DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS
Proc. Conf. Prague, August 30 – September 3, 2004
Charles University, Prague (Czech Republic), 2005, 591 – 604

Second variational derivative of gauge-natural invariant Lagrangians and conservation laws

M. Francaviglia, M. Palese and E. Winterroth

Abstract. We consider the second variational derivative of a given gauge-natural invariant Lagrangian taken with respect to (prolongations of) vertical parts of gauge-natural lifts of infinitesimal principal automorphisms. By requiring such a second variational derivative to vanish, *via* the Second Noether Theorem we find that a covariant strongly conserved current is canonically associated with the deformed Lagrangian obtained by contracting Euler–Lagrange equations of the original Lagrangian with (prolongations of) vertical parts of gauge-natural lifts of infinitesimal principal automorphisms lying in the kernel of the generalized gauge-natural Jacobi morphism.

1. Introduction

After the works by Bergmann (see *e.g.* [**1**] and references quoted therein) the general problem has been tackled of coherently defining the lifting of infinitesimal transformations of the basis manifold to bundles of fields (namely bundles of tensors or tensor densities which could be obtained as suitable representations of the action of infinitesimal space-time transformations on frame bundles of a given order [**26**]). Such theories were also called geometric or *natural* [**33**]. A physically important generalization of natural theories to gauge field theories passed through the concept of jet prolongation of a principal bundle and the introduction of a very important geometric construction, namely the *gauge-natural* bundle functor [**4, 19**].

We recall that within the above mentioned program generalized Bianchi identities for geometric field theories were introduced by Bergmann to get (after an integration by parts procedure) a consistent equation involving divergences within the first variation formula. It was also stressed that in the general theory of Relativity those identities coincide with the contracted Bianchi identities for the curvature tensor of the pseudo-Riemannian metric. Our present aim is to suitably generalize this setting to the gauge-natural framework.

Our general setting is the calculus of variations on finite order fibered bundles which will be assumed to be *gauge-natural bundles* (*i.e.* jet prolongations of fibered bundles associated to some gauge-natural prolongation of a principal bundle $\boldsymbol{P}$ [**4, 19**]). Such geometric structures have been widely recognized to suitably

2000 *Mathematics Subject Classification.* 58A20,58A32,58E30,58E40,58J10,58J70.
Key words and phrases. jet, gauge-natural bundle, second variational derivative, generalized Jacobi morphism.
This paper is in final form and no version of it will be published elswhere.

describe so-called gauge-natural field theories, *i.e.* physical theories in which right-invariant infinitesimal automorphisms of the structure bundle $\boldsymbol{P}$ uniquely define the transformation laws of the fields themselves (see *e.g.* [**5**] and references quoted therein). We shall in particular consider *finite order variational sequences* [**22**] on gauge-natural bundles. We represent the second variational derivative [**9, 10, 13**] of a given gauge-natural invariant Lagrangian taken with respect to (prolongations of) vertical parts of gauge-natural lifts of infinitesimal principal automorphisms. We define the generalized gauge-natural Jacobi morphism associated with a given gauge-natural invariant Lagrangian by taking as *variation vector fields* the Lie derivatives of sections of the gauge-natural bundle with respect to gauge-natural lifts of infinitesimal automorphisms of the underlying principal bundle $\boldsymbol{P}$. We stress that such variation vector fields are, in particular, generalized symmetries [**6, 15**]. We recall that, as a consequence of the Second Noether Theorem [**25**], within such a picture it is possible to relate the generalized Bianchi morphism to the second variational derivative of the Lagrangian and then to the associated generalized gauge-natural Jacobi morphism; details can be found in [**28, 29**].

Recall that, in the case of geodesics in a Riemannian manifold, vector fields which make the second variation to vanish identically modulo boundary terms are called *Jacobi fields* and they are solutions of a second–order differential equation known as *Jacobi equation* (of geodesics). The notion of Jacobi equation as an outcome of the second variation is in fact fairly more general: formulae for the second variation of a Lagrangian functional in higher order field theory and generalized Jacobi equations along critical sections have been already considered (see, *e.g.* the results of [**2, 3, 16**] and classical references quoted therein).

In particular, by requiring the gauge-natural second variational derivative to vanish, we find that a covariant Noether strongly conserved current is canonically associated with the generalized *deformed* Lagrangian ω obtained by contracting Euler–Lagrange equations of the original Lagrangian λ with (prolongations of) vertical parts of gauge-natural lifts of infinitesimal principal automorphisms which are in the kernel of the generalized gauge-natural Jacobi morphism.

Our results can also be compared with the ones reported in the recent paper [**15**], where conservation laws associated with generalized Lagrangian symmetries, and in particular with divergence symmetries of a Lagrangian are presented, *via* the so-called *characteristic equation*, whithin the framework of infinite order variational bicomplexes.

2. Finite order jets of gauge-natural bundles and variational sequences

We recall some basic facts about jet spaces, sheaves of forms on the s–th order jet space (standard references are *e.g.* [**19, 22, 23, 30, 34, 35**]) and we mainly set the notation.

Our framework is a fibered manifold $\pi : \boldsymbol{Y} \to \boldsymbol{X}$, with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n + m$. For $s \geqslant q \geqslant 0$ integers we are concerned with the s–jet space $J_s\boldsymbol{Y}$ of s–jet prolongations of (local) sections of π; in particular, we set $J_0\boldsymbol{Y} \equiv \boldsymbol{Y}$. We recall the natural fiberings $\pi^s_q : J_s\boldsymbol{Y} \to J_q\boldsymbol{Y}$, $s \geqslant q$, $\pi^s : J_s\boldsymbol{Y} \to \boldsymbol{X}$, and, among these, the *affine* fiberings π^s_{s-1}. We denote by $V\boldsymbol{Y}$ the vector subbundle of the tangent bundle $T\boldsymbol{Y}$ of vectors on $\boldsymbol{Y}$ which are vertical with respect to the fibering π. Greek indices $\sigma, \mu, \dots$ run from 1 to n and they label basis coordinates, while Latin indices $i, j, \dots$ run from 1 to m and label fibre coordinates, unless otherwise specified. We denote multi–indices of dimension n by boldface Greek letters such as $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, with $0 \leqslant \alpha_\mu$, $\mu = 1, \dots, n$; by an abuse of notation, we denote by σ the multi–index such that $\alpha_\mu = 0$, if $\mu \neq \sigma$, $\alpha_\mu = 1$, if $\mu = \sigma$. We also set $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n$ and $\boldsymbol{\alpha}! := \alpha_1! \dots \alpha_n!$. The charts induced on $J_s\boldsymbol{Y}$ are denoted by $(x^\sigma, y^i_{\boldsymbol{\alpha}})$, with $0 \leqslant |\boldsymbol{\alpha}| \leqslant s$; in particular, we set $y^i_{\boldsymbol{0}} \equiv y^i$. The local vector fields and forms of $J_s\boldsymbol{Y}$ induced by the above coordinates are denoted by $(\partial^{\boldsymbol{\alpha}}_i)$ and $(d^i_{\boldsymbol{\alpha}})$, respectively.

