Anisotropic goal-oriented error estimates for nonlinear problems

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Second progress meeting

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Nonlinear problem

Numerical solution

- nonlinear PDE $a(u, \phi) = 0 \ \forall \phi \in V$
- numerical solution $a_h(u_h, \phi_h) = 0 \ \forall \phi_h \in V_h$
- iterative solver
  $u_h^{n+1} = u_h^n + d_h^n, \quad a_h^{\mathrm{L}}(u_h^n; d_h^n, \phi_h) = -a_h(u_h^n, \phi_h) \ \forall \phi_h \in V_h$

Quantity of interest: Function $J(u) \in \mathbb{R}$

- goal to estimate $J(u) - J(u_h^n)$
- adjoint problem: $a_h'(u_h^n, \psi, z) = J'(u_h^n, \psi) \ \forall \psi \in V$
- $J(u - u_h^n) \approx \frac{1}{2}(r_h(u_h^n)(z - \Pi z) + r_h^*(z_h^n)(u - \Pi u))$

Approach based on linearization

- Idea: employ the information available from the iterative solver
- adjoint problem $a_h^{\mathrm{L}}(u_h^n, \psi, z) = J_h^{\mathrm{L}}(u_h^n, \psi) \ \forall \psi \in V$

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$$a_h^{\mathrm{L}}(u_h^n, \psi_h, z_h) = J_h^{\mathrm{L}}(u_h^n, \psi_h) \ \forall \psi_h \in V_h$$

Requirements

- let u, z be the weak solutions of primal and adjoint problems
- primal consistency: $a_h(u, \phi_h) = 0 \ \forall \phi_h \in V_h$
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Goal-oriented error estimates

Primal problem

- continuous primal problem $a(u, \phi) = 0 \ \forall \phi \in V$
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- continuous adjoint problem $a^{\mathrm{L}}(u, \psi, z) = J^{\mathrm{L}}(u, \psi) \ \forall \psi \in V$
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- iterative solver $u_h^{n+1} = u_h^n + d_h^n, \quad \mathbb{A}(u_h^n) d_h^n = f(u_h^n)$

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- systems are solved at once by BiCG [D., Tichy, JSC 2020]
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$u_h^+ := \mathscr{R}(u_h^n), \quad z_h^+ := \mathscr{R}(z_h^n),$

$$J(u - u_h^n) \approx \tfrac{1}{2} r_h(u_h^n)(z_h^+ - \Pi z_h^+) + \tfrac{1}{2} r_h^*(z_h^n)(u_h^+ - \Pi u_h^+) + r_h(u_h^n)(z_h^n)$$

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Anisotropic goal-oriented error estimates including

Error estimates of type I

$J(u - u_h^n) \approx \eta = \eta^{\mathrm{I}} + \eta_{\mathrm{A}},$

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Localization of the error estimates of type I

$\eta^{\mathrm{I}} = \sum_{K \in \mathscr{T}_h} \eta_K^{\mathrm{I}},$

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Error estimates of type II

$\eta_K^{\mathrm{I}} \leq \sum_\star R_\star(u_h) \left\| z_h^+ - \Pi z_h^+ \right\|_\star + R_\star^*(z_h) \left\| u_h^+ - \Pi u_h^+ \right\|_\star, \; K \in \mathscr{T}_h$

Interpolation error estimates [D. May, Birghäuser, 2022]

$\left\| w - \Pi w \right\|_\star \leq G_\star(D^{p+1} w, \mu_K, \sigma_K, \phi_K)$

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$J(u - u_h^n) \approx \eta = \eta^{\mathrm{I}} + \eta_{\mathrm{A}},$

$\eta^{\mathrm{I}} := \frac{1}{2} r_h(u_h^n)(z_h^+ - \Pi z_h^+) + \frac{1}{2} r_h^*(z_h^n)(u_h^+ - \Pi u_h^+),$

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Localization of the error estimates of type I

$\eta^{\mathrm{I}} = \sum_{K \in \mathscr{T}_h} \eta_K^{\mathrm{I}},$

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Error estimates of type II

$\eta_K^{\mathrm{I}} \leq \sum_\star R_\star(u_h) \left\| z_h^+ - \Pi z_h^+ \right\|_\star + R_\star^*(z_h) \left\| u_h^+ - \Pi u_h^+ \right\|_\star, \quad K \in \mathscr{T}_h$

Interpolation error estimates [D. May, Birghäuser, 2022]

$\left\| w - \Pi w \right\|_\star \leq G_\star(D^{p+1} w, \mu_K, \sigma_K, \phi_K)$

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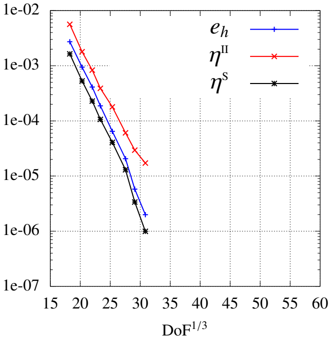
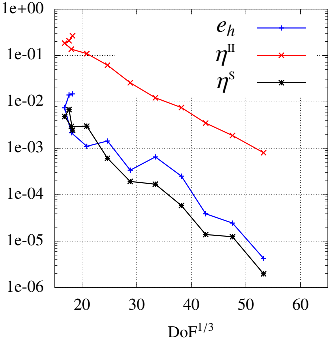
Examples: NACA 0012 profile, $M = 0.5$, $\alpha = 1.25^\circ$

convergence of the error and estimators



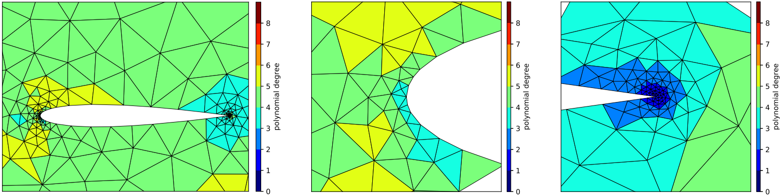
$J =$ drag coefficient

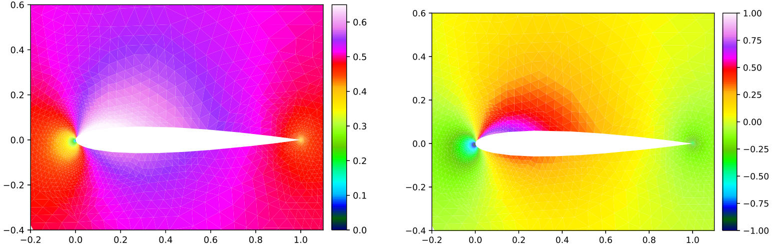


$J =$ lift coefficient

Examples: NACA 0012 profile – drag coefficient

hp-mesh





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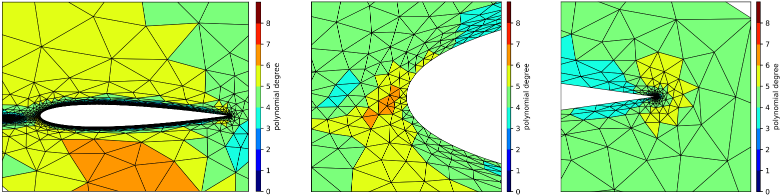
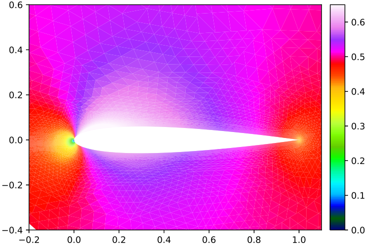


Mach number



adjoint solution