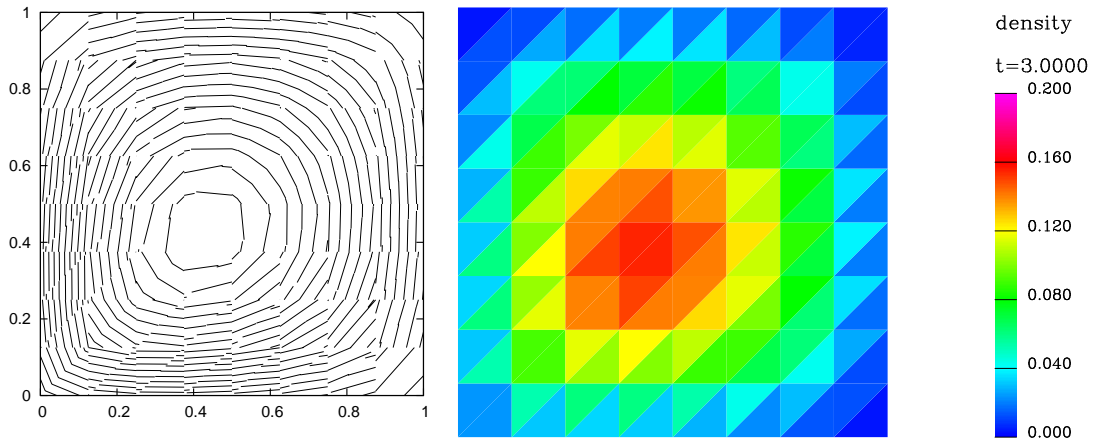


Figure 1.10: The basis graphical outputs: isolines (left) and a color map (right)

### 1.6.1 Types of visualization

There are several usual graphical outputs, which can be required based on the considered problem. Namely

- *isolines/isosurfaces of the solutions* are the lines (for  $d = 2$ ) or surfaces (for  $d = 3$ ) consisting of points of the computational domains where the approximate solution equals to a given value, i.e., the set  $\{\mathbf{x} \in \Omega; u_h(\mathbf{x}) = g\}$  is the isoline (isosurface) corresponding to the value  $g$ . Figure 1.10, left shows the isolines of the piecewise linear approximation of the function

$$u(x_1, x_2) = 2(x_1^2 + x_2^2)^{-1/4} x_1 x_2 (1 - x_1)(1 - x_2), \quad (x_1, x_2) \in (0, 1) \times (0, 1) \quad (6.128)$$

for the values  $g = 0.01n$ ,  $n = 0, 1, \dots, 18$ .

- *color maps of the solutions* is the visualization technique where each element of the mesh is drawn by a color which corresponds to a given value. Figure 1.10, right shows the color map of the function (6.128). Let us note that color maps very often smear details of approximate solution.
- *3D plot of the solution* (only for  $d = 2$ ) are the graph of the surface

$$\{(x_1, x_2, x_3); x_3 = u_h(x_1, x_2), (x_1, x_2) \in \Omega\}.$$

Figure 1.11, left shows the 3D plot function (6.128). This graphical output contains the whole information of the approximate solution. On the other hand if the approximate solution is very complex this type of output is hardly to see.

- *boundaries visualization* are important, e.g., in aerodynamics, when we need to know a pressure distribution on the airplane surface.
- *cuts of the solution* help to see the solution inside of the computational domain, Figure 1.11, right shows the diagonal cut from the left-below to right-up of the function (6.128).