For $s \geqslant 1$, we consider the natural complementary fibered morphisms over $J_s\boldsymbol{Y} \to J_{s-1}\boldsymbol{Y}$ (see [**22, 23, 34**]):

$$\mathcal{D} : J_s\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X} \to TJ_{s-1}\boldsymbol{Y}\,, \qquad \vartheta : J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} TJ_{s-1}\boldsymbol{Y} \to VJ_{s-1}\boldsymbol{Y}\,,$$

with coordinate expressions, for $0 \leqslant |\boldsymbol{\alpha}| \leqslant s-1$, given by

$$\mathcal{D} = d^\lambda \otimes \mathcal{D}_\lambda = d^\lambda \otimes (\partial_\lambda + y^j_{\boldsymbol{\alpha}+\lambda}\partial^{\boldsymbol{\alpha}}_j)\,, \quad \vartheta = \vartheta^j_{\boldsymbol{\alpha}} \otimes \partial^{\boldsymbol{\alpha}}_j = (d^j_{\boldsymbol{\alpha}} - y^j_{\boldsymbol{\alpha}+\lambda} d^\lambda) \otimes \partial^{\boldsymbol{\alpha}}_j\,.$$

The morphisms above induce the following natural splitting (and its dual):

$$J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} T^*J_{s-1}\boldsymbol{Y} = \left(J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} T^*\boldsymbol{X}\right) \oplus \mathcal{C}^*_{s-1}[\boldsymbol{Y}]\,, \tag{1}$$

where $\mathcal{C}^*_{s-1}[\boldsymbol{Y}] := \operatorname{im} \vartheta^*_s$ and $\vartheta^*_s : J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} V^*J_{s-1}\boldsymbol{Y} \to J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} T^*J_{s-1}\boldsymbol{Y}$.

If $f : J_s\boldsymbol{Y} \to I\!R$ is a function, then we set $D_\sigma f := \mathcal{D}_\sigma f$, $D_{\boldsymbol{\alpha}+\sigma} f := D_\sigma D_{\boldsymbol{\alpha}} f$, where D_σ is the standard *formal derivative*. Given a vector field $u : J_s\boldsymbol{Y} \to TJ_s\boldsymbol{Y}$, the splitting (1) yields $u \circ \pi^{s+1}_s = u_H + u_V$ where, if $u = u^\gamma \partial_\gamma + u^i_{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}}_i$, then we have $u_H = u^\gamma D_\gamma$ and $u_V = (u^i_{\boldsymbol{\alpha}} - y^i_{\boldsymbol{\alpha}+\gamma} u^\gamma)\partial^{\boldsymbol{\alpha}}_i$. We shall call u_H and u_V the horizontal and the vertical part of u, respectively.

The splitting (1) induces also a decomposition of the exterior differential on $\boldsymbol{Y}$, $(\pi^s_{s-1})^* \circ d = d_H + d_V$, where d_H and d_V are defined to be the *horizontal* and *vertical differential*. The action of d_H and d_V on functions and 1–forms on $J_s\boldsymbol{Y}$ uniquely characterizes d_H and d_V (see, *e.g.*, [**22, 23, 30, 34**] for more details). A *projectable vector field* on $\boldsymbol{Y}$ is defined to be a pair (Ξ, ξ), where $\Xi : \boldsymbol{Y} \to T\boldsymbol{Y}$ and $\xi : \boldsymbol{X} \to T\boldsymbol{X}$ are vector fields and Ξ is a fibered morphism over ξ. If there is no danger of confusion, we will denote simply by Ξ a projectable vector field (Ξ, ξ). A projectable vector field (Ξ, ξ) can be prolonged by the flow functor to a projectable vector field $(j_s\Xi, \xi)$, the coordinate expression of which can be found *e.g.* in [**22, 30, 34**]; in particular, if locally $\Xi = \xi^\sigma(x^\mu)\partial_\sigma + \xi^i(x^\mu, y^j)\partial_i$, we have the following expressions $(j_s\Xi)_H = \xi^\sigma D_\sigma$, $(j_s\Xi)_V = D_{\boldsymbol{\alpha}}(\Xi_V)^i \, \partial^{\boldsymbol{\alpha}}_i$, with $(\Xi_V)^i = \xi^i - y^i_\sigma \xi^\sigma$, for the horizontal and the vertical part of $j_s\Xi$, respectively. From now on, by an abuse of notation, we will write simply $j_s\Xi_H$ and $j_s\Xi_V$. In particular, we stress that $j_s\Xi_V$ can be seen as a fibered morphism: $j_s\Xi_V : J_{s+1}\boldsymbol{Y} \underset{J_s\boldsymbol{Y}}{\times} J_s\boldsymbol{Y} \to J_{s+1}\boldsymbol{Y} \underset{J_s\boldsymbol{Y}}{\times} J_sV\boldsymbol{Y}$.

2.1. Gauge-natural bundles. Let $\boldsymbol{P} \to \boldsymbol{X}$ be a principal bundle with structure group $\boldsymbol{G}$. Let $r \leqslant k$ be integers and $\boldsymbol{W}^{(r,k)}\boldsymbol{P} := J_r\boldsymbol{P} \underset{\boldsymbol{X}}{\times} L_k(\boldsymbol{X})$, where $L_k(\boldsymbol{X})$ is the bundle of k–frames in $\boldsymbol{X}$ [**4, 19**], $\boldsymbol{W}^{(r,k)}\boldsymbol{G} := J_r\boldsymbol{G} \odot GL_k(n)$ the semidirect product with respect to the action of $GL_k(n)$ on $J_r\boldsymbol{G}$ given by the jet composition and $GL_k(n)$ is the group of k–frames in $I\!R^n$. Here we denote by $J_r\boldsymbol{G}$ the space of (r,n)-velocities on $\boldsymbol{G}$. The bundle $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ is a principal bundle over $\boldsymbol{X}$ with structure group $\boldsymbol{W}^{(r,k)}\boldsymbol{G}$. Let $\boldsymbol{F}$ be any manifold and $\zeta : \boldsymbol{W}^{(r,k)}\boldsymbol{G} \times \boldsymbol{F} \to \boldsymbol{F}$ be a left action of $\boldsymbol{W}^{(r,k)}\boldsymbol{G}$ on $\boldsymbol{F}$. There is a naturally defined right action of $\boldsymbol{W}^{(r,k)}\boldsymbol{G}$

on $\boldsymbol{W}^{(r,k)}\boldsymbol{P} \times \boldsymbol{F}$ so that we can associate in a standard way to $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ the bundle, on the given basis $\boldsymbol{X}$, $\boldsymbol{Y}_\zeta := \boldsymbol{W}^{(r,k)}\boldsymbol{P} \times_\zeta \boldsymbol{F}$.

DEFINITION 1. We say $(\boldsymbol{Y}_\zeta, \boldsymbol{X}, \pi_\zeta; \boldsymbol{F}, \boldsymbol{G})$ to be the *gauge-natural bundle* of order (r,k) associated to the principal bundle $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ by means of the left action ζ of the group $\boldsymbol{W}^{(r,k)}\boldsymbol{G}$ on the manifold $\boldsymbol{F}$ [**4, 19**].

