Task 2.5: Analysis of Degenerate Parabolic Problems

GA ČR Progress meeting

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Presentation Outline

1. Richards' Equation
2. Numerical Solution
3. Summary & Outlook

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Richards' Equation

$$\frac{\partial \vartheta(\Psi - z)}{\partial t} - \nabla \cdot (\mathbf{K}(\Psi - z)\nabla \Psi) = 0 \qquad (1)$$

- $\Psi = \psi + z$,
- Ψ – hydraulic head
- ψ – pressure head
- z – geodetic head
- $\boldsymbol{K}$ – hydraulic conductivity

- $\vartheta(\psi) = \theta(\psi) + \frac{S_s}{\theta_S} \int_{-\infty}^{\psi} \theta(s) \mathrm{d}s.$
- ϑ – active pure volume
- θ – water content
- θ_S – saturated water content
- S_S – specific aquific storage

Degeneracy of Parabolic PDE

$$\frac{\partial \vartheta(\Psi - z)}{\partial t} - \nabla \cdot (\mathbf{K}(\Psi - z)\nabla\Psi) = 0 \tag{2}$$

where

$$\vartheta(\psi) = \theta(\psi) + \frac{S_s}{\theta_S}\int_{-\infty}^{\psi} \theta(s)\mathrm{d}s. \tag{3}$$

Degeneracy of Parabolic PDE

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- fast-diffusion type of degeneracy

$$S_s = 0 \quad \& \quad \psi > 0 (\implies \theta'(\psi) = 0)$$

Degeneracy of Parabolic PDE

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- fast-diffusion type of degeneracy

$$S_s = 0 \quad \& \quad \psi > 0(\implies \theta'(\psi) = 0)$$

- slow-diffusion type of degeneracy
  - $\vartheta'(\psi) \to \infty$ when $\psi \to 0$
  - $\mathbf{K}(\psi) \to 0$ when $\psi \to -\infty$

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Initial Boundary Value Problem

Find $\Psi = \Psi(x, t) : \Omega \times (0, T) \to \mathbb{R}$ such that

$$\frac{\partial \vartheta(\Psi - z)}{\partial t} - \nabla \cdot (\mathbf{K}(\Psi - z)\nabla\Psi) = 0 \qquad \text{in } \Omega \times (0, T), \tag{4a}$$

$$\Psi = \Psi_D \qquad \text{on } \Gamma_D \times (0, T), \tag{4b}$$

$$\mathbf{K}(\Psi)\nabla\Psi \cdot \boldsymbol{n} = \boldsymbol{q}_N \qquad \text{on } \Gamma_N \times (0, T), \tag{4c}$$

$$\left.\begin{array}{r} \nabla\Psi \cdot \boldsymbol{n} = 0 \text{ if } \Psi \le 0 \\ \Psi = 0 \text{ if } \nabla\Psi \cdot \boldsymbol{n} \le 0 \end{array}\right\} \qquad \text{on } \Gamma_E \times (0, T), \tag{4d}$$

$$\Psi(x, t = 0) = \Psi_0 \qquad \text{in } \Omega. \tag{4e}$$

Higher order STDG solution (Dolejší et. al. 2019)

hp-STDG scheme

$$\int_{I_m} \left((\partial_t \vartheta(\Psi - z), \varphi) + A_{h,m}(\Psi, \varphi) \right) \mathrm{d}t + \left(\{\vartheta(\Psi - z)\}_{m-1}, \varphi|_{m-1}^{+} \right) = 0 \tag{5}$$

for all $\varphi \in S_{h,\mathbf{p}}^{\tau,q}$, $m = 1, \ldots, r$ with $\Psi_{h\tau}|_0^- := \Psi_0$

- $\Psi_{h\tau} \in S_{h,\mathbf{p}}^{\tau,q}$ an approximate solution of (4)
- $S_{h,\mathbf{p}}^{\tau,q}$ space-time polynomials with varying degree $\boldsymbol{p}$ in space

A priori Analysis

GOAL

Derive an a priori error estimate – an estimate in terms of degrees of freedom

$$\|\Psi - \Psi_{h\tau}\| \leq C(\Psi)(h^{\alpha} + \tau^{\beta}).$$

- we obtain the convergence as a consequence ($\lim_{\mathrm{dof}\to\infty} \|\Psi - \Psi_{h\tau}\| = 0, \quad \mathrm{dof} = \dim(S_{h,\mathbf{p}}^{\tau,q})$)
- [Dolejší, Feistauer 2015] a priori error estimates for STDG method in case of a convection-diffusion equation with nonlinear convection and nonlinear nonvanishing bounded from above diffusion

Space Semi–Discrete Analysis

Space semi–discrete scheme

$$(\partial_t \vartheta(\Psi), \varphi)_\Omega + A_{h,m}(\Psi, \varphi) = 0, \quad \forall \varphi \in S_{h,p} \tag{6}$$

$$A_{h,m}(\Psi,\varphi) := \sum_{K\in\mathscr{T}_{h,m}} (\mathbf{K}(\Psi - z)\nabla\Psi, \nabla\varphi)_K - \sum_{\gamma\in\Gamma^D_{h,m}} (\langle \mathbf{K}(\Psi - z)\nabla\Psi\rangle \cdot \boldsymbol{n}, [\varphi])_\gamma + \sum_{\gamma\in\Gamma^D_{h,m}} (\sigma[\varphi], [\Psi])_\gamma$$

- Standard assumptions for Richards' equation
  - ϑ monotone nondecreasing, Lipschitz continuous with possibly $\vartheta'(\Psi) \to 0$
  - **K** uniformly bounded, SPD & Lipschitz continuous

Space Semi–Discrete Analysis - Solution Strategy

- Extend FEM approach from [Woodward, Dawson 2000; Arbogast 1990] to DG method allowing $\vartheta' \to 0$
  - Assume $(\partial_u \vartheta) \circ \vartheta^{-1}$ Hölder continuous with order β
  - Two test functions from $S_{h,p}$
  - $\varphi_1 = \int_t^{\bar{t}} \xi e^{Q\tau} \mathrm{d}\tau$ (mimics Gronwall argument) & $\varphi_2 = \xi$
  - Continuous induction approach
  - Estimates in terms of $\|\vartheta(\Psi) - \vartheta(\Psi_h)\|_{L^\infty(L^2)}$, $\|\Psi - \Psi_h\|_{L^2(H^1)}$ & $\int_0^T (\vartheta(\Psi) - \vartheta(\Psi_h), \Psi - \Psi_h)\,\mathrm{d}t$
- However, presence of interior and penalty terms in elliptic part of (6) aggravates the analysis
  - Modify "$\mathbf{K}(\nabla\Psi)\Psi$" by "$\mathbf{K}(\Psi, \nabla\Psi)$" [Dolejší 2008] or "$\mathbf{K}(\theta(\Psi)), \nabla\Psi)$"
  - Estimates in terms of DG-norm

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Outlook

- Complete analysis of the semidiscrete DG scheme & perform numerical experiments
- Analysis of fully discrete STDG scheme
- Analysis of the *hp* variant of STDG method

Thank you!