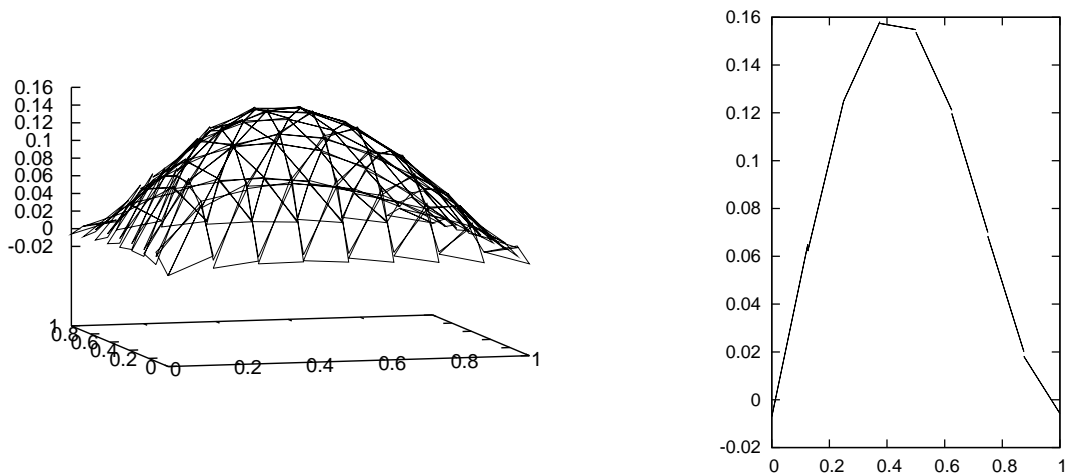


Figure 1.11: The basis graphical outputs: a 3D plot (left) and a diagonal cut (right)

## 1.6.2 Software for visualization

Within this section we briefly describe types of software which can be used for the visualization of finite element solutions. Our aim is not to give a complete list of all available software but give to the readers some overview about possibilities.

We suppose that the software for the visualization can be split into the following groups.

- *Commercial graphical software* usually represent a complete tool for visualization. Let us mention as an example the code `Techplot` which is able to draw a continuous piecewise linear or piecewise constant approximate solution for  $d = 2$  as well as  $d = 3$  meshes. It allows a wide range of graphical tools, e.g., slices, isosurfaces, cuts, animations etc. However, `Techplot` uses some postprocessing when smooths the given approximate solution. This is advantageous for effect of visualization but it brings some additional perturbation and/or “errors”. A use of the commercial software does not require any pre-computing, in generally. However, most of the commercial software do not allow a visualization of a higher order approximation. In this case, some additional technique has to be used, see Section 1.6.3.
- *Free graphical libraries* represented, e.g., by `PGPLOT`, which is a set of Fortran 77 subroutines which are able to draw a given line, to fill a given polygon by a give color, etc. Therefore, a use of this library requires a creation of an additional code calling the `PGPLOT` subroutines. This represents an additional work but you can draw exactly what you want. I.e., no postprocessing perturbs the results. Figure 1.10, right was obtained by `PGPLOT`.
- *Basic graphical code* as, e.g., `gnuplot` are able to carried out basis graphical task as connecting a given set of nodes. This type of software can be used with

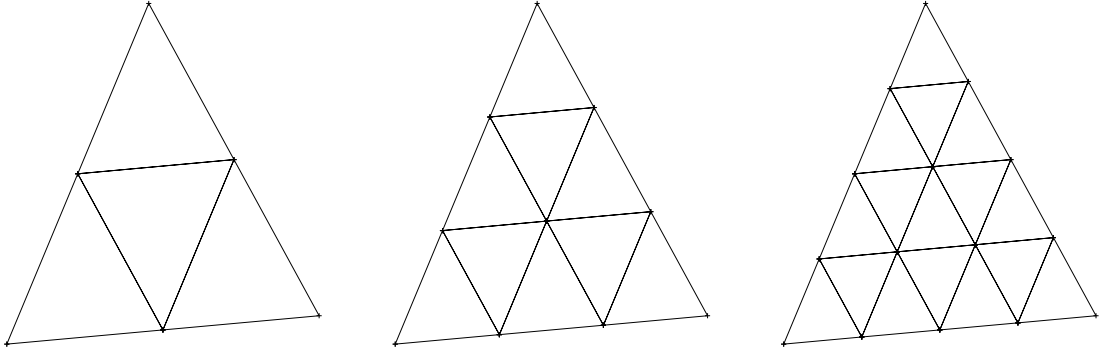


Figure 1.12: The sub-grids for the visualization on a triangle  $T$  for  $p = 2$  (left),  $p = 3$  (center) and  $p = 4$  (right)

success for wall distribution, cuts and isolines. However, a visualization of isolines on unstructured grids requires their pre-computation by some additional code. Figures 1.10, left and 1.11 were created by `gnuplot`.