REMARK 1. A principal automorphism Φ of $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ induces an automorphism of the gauge-natural bundle by:

$$\Phi_\zeta : \boldsymbol{Y}_\zeta \to \boldsymbol{Y}_\zeta : [(j^x_r\gamma, j^0_k t), \hat{f}]_\zeta \mapsto [\Phi(j^x_r\gamma, j^0_k t), \hat{f}]_\zeta \,, \tag{2}$$

where $\hat{f} \in \boldsymbol{F}$ and $[\cdot,\cdot]_\zeta$ is the equivalence class induced by the action ζ.

DEFINITION 2. We also define the *vector* bundle over $\boldsymbol{X}$ of right invariant infinitesimal automorphisms of $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$ by setting $\mathcal{A}^{(r,k)} := T\boldsymbol{W}^{(r,k)}\boldsymbol{P}/\boldsymbol{W}^{(r,k)}\boldsymbol{G}$ $(r \leqslant k)$.

Denote by $\mathcal{T}_{\boldsymbol{X}}$ and $\mathcal{A}^{(r,k)}$ the sheaf of vector fields on $\boldsymbol{X}$ and the sheaf of right invariant vector fields on $\boldsymbol{W}^{(r,k)}\boldsymbol{P}$, respectively. A functorial mapping $\mathfrak{G}$ is defined which lifts any right–invariant local automorphism (Φ, φ) of the principal bundle $W^{(r,k)}\boldsymbol{P}$ into a unique local automorphism (Φ_ζ, φ) of the associated bundle $\boldsymbol{Y}_\zeta$. Its infinitesimal version associates to any $\bar{\Xi} \in \mathcal{A}^{(r,k)}$, projectable over $\xi \in \mathcal{T}_{\boldsymbol{X}}$, a unique *projectable* vector field $\hat{\Xi} := \mathfrak{G}(\bar{\Xi})$ (called the gauge-natural lift) on $\boldsymbol{Y}_\zeta$ in the following way:

$$\mathfrak{G} : \boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} \mathcal{A}^{(r,k)} \to T\boldsymbol{Y}_\zeta \,:\, (\boldsymbol{y}, \bar{\Xi}) \mapsto \hat{\Xi}(\boldsymbol{y}) \,, \tag{3}$$

where, for any $\boldsymbol{y} \in \boldsymbol{Y}_\zeta$, one sets: $\hat{\Xi}(\boldsymbol{y}) = \frac{d}{dt}\left((\Phi_{\zeta\, t})(\boldsymbol{y})\right)_{t=0}$, and $\Phi_{\zeta\, t}$ denotes the (local) flow corresponding to the gauge-natural lift of Φ_t.

This mapping fulfils the following properties (see [**19**]):

(1) $\mathfrak{G}$ is linear over $id_{\boldsymbol{Y}_\zeta}$;
(2) we have $T\pi_\zeta \circ \mathfrak{G} = id_{T\boldsymbol{X}} \circ \bar{\pi}^{(r,k)}$, where $\bar{\pi}^{(r,k)}$ is the natural projection $\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} \mathcal{A}^{(r,k)} \to T\boldsymbol{X}$;
(3) for any pair $(\bar{\Lambda}, \bar{\Xi}) \in \mathcal{A}^{(r,k)}$ we have $\mathfrak{G}([\bar{\Lambda}, \bar{\Xi}]) = [\mathfrak{G}(\bar{\Lambda}), \mathfrak{G}(\bar{\Xi})]$ for commutators.

2.2. Lie derivative of gauge-natural sections.

DEFINITION 3. Let γ be a (local) section of $\boldsymbol{Y}_\zeta$, $\bar{\Xi} \in \mathcal{A}^{(r,k)}$ and $\hat{\Xi}$ its gauge-natural lift. Following [**19**] we define the *generalized Lie derivative* of γ along the vector field $\hat{\Xi}$ to be the (local) section $\pounds_{\bar{\Xi}}\gamma : \boldsymbol{X} \to V\boldsymbol{Y}_\zeta$, given by

$$\pounds_{\bar{\Xi}}\gamma = T\gamma \circ \xi - \hat{\Xi} \circ \gamma \,.$$

REMARK 2. The Lie derivative operator acting on sections of gauge-natural bundles satisfies the following properties:

(1) for any vector field $\bar{\Xi} \in \mathcal{A}^{(r,k)}$, the mapping $\gamma \mapsto \pounds_{\bar{\Xi}}\gamma$ is a first–order quasilinear differential operator;
(2) for any local section γ of $\boldsymbol{Y}_\zeta$, the mapping $\bar{\Xi} \mapsto \pounds_{\bar{\Xi}}\gamma$ is a linear differential operator;

(3) we can regard $\pounds_{\Xi} : J_1\boldsymbol{Y}_\zeta \to V\boldsymbol{Y}_\zeta$ as a morphism over the basis $\boldsymbol{X}$. By using the canonical isomorphisms $VJ_s\boldsymbol{Y}_\zeta \simeq J_sV\boldsymbol{Y}_\zeta$ for all s, we have $\pounds_{\Xi}[j_s\gamma] = j_s[\pounds_{\Xi}\gamma]$, for any (local) section γ of $\boldsymbol{Y}_\zeta$ and for any (local) vector field $\bar{\Xi} \in \mathcal{A}^{(r,k)}$. Furthermore, the fundamental relation holds true:

$$\hat{\Xi}_V := \mathfrak{G}(\bar{\Xi})_V = -\pounds_{\Xi}\,.$$

2.3. Variational sequences. For the sake of simplifying notation, we will sometimes omit the subscript ζ, so that all our considerations shall refer to $\boldsymbol{Y}$ as a gauge-natural bundle as defined above.

We shall be here concerned with some distinguished sheaves of forms on jet spaces [**22, 30, 34**]. We shall in particular follow notation given in [**34**] to which the reader is referred for details. For $s \geqslant 0$, we consider the standard sheaves Λ^p_s of p–forms on $J_s\boldsymbol{Y}$. For $0 \leqslant q \leqslant s$, we consider the sheaves $\mathcal{H}^p_{(s,q)}$ and $\mathcal{H}^p_s$ of *horizontal forms* with respect to the projections π^s_q and π^s_0, respectively. For $0 \leqslant q < s$, we consider the subsheaves $\mathcal{C}^p_{(s,q)} \subset \mathcal{H}^p_{(s,q)}$ and $\mathcal{C}^p{}_s \subset \mathcal{C}^p_{(s+1,s)}$ of *contact forms*, *i.e.* horizontal forms valued into $\mathcal{C}^*_s[\boldsymbol{Y}]$(they have the property of vanishing along any section of the gauge-natural bundle).

