Numerical Solution of Degenerate Parabolic Problems

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Richards Equation

We consider the Richards equation for modelling flows in variably-saturated porous medium, based on the law of mass conservation with Darcy-Buckingham for fluxes:

$$C(\psi)\frac{\partial\Psi}{\partial t} - \nabla \cdot (\boldsymbol{K}(\psi)\nabla\Psi) = S.$$

- Ψ — Hydraulic head; ψ — Pressure head ($\Psi = \psi + z$)
- $K(\psi)$ — Unsaturated hydraulic conductivity
- $C(\psi)$ — Water retention capacity; usually:

  $$C(\psi) = \frac{d\theta(\psi)}{d\psi} + \frac{\theta(\psi)}{\theta_s}S_s,$$

  where $\theta(\psi)$ is the water content function, S_s is the specific aquifer storage, and θ_s is the saturated water content.
- S — Source (positive) or sink (negative) term.

Boundary Conditions

Three types of boundary conditions:

- Dirichlet
- Neumann
- Seepage face (or atmospheric condition/outflow BC) — interface between porous medium and atmosphere (free flow domain):

$$\begin{aligned} \psi := \Psi - z &\leq 0 \quad \text{(pressure head cannot be positive)} \\ -\boldsymbol{K}(\Psi - z)\nabla\Psi \cdot \boldsymbol{n} &\geq 0 \quad \text{(fluid cannot enter medium)} \\ \psi(\nabla\Psi \cdot \boldsymbol{n}) &= 0 \quad \text{(fluid only exit if pressure head } \psi = 0\text{)} \end{aligned}$$

Can be treated as Signorini-type BC:

$$\begin{aligned} \nabla\Psi \cdot \boldsymbol{n} = 0 \text{ if } \psi < 0 &\quad \text{(unsaturated — no-flow Neumann)} \\ \psi = 0 \text{ if } \nabla\Psi \cdot \boldsymbol{n} < 0 &\quad \text{(saturated — zero pressure head Dirichlet)} \end{aligned}$$

Degeneracy

Richards equation is a quasilinear degenerate parabolic equation; namely,

- $C(\psi)$ vanishes for $S_s = 0$ and fully saturated medium ($\psi \geq 0$) — degenerates to elliptic problem
- $C(\psi)$ can 'blow-up' near the saturated/unsaturated transition ($\psi = 0$)
- $C(\psi)$ and $\boldsymbol{K}(\psi)$ go to zero as ψ goes to $-\infty$

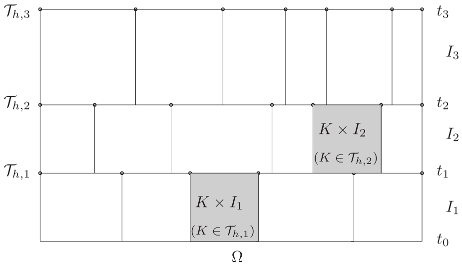
From definition of $C(\psi)$, $S=0$, and $\vartheta(\psi) := \theta(\psi) + \frac{S_s}{\theta_s}\int_{-\infty}^{\psi}\theta(s)\,ds$:

$$
\begin{aligned}
\frac{\partial \vartheta(\Psi - z)}{\partial t} - \nabla\cdot(\boldsymbol{K}(\Psi - z)\nabla\Psi) &= 0 && \text{in } \Omega\times(0,T),\\
\Psi &= \Psi_D && \text{on } \Gamma_D\times(0,T),\\
\boldsymbol{K}(\Psi)\nabla\Psi\cdot\boldsymbol{n} &= \boldsymbol{q}_N && \text{on } \Gamma_N\times(0,T),\\
\left.\begin{array}{lll}\nabla\Psi\cdot\boldsymbol{n} & = 0 & \text{if } \Psi\le 0\\ \Psi & = 0 & \text{if } \nabla\Psi\cdot\boldsymbol{n}\le 0\end{array}\right\} & && \text{on } \Gamma_E\times(0,T),\\
\Psi(x,0) &= \Psi_0 && \text{in } \Omega.
\end{aligned}
$$

- $\vartheta : \mathbb{R}\to\mathbb{R}$ is Hölder continuous and nondecreasing, with non-negative derivative; if $S_s = 0$ and $\psi > 0$ then $\vartheta'(\psi) = 0$ and the equation degenerates (fast-diffusion type degeneracy).
- For particular choice of material parameters $\vartheta'(\psi)\to\infty$ as $\psi\to 0$ (slow-diffusion type degeneracy).
- $\boldsymbol{K} : \mathbb{R}\to\mathbb{R}^{2\times 2}$, positive, Lipschitz-continuous and non-decreasing and can vanish, typically $\boldsymbol{K}(\psi)\to 0$ as $\psi\to-\infty$ (slow-diffusion deg.).