### 1.6.3 Visualization of higher order polynomial function

In this section we discuss some possibilities how to visualize a higher order piecewise polynomial solution  $u_h$  of an abstract problem. These *visualization techniques* are in fact independent, if  $u_h \in V_h$  is a continuous or  $u_h \in S_{hp}$  is discontinuous.

In order to visualize the any piecewise polynomial function, we need to evaluate its value in given nodes from  $\Omega$ . If we use the data structure introduced in Section 1.5.4, we can simply evaluate the solution in integrations nodes. However, it is not usually enough since the Gauss and the Dunavant quadrature nodes do not lay at end points of  $(0, 1)$  or at the boundary of the reference triangle, respectively. Therefore, for the purposes of the visualization it is suitable to evaluate the solution at the Lagrangian nodes introduced in Section 1.3.1. This means, the we have to evaluate and store also

(D2a) the shape functions  $\hat{\varphi}_i$ ,  $i = 1, \dots, \hat{N}$  on  $\hat{T}$  in the nodes used for a visualization.

All graphical codes are able to draw a line between two nodes. Therefore, it is possible to visualize without problems a piecewise linear approximate solution. A visualization of higher order approximation is more complicated. It is possible to calculate exactly the corresponding solution dot per dot, but it is usually very time consuming.

More efficient and sufficiently accurate approach is the following. Let  $T \in \mathcal{T}_h$  be an element and  $u_h|_T$  a polynomial function of degree  $p$ . Then we split  $T$  at several sub-elements by connecting the Lagrangian nodes  $F_T(\hat{\mathbf{z}}_i^p)$ ,  $i = 1, \dots, \hat{N}^p$ , see Figure 1.12. We evaluate  $u_h|_T$  at each of the Lagrangian nodes  $F_T(\hat{\mathbf{z}}_i^p)$ ,  $i = 1, \dots, \hat{N}^p$  and carried out a piecewise linear visualization over this sub-grid.

Figure 1.13 shows the exact solution of a given problem which is visualized by the isolines and the 3D plot. This problem was solved on a mesh consisting of four triangles

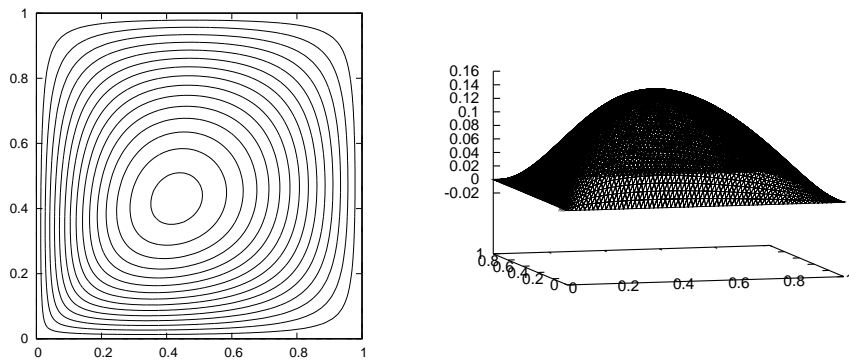


Figure 1.13: The visualization of the given function, isolines (left) and 3D plot (right)