According to [**22, 34**] the fibered splitting (1) yields the *sheaf splitting* $\mathcal{H}^p_{(s+1,s)} = \bigoplus_{t=0}^p \mathcal{C}^{p-t}_{(s+1,s)} \wedge \mathcal{H}^t_{s+1}$, which restricts to the inclusion $\Lambda^p_s \subset \bigoplus_{t=0}^p \mathcal{C}^{p-t}{}_s \wedge \mathcal{H}^{t,h}{}_{s+1}$, where $\mathcal{H}^{p,h}{}_{s+1} := h(\Lambda^p_s)$ for $0 < p \leqslant n$ and the surjective map h is defined to be the restriction to Λ^p_s of the projection of the above splitting onto the non–trivial summand with the highest value of t. By an abuse of notation, let us denote by $d\ker h$ the sheaf generated by the presheaf $d\ker h$ in the standard way. We set $\Theta^*_s := \ker h + d\ker h$.

In [**22**], to which the reader is referred for details, it was proved that the following sequence is an exact resolution of the constant sheaf $\mathbb{R}_{\boldsymbol{Y}}$ over $\boldsymbol{Y}$:

$$0 \longrightarrow \mathbb{R}_Y \longrightarrow \Lambda^0_s \xrightarrow{\mathcal{E}_0} \Lambda^1_s/\Theta^1_s \xrightarrow{\mathcal{E}_1} \Lambda^2_s/\Theta^2_s \xrightarrow{\mathcal{E}_2} \cdots \xrightarrow{\mathcal{E}_{I-1}} \Lambda^I_s/\Theta^I_s \xrightarrow{\mathcal{E}_I} \Lambda^{I+1}_s \xrightarrow{d} 0$$

DEFINITION 4. The above sequence, where the highest integer I depends on the dimension of the fibers of $J_s\boldsymbol{Y} \to \boldsymbol{X}$ (see, in particular, [**22**]), is said to be the s–th order *variational sequence* associated with the fibered manifold $\boldsymbol{Y} \to \boldsymbol{X}$.

For practical purposes we shall limit ourselves to consider the truncated variational sequence:

$$0 \longrightarrow \mathbb{R}_Y \longrightarrow \mathcal{V}^0_s \xrightarrow{\mathcal{E}_0} \mathcal{V}^1_s \xrightarrow{\mathcal{E}_1} \cdots \xrightarrow{\mathcal{E}_n} \mathcal{V}^{n+1}_s \xrightarrow{\mathcal{E}_{n+1}} \mathcal{E}_{n+1}(\mathcal{V}^{n+1}_s) \xrightarrow{\mathcal{E}_{n+2}} 0\,,$$

where, following [**34**], the sheaves $\mathcal{V}^p_s := \mathcal{C}^{p-n}_s \wedge \mathcal{H}^{n,h}{}_{s+1}/h(d\ker h)$, with $0 \leqslant p \leqslant n+2$, are suitable representations of the corresponding quotient sheaves in the variational sequence by means of sheaves of sections of tensor bundles. Representations of Krupka's long variational sequence by means of differential forms have been provided *e.g.* in [**21, 31, 35**].

We recall now some intrinsic decomposition involved with the first and the second variation formulae.

- Let $\alpha \in \mathcal{C}^1_s \wedge \mathcal{H}^{n,h}{}_{s+1} \subset \mathcal{V}^{n+1}_{s+1}$. Then there is a unique pair of sheaf morphisms ([**17, 20, 34**])

$$E_\alpha \in \mathcal{C}^1_{(2s,0)} \wedge \mathcal{H}^{n,h}{}_{2s+1}\,, \qquad F_\alpha \in \mathcal{C}^1_{(2s,s)} \wedge \mathcal{H}^{n,h}{}_{2s+1}\,, \tag{4}$$

such that $(\pi^{2s+1}_{s+1})^*\alpha = E_\alpha - F_\alpha$ and F_α is *locally* of the form $F_\alpha = d_H p_\alpha$, with $p_\alpha \in \mathcal{C}^1_{(2s-1,s-1)} \wedge \mathcal{H}^{n-1}{}_{2s}$.

- Let $\eta \in \mathcal{C}^1_s \wedge \mathcal{C}^1_{(s,0)} \wedge \mathcal{H}^{n,h}{}_{s+1} \subset \mathcal{V}^{n+2}_{s+1}$; then there is a unique morphism

$$K_\eta \in \mathcal{C}^1_{(2s,s)} \otimes \mathcal{C}^1_{(2s,0)} \wedge \mathcal{H}^{n,h}{}_{2s+1}$$

such that, for all $\Xi : \boldsymbol{Y} \to V\boldsymbol{Y}$, $E_{j_s\Xi\rfloor\eta} = C^1_1(j_{2s}\Xi \otimes K_\eta)$, where C^1_1 stands for tensor contraction on the first factor and $\rfloor$ denotes inner product (see [**20, 34**]). Furthermore, there is a unique pair of sheaf morphisms

$$H_\eta \in \mathcal{C}^1_{(2s,s)} \wedge \mathcal{C}^1_{(2s,0)} \wedge \mathcal{H}^{n,h}{}_{2s+1}\,, \quad G_\eta \in \mathcal{C}^2_{(2s,s)} \wedge \mathcal{H}^{n,h}{}_{2s+1}\,, \tag{5}$$

such that $(\pi^{2s+1}_{s+1})^*\eta = H_\eta - G_\eta$ and $H_\eta = \frac{1}{2}A(K_\eta)$, where A stands for antisymmetrisation. Moreover, G_η is *locally* of the type $G_\eta = d_H q_\eta$, where $q_\eta \in \mathcal{C}^2_{(2s-1,s-1)} \wedge \mathcal{H}^{n-1}{}_{2s}$; hence $[\eta] = [H_\eta]$ (see [**20, 34**]).

REMARK 3. A section $\lambda \in \mathcal{V}^n_s$ is just a Lagrangian of order $(s+1)$ of the standard literature, while $\mathcal{E}_n(\lambda) \in \mathcal{V}^{n+1}_s$ coincides with the standard higher order Euler–Lagrange morphism associated with λ. The kernel of the morphism H coincides with standard Helmholtz conditions of local variationality.

EXAMPLE. Let $\lambda \in \mathcal{V}^n_s$. It is known (see *e.g.* [**17**]) that

$$d_V\lambda = (d_V\lambda)^{\boldsymbol{\alpha}}_i \vartheta^i_{\boldsymbol{\alpha}} \wedge \omega\,,\; E_{d_V\lambda} = \mathcal{E}_n(\lambda)_i\vartheta^i \wedge \omega\,,\; p_{d_V\lambda} = p(\lambda)^{\boldsymbol{\alpha}\mu}_i \vartheta^i_{\boldsymbol{\alpha}} \wedge \omega_\mu\,,$$
$$p(\lambda)^{\boldsymbol{\beta}\mu}_i = (d_V\lambda)^{\boldsymbol{\alpha}}_i \qquad \boldsymbol{\beta} + \mu = \boldsymbol{\alpha}, |\boldsymbol{\alpha}| = s\,,$$
$$p(\lambda)^{\boldsymbol{\beta}\mu}_i = (d_V\lambda)^{\boldsymbol{\alpha}}_i - D_\nu p(\lambda)^{\boldsymbol{\alpha}\nu}_i \qquad \boldsymbol{\beta} + \mu = \boldsymbol{\alpha}, |\boldsymbol{\alpha}| = s-1\,,$$
$$\mathcal{E}_n(\lambda)^{\boldsymbol{\alpha}}_i = (d_V\lambda)^{\boldsymbol{\alpha}}_i - D_\nu p(\lambda)^{\boldsymbol{\alpha}\nu}_i \qquad |\boldsymbol{\alpha}| = 0\,.$$

Furthermore, $\mathcal{E}_n(\lambda)_i = \sum_{|\boldsymbol{\alpha}|\leqslant s}(-1)^{|\boldsymbol{\alpha}|}D_{\boldsymbol{\alpha}}(d_V\lambda)^{\boldsymbol{\alpha}}_i$.