[Dolejšì, Kuraz, Solin 2019]

- Partition time $0 = t_0 < t_1 < \cdots < t_r = T$, with intervals $I_m = (t_{m-1}, t_m)$, $m = 1, \ldots, r$.
- For each time interval I_m, $m = 1, \ldots, r$ consider a different space partition $\mathcal{T}_{h,m}$ of simplices.
- Assign a fixed polynomial degree $q \in \mathbb{N}$ w.r.t time, and varying polymomial degree p_K, $K \in \mathcal{T}_{h,m}$, in space.



DGFEM can naturally handle the seepage BC by a function σ^*:

$$a_{h,m}(\Psi,\psi) := \sum_{K\in\mathcal{T}_{h,m}} (\boldsymbol{K}(\Psi - z)\nabla\Psi, \nabla\psi)_K - \sum_{\gamma\in\Gamma_{h,m}^{D,I}} (\langle\boldsymbol{K}(\Psi - z)\nabla\Psi\rangle\cdot\boldsymbol{n}, [\psi])_\gamma$$
$$-\sum_{\gamma_{h,m}^E} (\sigma^*\boldsymbol{K}(\Psi - z)\nabla\Psi\cdot\boldsymbol{n}, \psi)_\gamma + \sum_{\gamma\in\Gamma_{h,m}^I} (\sigma[\psi], [\Psi])_\gamma$$
$$+ \sum_{\gamma\in\Gamma_{h,m}^D} (\sigma\psi, \Psi - \Psi_D)_\gamma + \sum_{\gamma_{h,m}^E} (\sigma^*\sigma\psi, \Psi - z)_\gamma - (\boldsymbol{q}_n, \psi)_{\Gamma_N}$$

$\sigma^* = 1$, Dirichlet BC is prescribed; $\sigma^* = 0$ the Neumann BC is prescribed.

STDG Formulation

Find $\Psi \in S_{h,p}^{\tau,q}$ such that

$$A_{h,m}(\Psi,\psi) = 0 \qquad \forall \psi \in S_{h,p}^{\tau,q}$$

where

$$A_{h,m}(\Psi,\psi) := \int_{I_m} \left(-\left(\vartheta(\Psi - z), \partial_t \psi\right)_\Omega + a_{h,m}(\Psi,\psi)\right) dt + \left(\vartheta(\Psi - z)|_m^-, \psi|_m^-\right)_\Omega - \left(\vartheta(\Psi - z)|_{m-1}^-, \psi|_{m-1}^+\right)_\Omega$$

Outstanding issues/tasks:

- Unique/existence for linear case exists [Dolejšì, Feistauer 2015], straighforward for Lipschitz continuous, bounded and strictly positive ϑ' and $\boldsymbol{K}$. Unique and existence for general case open.
- Analysis in the degenerate cases
- *A priori* error analysis of the method
- *A posteriori* error analysis to lead to optimal *hp*-mesh

Nonlinear Solver

- We solve a strongly nonlinear algebraic system.
- Newton methods often fail for parabolic degenerate problems, since the Jacobian becomes singular.
- Evaluation of derivative of conductivity term can be avoidided by a modified Picard method — better convergence w.r.t temporal discretization, but nonlinear operator may still not converge.
- Can use L-scheme [Soldicka 2002; Pop, Radu, Knabner 2004] — modified Picard with derivative ϑ' replaced with constant $L \geq \max_\psi |\vartheta'(\psi)|$.
- Can adapt scheme from [Dolejší, Holik, Hozman 2011] (for Navier-Stokes) to linearise the system.

For solving the resulting system with reasonable convergence rates [Dolejší, Kuraz, Solin 2019] proposes two methods: a Newton-like method and a Anderson acceleration.

Prove of convergence of these schemes would be an aim of this topic.