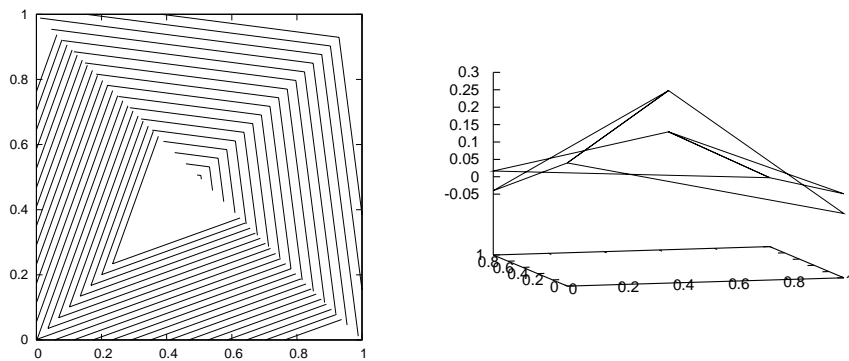


Figure 1.14: Visualization of the  $P_1$  approximation, isolines (left) and 3D plot (right)

(arising by splitting the square by its diagonals) using  $P_k$ ,  $k = 1, \dots, 7$  approximations. Figures 1.14 – 1.20 show these approximate solutions using the visualization on the corresponding subgrids.



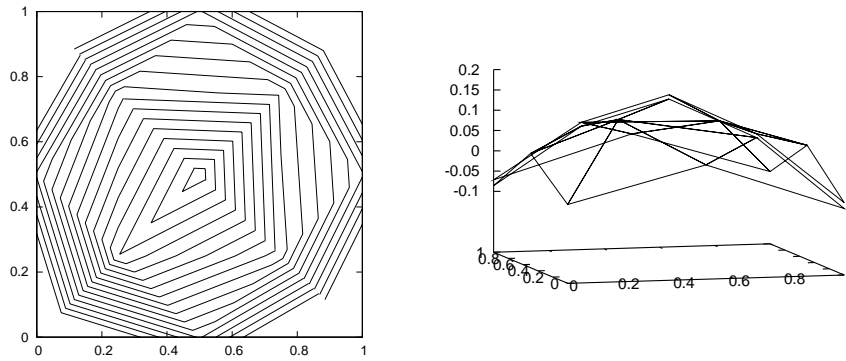


Figure 1.15: Visualization of the  $P_2$  approximation, isolines (left) and 3D plot (right)

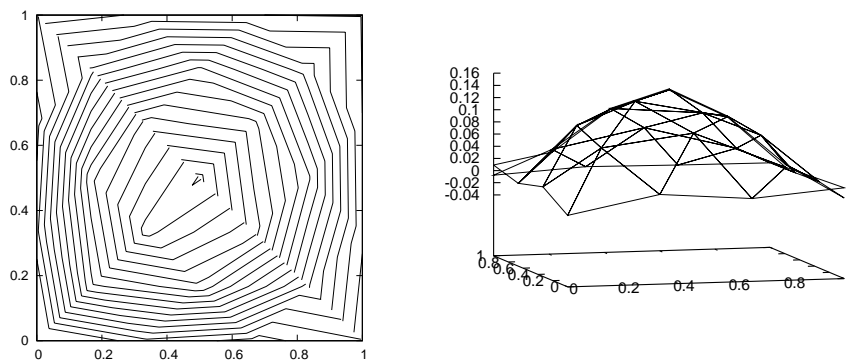


Figure 1.16: Visualization of the  $P_3$  approximation, isolines (left) and 3D plot (right)

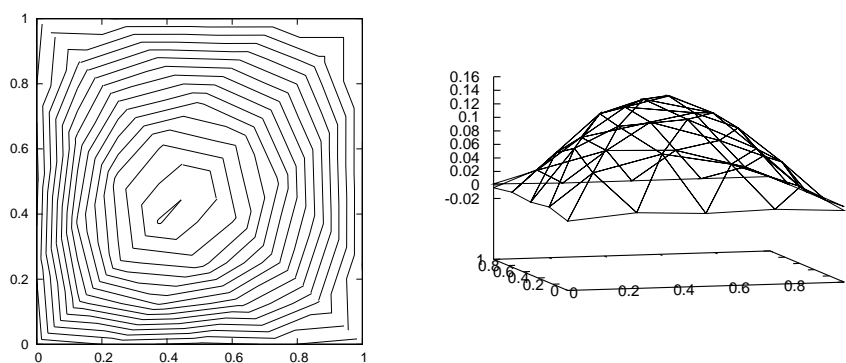


Figure 1.17: Visualization of the  $P_4$  approximation, isolines (left) and 3D plot (right)