3. Variations and generalized Jacobi morphisms

We will represent generalized *gauge-natural* Jacobi morphisms in variational sequences and establish their relation with the second variational derivative of a generalized gauge-natural invariant Lagrangian. We consider *formal variations* of a fibered morphism as *multiparameter deformations* and relate the second variational derivative of the Lagrangian λ to the variational Lie derivative of the associated Euler–Lagrange morphism and to the generalized Bianchi morphism; see [**28**] for details.

DEFINITION 5. Let $\pi : \boldsymbol{Y} \to \boldsymbol{X}$ be any bundle and let $\alpha : J_s\boldsymbol{Y} \to \overset{p}{\wedge} T^*J_s\boldsymbol{Y}$ and $L_{j_s\Xi_k}$ be the Lie derivative operator acting on differential fibered morphisms. Let Ξ_k, $1 \leqslant k \leqslant i$, be variation vector fields on $\boldsymbol{Y}$ in the sense of [**9, 10, 13, 28**]. In particular in the sequel we will assume that they are vertical parts of projectable vector fields. Inspired by the classical approach [**16**] we define the i–th formal variation of the morphism α to be the operator: $\delta^i\alpha = L_{j_s\Xi_1}\dots L_{j_s\Xi_i}\alpha$.

REMARK 4. Let now the morphism α be a representative of an equivalence class in the variational sequence, *i.e.* $\alpha \in (\mathcal{V}^n_s)_{\boldsymbol{Y}}$.

Since the Lie derivative operator $L_{j_s\Xi}$ with respect to a projectable vector field $j_s\Xi$ preserves the contact structure [**11, 23**], by an abuse of notation we can write:

$\delta^i[\alpha] := [\delta^i\alpha] = [L_{\Xi_i} \dots L_{\Xi_1}\alpha] = \mathcal{L}_{\Xi_i} \dots \mathcal{L}_{\Xi_1}[\alpha]$, where $\mathcal{L}_{\Xi_i}$ is the *variational Lie derivative* acting on quotient variational morphisms, an operator defined and conveniently represented in [**11**]; see also [**23**].

DEFINITION 6. We call the operator δ^i the *i–th variational derivative.*

The following characterization of the second variational derivative of a generalized Lagrangian in the variational sequence holds true [**28**].

PROPOSITION 1. *Let $\lambda \in (\mathcal{V}^n_s)_{\boldsymbol{Y}}$ and let Ξ be a variation vector field; then we have*

$$\delta^2\lambda = [\mathcal{E}_n(j_{2s}\Xi\rfloor h\delta\lambda) + C^1_1(j_{2s}\Xi \otimes K_{hd\delta\lambda})]\,. \tag{6}$$

3.1. Generalized *gauge-natural* Jacobi morphisms. Let now specify $\boldsymbol{Y}$ to be a gauge-natural bundle and let $\hat{\Xi}_V \equiv \mathfrak{G}(\bar{\Xi})_V$ be a variation vector field. Let us consider $j_s\bar{\Xi}_V$, *i.e.* the vertical part according to the splitting (1). The set of all sections of the vector bundle $\mathcal{A}^{(r+s,k+s)}$ of this kind defines a vector subbundle of $J_s\mathcal{A}^{(r,k)}$, which (since we are speaking about vertical parts with respect to the splitting (1)) by a slight abuse of notation will be denoted by $VJ_s\mathcal{A}^{(r,k)}$.

By applying an abstract result due to Kolář (see [**17**]) concerning a global decomposition formula for vertical morphisms, using Proposition 1 we can prove the following.

LEMMA 1. *Let λ be a Lagrangian and $\hat{\Xi}_V$ a variation vecrtor field. Let us set*

$$\chi(\lambda, \mathfrak{G}(\bar{\Xi})_V) := C^1_1(j_{2s}\hat{\Xi} \otimes K_{hd\mathcal{L}_{j_{2s}\bar{\Xi}_V}\lambda}) \equiv E_{j_s\hat{\Xi}\rfloor hd\mathcal{L}_{j_{2s+1}\bar{\Xi}_V}\lambda}.$$

Let D_H be the horizontal differential on $J_{4s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s}\mathcal{A}^{(r,k)}$. Then we have:

$$(\pi^{4s+1}_{2s+1})^*\chi(\lambda, \mathfrak{G}(\bar{\Xi})_V) = E_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V)} + F_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V)}\,, \tag{7}$$

where

$$E_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V} : J_{4s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X})\,,$$

and locally $F_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V)} = D_H M_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V)}$ with

$$M_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V)} : J_{4s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_{2s-1}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n-1}{\wedge} T^*\boldsymbol{X})\,.$$

PROOF. As a consequence of linearity properties of both $\chi(\lambda, \Xi)$ and the Lie derivative operator $\pounds$ we have $\chi(\lambda, \mathfrak{G}(\bar{\Xi})_V) : J_{2s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{2s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X})$ and $D_H\chi(\lambda, \mathfrak{G}(\bar{\Xi})_V) = 0$. Thus Kolář's decomposition formula can be applied. Coordinate expressions of both terms in the above invariant decomposition (7) can be easily evaluated by a backwards procedure analogously to Example 2.3. □

DEFINITION 7. We call the morphism $\mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V) := E_{\chi(\lambda,\mathfrak{G}(\bar{\Xi})_V)}$ the *gauge-natural generalized Jacobi morphism* associated with the Lagrangian λ and the variation vector field $\hat{\Xi}_V$.

Notice that the morphism $\mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ is a *linear* morphism with respect to the projection $J_{4s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s}\mathcal{A}^{(r,k)} \to J_{4s}\boldsymbol{Y}_\zeta$.

We have the following relations for the second variational gauge-natural derivative (see [**28**]).

THEOREM 1. *Let $\delta^2_{\mathfrak{G}}\lambda$ be the second variational derivative of λ with respect to (prolongations of) vertical parts of gauge-natural lifts of infinitesimal principal automorphisms. The two equalities hold:*

$$\delta^2_{\mathfrak{G}}\lambda = \mathfrak{G}(\bar{\Xi})_V \rfloor \mathcal{E}_n\left(\mathfrak{G}(\bar{\Xi})_V \rfloor \mathcal{E}_n(\lambda)\right) = \mathcal{E}_n\left(\mathfrak{G}(\bar{\Xi})_V \rfloor h(d\delta\lambda)\right) . \tag{8}$$

4. Canonical covariant conserved currents

In the following we assume that the field equations are generated by means of a variational principle from a Lagrangian which is gauge-natural invariant, *i.e.* invariant with respect to any gauge-natural lift of infinitesimal right invariant vector fields.

DEFINITION 8. Let $(\hat{\Xi}, \xi)$ be a projectable vector field on $\boldsymbol{Y}_\zeta$. Let $\lambda \in \mathcal{V}^n_s$ be a generalized Lagrangian. We say $\hat{\Xi}$ to be a *symmetry* of λ if $\mathcal{L}_{j_{s+1}\hat{\Xi}}\lambda = 0$.

We say λ to be a *gauge-natural invariant Lagrangian* if the gauge-natural lift $(\hat{\Xi}, \xi)$ of *any* vector field $\bar{\Xi} \in \mathcal{A}^{(r,k)}$ is a symmetry for λ, *i.e.* if $\mathcal{L}_{j_{s+1}\bar{\Xi}}\lambda = 0$. In this case the projectable vector field $\hat{\Xi} \equiv \mathfrak{G}(\bar{\Xi})$ is called a *gauge-natural symmetry* of λ.

The First Noether Theorem [**25**] takes a particularly interesting form in the case of gauge-natural Lagrangians as shown in the following.

PROPOSITION 2. *Let $\lambda \in \mathcal{V}^n_s$ be a gauge-natural Lagrangian and $(\hat{\Xi}, \xi)$ a gauge-natural symmetry of λ. Then we have*

$$0 = -\pounds_{\bar{\Xi}} \rfloor \mathcal{E}_n(\lambda) + d_H(-j_s\pounds_{\bar{\Xi}} \rfloor p_{d_V\lambda} + \xi \rfloor \lambda).$$

Suppose that $(j_{2s+1}\sigma)^*\left(-\pounds_{\bar{\Xi}} \rfloor \mathcal{E}_n(\lambda)\right) = 0$. Then, the $(n-1)$–form

$$\varepsilon = -j_s\pounds_{\bar{\Xi}} \rfloor p_{d_V\lambda} + \xi \rfloor \lambda$$

fulfills the equation $d\left((j_{2s}\sigma)^*(\varepsilon)\right) = 0$.

If σ is a critical section for $\mathcal{E}_n(\lambda)$, *i.e.* $(j_{2s+1}\sigma)^*\mathcal{E}_n(\lambda) = 0$, the above equation admits a physical interpretation as a so-called *weak conservation law* for the density associated with ε.

DEFINITION 9. Let $\lambda \in \mathcal{V}^n_s$ be a gauge-natural Lagrangian and $\bar{\Xi} \in \mathcal{A}^{(r,k)}$. Then the sheaf morphism $\varepsilon : J_{2s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{2s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n-1}{\wedge} T^*\boldsymbol{X})$ is said to be a *gauge-natural weakly conserved current*.

REMARK 5. In general, this conserved current is not uniquely defined. In fact, it depends on the choice of $p_{d_V\lambda}$, which is not unique (see *e.g.* [**34**] and references quoted therein).

4.1. The deformed Lagrangian and Bianchi morphism. In gauge-natural Lagrangian theories a well known procedure suggests to perform suitable integrations by parts to decompose the conserved current ε into the sum of a conserved current vanishing along solutions of the Euler–Lagrange equations, the so–called *reduced current* $\tilde{\varepsilon}$, and the formal divergence of a skew–symmetric (tensor) density called a *superpotential* (which is defined modulo a divergence). Within such a procedure, the generalized Bianchi identities are in fact necessary and (locally) sufficient conditions for the conserved current ϵ to be not only closed but also the divergence of a skew-symmetric (tensor) density along solutions of the Euler–Lagrange equations.

Let then λ be a gauge-natural invariant Lagrangian. We set

(9) $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) \equiv -\pounds_{\bar{\Xi}} \rfloor \mathcal{E}_n(\lambda) : J_{2s}\boldsymbol{Y}_\zeta \to \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X})$.

The morphism $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ so defined is a generalized Lagrangian associated with the field equations of the original Lagrangian λ. It has been considered in applications *e.g.* in General Relativity (see [**8**] and references quoted therein). We have $D_H\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) = 0$ and by the linearity of the operator $\pounds$ we can regard $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ as the extended morphism

$$\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) : J_{2s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{2s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X}).$$

The following Lemma (see [**28**]) is a geometric version of the integration by parts procedure quoted above and it is based on a global decomposition formula of vertical morphisms due to Kolář [**17**].

LEMMA 2. *Let* $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ *be as above. On the domain of* $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ *we have:*

$$(\pi^{4s+1}_{s+1})^*\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V) = \beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) + F_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)},$$

where

$$\beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) \equiv E_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)} : \tag{10}$$

$$J_{4s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s}\mathcal{A}^{(r,k)} \to \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X}) \tag{11}$$

and, locally, $F_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)} = D_H M_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)}$, *with*

$$M_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)} :$$

$$J_{4s-1}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s-1}\mathcal{A}^{(r,k)}) \to \mathcal{C}^*_{2s}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_{2s-1}[\mathcal{A}^{(r,k)}] \otimes \mathcal{C}^*_0[\mathcal{A}^{(r,k)}] \wedge (\overset{n-1}{\wedge} T^*\boldsymbol{X}).$$

DEFINITION 10. We call the global morphism $\beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) := E_{\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)}$ the *generalized Bianchi morphism* associated with the Lagrangian λ and the variation vector field $\hat{\Xi}_V$.

REMARK 6. For any $(\bar{\Xi}, \xi) \in \mathcal{A}^{(r,k)}$, as a consequence of the gauge-natural invariance of the Lagrangian, the morphism $\beta(\lambda, \mathfrak{G}(\bar{\Xi})_V) \equiv \mathcal{E}_n(\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V))$ is *locally* identically vanishing. We stress that these are just *local generalized Bianchi identities* [**1**].

Let $\mathfrak{K}$ be the *kernel* of $\mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V)$. As a consequence of Theorem 1 and the considerations above we have the following result, the detailed proof of which will appear in [**28**].

THEOREM 2. *The generalized Bianchi morphism is globally vanishing for the variation vector field* $\bar{\Xi}$ *if and only if* $\delta^2_{\mathfrak{G}}\lambda \equiv \mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V) = 0$, i.e. *if and only if* $\mathfrak{G}(\bar{\Xi})_V \in \mathfrak{K}$.

4.2. A strong conservation law equivalent to the Jacobi equations. From now on we shall write $\omega(\lambda, \mathfrak{K})$ to denote $\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)$ when $\mathfrak{G}(\bar{\Xi})_V$ belongs to $\mathfrak{K}$. Analogously for β and other morphisms.

First of all let us make the following important considerations (partial results in this direction were already given in [**14**], where the consequences on the concept of curvature of a gauge-natural invariant principle have been stressed).