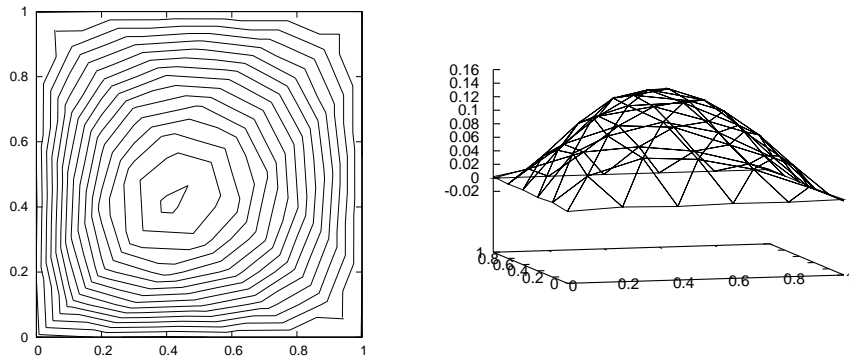


Figure 1.18: Visualization of the  $P_5$  approximation, isolines (left) and 3D plot (right)

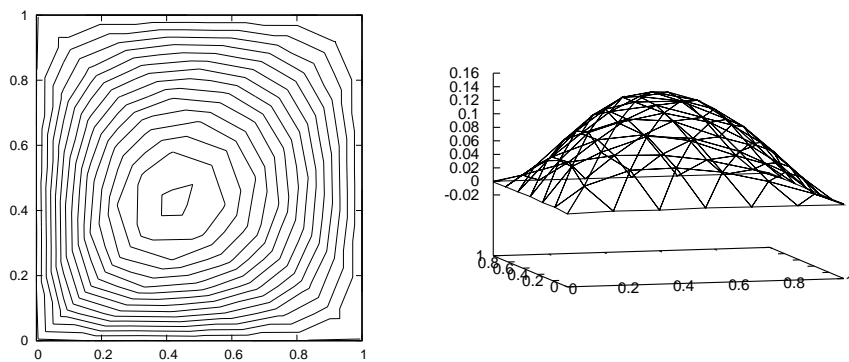


Figure 1.19: Visualization of the  $P_6$  approximation, isolines (left) and 3D plot (right)

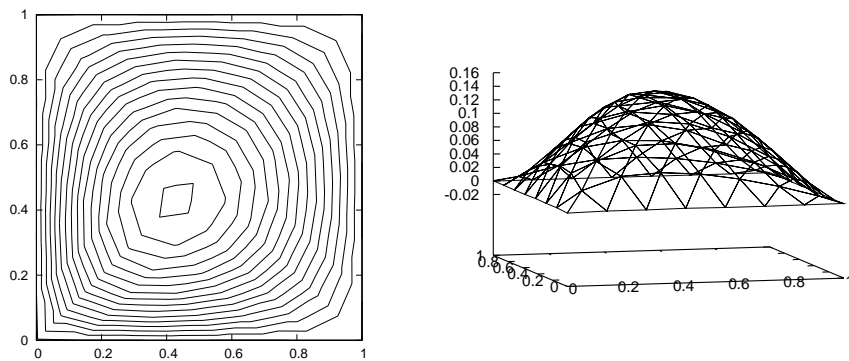


Figure 1.20: Visualization of the  $P_7$  approximation, isolines (left) and 3D plot (right)

# Bibliography

- [1] L. Dubcová. *hp-FEM for coupled problems in fluid mechanics*. PhD thesis, Charles University Prague, 2010.
- [2] D. A. Dunavant. High degree efficient symmetrical gaussian quadrature rules for the triangle. *Int. J. Numer. Methods Eng.*, 21:1129–1148, 1985.
- [3] P. Šolín, K. Segeth, and I. Doležel. *Higher-Order Finite Element Methods*. Chapman & Hall/CRC Press, 2003.