PROPOSITION 3. *Let D_V be the vertical differential on $J_{4s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{4s}\mathcal{A}^{(r,k)}$. For each $\bar{\Xi} \in \mathcal{A}^{(r,k)}$ such that $\bar{\Xi}_V \in \mathfrak{K}$, we have*

$$\mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda,\mathfrak{K}) = -D_H(-j_s\pounds_{\bar{\Xi}_V}\rfloor p_{D_V\omega(\lambda,\mathfrak{K})})\,. \tag{12}$$

PROOF. The horizontal splitting gives us

$$\mathcal{L}_{j_s\bar{\Xi}}\omega(\lambda,\mathfrak{K}) = \mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda,\mathfrak{K}) + \mathcal{L}_{j_s\bar{\Xi}_V}\omega(\lambda,\mathfrak{K}).$$

Furthermore, $\omega(\lambda,\mathfrak{K}) \equiv -\pounds_{\bar{\Xi}}\rfloor\mathcal{E}_n(\lambda) = \mathcal{L}_{j_s\bar{\Xi}}\lambda - d_H(-j_s\pounds_{\bar{\Xi}}\rfloor p_{d_V\lambda} + \xi\rfloor\lambda)$; so that

$$\mathcal{L}_{j_s\bar{\Xi}_V}\omega(\lambda,\mathfrak{K}) = \mathcal{L}_{j_s\bar{\Xi}_V}\mathcal{L}_{j_s\bar{\Xi}}\lambda = \mathcal{L}_{j_s[\bar{\Xi}_V,\bar{\Xi}_H]}\lambda\,.$$

On the other hand we have

$$\mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda,\mathfrak{K}) = \mathcal{L}_{j_s[\bar{\Xi}_H,\bar{\Xi}_V]}\lambda = -\mathcal{L}_{j_s\bar{\Xi}_V}\omega(\lambda,\mathfrak{K})\,.$$

Recall now that from the Theorem above we have $\bar{\Xi}_V \in \mathfrak{K}$ if and only if $\beta(\lambda,\mathfrak{K}) = 0$. Since

$$\begin{aligned}&\mathcal{L}_{j_s\bar{\Xi}_V}\omega(\lambda,\mathfrak{K}) = -\pounds_{\bar{\Xi}_V}\rfloor\mathcal{E}_n(\omega(\lambda,\mathfrak{K})) + D_H(-j_s\pounds_{\bar{\Xi}_V}\rfloor p_{D_V\omega(\lambda,\mathfrak{K})}) = \\ &= \beta(\lambda,\mathfrak{K}) + D_H(-j_s\pounds_{\bar{\Xi}_V}\rfloor p_{D_V\omega(\lambda,\mathfrak{K})})\,,\end{aligned}$$

we get the assertion. □

It is easy to realize that, because of the gauge-natural invariance of the generalized Lagrangian λ, the new generalized Lagrangian $\omega(\lambda,\mathfrak{K})$ is gauge-natural invariant too, *i.e.* $\mathcal{L}_{j_s\bar{\Xi}}\,\omega(\lambda,\mathfrak{K}) = 0$.

Even more, we can state the following:

PROPOSITION 4. *Let $\bar{\Xi}_V \in \mathfrak{K}$. We have*

$$\mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda,\mathfrak{K}) = 0\,. \tag{13}$$

PROOF. In fact, when $\bar{\Xi}_V \in \mathfrak{K}$, since

$$\mathcal{L}_{j_s\bar{\Xi}_V}\omega(\lambda,\mathfrak{K}) = [\bar{\Xi}_V,\bar{\Xi}_V]\rfloor\mathcal{E}_n(\lambda) + \bar{\Xi}_V\rfloor\mathcal{L}_{j_s\bar{\Xi}_V}\mathcal{E}_n(\lambda) = 0$$

, we have

$$0 = \mathcal{L}_{j_s\bar{\Xi}}\omega(\lambda,\mathfrak{K}) = \mathcal{L}_{j_s\bar{\Xi}_V}\omega(\lambda,\mathfrak{K}) + \mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda,\mathfrak{K}) = \mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda,\mathfrak{K})\,.$$

□

As a quite relevant byproduct we get also the following (this result can be compared with [**3**], where some preliminary results have been obtained for second order Lagrangians; see also [**8**] for applications to General Relativity).

COROLLARY 1. *Let $\bar{\Xi}_V \in \mathfrak{K}$. We have the* covariant *strong conservation law*

$$D_H(-j_s\pounds_{\bar{\Xi}_V}\rfloor p_{D_V\omega(\lambda,\mathfrak{K})}) = 0\,. \tag{14}$$

PROOF.

$$\begin{aligned}0 \;&=\; \mathcal{L}_{j_s\bar{\Xi}_H}\omega(\lambda,\mathfrak{K}) = -\beta(\lambda,\mathfrak{K}) - D_H(-j_s\pounds_{\bar{\Xi}_V}\rfloor p_{D_V\omega(\lambda,\mathfrak{K})}) = \\ &=\; -D_H(-j_s\pounds_{\bar{\Xi}_V}\rfloor p_{D_V\omega(\lambda,\mathfrak{K})})\,.\end{aligned}$$

□

DEFINITION 11. We define the covariant n-form

$$\mathcal{H}(\lambda, \mathfrak{K}) = -j_s\mathcal{L} \rfloor p_{D_V\omega(\lambda,\mathfrak{K})} , \tag{15}$$

to be a Hamiltonian form for $\omega(\lambda, \mathfrak{K})$ on the Legendre bundle $\Pi \equiv V^*(J_{2s}\boldsymbol{Y}_\zeta \underset{\boldsymbol{X}}{\times} VJ_{2s}\mathcal{A}^{(r,k)}) \wedge (\overset{n-1}{\wedge} T^*\boldsymbol{X})$ (see [**24**]).

Let Ω be the multisimplectic form on Π. It is well known that every Hamiltonian form $\mathcal{H}$ admits a Hamiltonian connection $\gamma_{\mathcal{H}}(\lambda, \mathfrak{K})$ such that $\gamma_{\mathcal{H}}(\lambda, \mathfrak{K})\rfloor\Omega = d\mathcal{H}(\lambda, \mathfrak{K})$. Let then $\gamma_{\mathcal{H}}(\lambda, \mathfrak{K})$ be the corresponding Hamiltonian connection form (see [**24**]).

As it has been stressed in [**10**] the Euler–Lagrange equations together with the Jacobi equations for a given Lagrangian λ can be obtained out of a unique variational problem for the Lagrangian $\delta\lambda$. This can be performed by requiring the invariance of $\delta\lambda$ with respect to vertical parts of gauge-natural lifts of infinitesimal principal automorphisms, which are solutions of the classical Jacobi equations along critical sections, thus recovering a well known classical result (see *e.g.* [**2, 3, 16, 32**]).

We can finally formulate and prove the following important new result:

THEOREM 3. *For all $\mathfrak{G}(\Xi)_V \in \mathfrak{K}$ the Hamilton equations for the Hamiltonian connection form $\gamma_{\mathcal{H}}(\lambda, \mathfrak{K})$ coincide with the kernel of the generalized gauge-natural Jacobi morphism.*

PROOF. The Hamilton equations for the Hamiltonian connection $\gamma_{\mathcal{H}}(\lambda, \mathfrak{K})$ are identically satisfied being equivalent to the kernel of the Euler lagrange morphism of the Lagrangian $\omega(\lambda, \mathfrak{K})$ (see *e.g.* [**24**]), which coincides with $\mathfrak{K}$ because of Theorem 2. Generalized Bianchi identities appear then as constraints for such an equivalence to hold true (see also [**16**]). $\square$

EXAMPLE. Let Q be a n–dimensional manifold and (TQ, Q, τ_Q) its tangent bundle. It is well known that TQ is a natural bundle associated to the principal bundle $L(I\!R)$ of frames in $I\!R$; the gauge-natural lift order is $(r, k) = (0, 1)$. In this case the natural lift of tangent vector fields to $I\!R$ is the tangent lift and will be denoted by a dot. Let g be a Riemannian metric on Q. The *geodesics* of (Q, g) are those curves $\gamma : I\!R \to Q$ whose tangent vector $\dot\gamma$ is parallel along γ, *i.e.* it satisfies $\nabla_{\dot\gamma}\dot\gamma = 0$. We assume that the *Jacobi fields* of (Q, g) are those vectorfields $\eta = \eta^i\partial^i_q$, with $\eta^i = \xi^i - \dot q^i\xi^0$, *i.e.* vertical parts of tangent vector fields $\xi = \xi^0\partial_t + \xi^i\partial_{q^i}$ and characterized along geodesics γ by the differential equation:

$$\nabla^2_{\dot\gamma}\eta + Riem_g(\eta, \dot\gamma, \dot\gamma) = 0 ,$$

where $\nabla^2_{\dot\gamma}$ denotes the second–order covariant derivative along the curve γ and $Riem_g$ is the tri–linear mapping defining the Riemannian curvature of g. Jacobi fields generically define infinitesimal deformations of geodesics into families of nearby geodesics. The metric g can be lifted to a metric g^C on the manifold TQ, called the "complete lift", as follows.

Let $g = g_{ij}dq^idq^j$ in a local chart (U, q^i); then the corresponding local expression of g^C in $(TU; q^i, u^i)$ is $g^C = 2g_{ij}\delta u^idq^j$, where δu^i stands for $\delta u^i = du^i + \boldsymbol{\Gamma}^i_{mk}u^mdq^k$.

For any function $f : Q \to I\!R$ a new function $\partial f : TQ \to I\!R$ is defined by setting locally: $(\partial f)(q^i, u^i) \equiv (\partial_j f)u^j$. With this notation g^C can be locally expressed by: $g^C = (\partial g_{ij})du^idu^j + 2g_{ij}du^idq^j$. We can easyly see that the system formed by the geodesic equation of g in Q and the Jacobi equation associated with g in TQ is the geodesic equation in TQ of the complete lift metric g^C. Therefore this system follows from a variational principle on TQ based on the energy functional defined by the lifted metric g^C. The energy functional of g is based on the Lagrangian $\lambda = \frac{1}{2} g_{ij} u^i u^j$; the associated first–order deformed Lagrangian is thence given by $\omega = g_{ij}[\dot{\eta}^i + \mathbf{\Gamma}^i_{mk} u^m \eta^k] u^j$.

Then ω is in fact the energy Lagrangian of the lifted metric $g^C = 2 g_{ij} \delta u^i dq^j$. Notice that similar results have been obtained in [**2, 3**]; however, we stress that here Jacobi fields, on which ω depends, are assumed to be of a specific nature, *i.e.* vertical parts of tangent lifts and they have to satisfy identically the Jacobi equation associated with g^C in TTQ defining generalized Bianchi identities for $Riem_{g^C}$.

4.3. Concluding remarks. It is interesting and useful to compare our results with those of the recent interesting paper [**15**], where conservation laws associated with generalized Lagrangian symmetries, and in particular with divergence symmetries of a Lagrangian, are presented in the framework of infinite order variational bicomplexes (for finite order variational sequences partial results in the same direction have been already obtained in [**23**]).

We notice in particular that variation vector fields $\mathfrak{G}(\bar{\Xi})_V$ which we are considering here are in fact generalized symmetries of $\mathcal{E}(\lambda)$ in the sense of [**6**] and [**15**]. Moreover, let us once more recall that one can represent second variational derivatives as iterated variational Lie derivatives of the Lagrangian and in particular relate them to the Lie derivative of Euler Lagrange morphisms (see also [**9, 10, 13, 28, 29**]). As a consequence of Theorem 1, we can now provide an intrinsic interpretation of the last term appearing in the *characteristic equation* for generalized symmetries derived in [**15**]. From Eq. (18) of [**13**], in fact, it is easy to realize that such term is nothing but the difference between the second variational derivative of λ and the Jacobi morphism (the latter being precisely characterized as the vertical differential of $\mathcal{E}_n(\lambda)$), *up to divergences* - and it vanishes along critical sections.

In other words, the characteristic equation in our case can be written, up to divergences, as follows:

$$\delta_{\mathfrak{G}} \mathcal{E}(\lambda) - \mathcal{E}(\delta_{\mathfrak{G}} \lambda) = [\delta^2_{\mathfrak{G}} \lambda - \mathcal{J}(\lambda, \mathfrak{G}(\bar{\Xi})_V)] \equiv 0 \,.$$

where [ ] denotes the equivalence class in the variational sequence. In the finite order variational sequence such an equivalence class is vanishing by definition also along non critical sections, being the equivalence class of the horizontal differential of contact forms of higher degree (this fact has been already stressed in [**28**]).

We also notice that by adapting the results of [**15**] to our case we would have that *requiring* that $\mathcal{E}(\omega(\lambda, \mathfrak{G}(\bar{\Xi})_V)) = 0$, then $\mathfrak{G}(\bar{\Xi})_V$ should be a divergence symmetry of λ, *i.e.* $\mathcal{L}_{\mathfrak{G}(\bar{\Xi})_V} \lambda$ should be a total divergence. We remark that - because of the gauge-natural invariance of λ - one finds $\mathcal{L}_{\mathfrak{G}(\bar{\Xi})_V} \mathcal{E}(\lambda) = \mathcal{L}_{\mathfrak{G}(\bar{\Xi})} \mathcal{E}(\lambda) = 0$. Since - by Theorem 2 - $\forall\ \mathfrak{G}(\Xi)_V \in \mathfrak{K}$, $\mathcal{E}(\omega(\lambda, \mathfrak{K})) = 0$, we can conclude that $\mathcal{L}_{\mathfrak{G}(\bar{\Xi})_V} \lambda$ is always a total divergence on the kernel of the generalized gauge-natural Jacobi morphism, so providing new important insights on the results of [**15**].

Acknowledgements. Thanks are due to I. Kolář for useful discussions. Special thanks are due to R. Vitolo, collaborator and member of a common research program on higher variations, for helpful remarks.

The second author (M.P.) especially thanks I. Kolář for the kind invitation to take part to the Conference.

The leading idea of the paper has took its first form during the stay of M.P. at the Mathematical Institute of the Silesian University in Opava (Czech Republic), May-October 2000, supported by the Italian Council of Researches through grant n. 203.01.71/03.01.02 and under the supervision of D. Krupka; the scientific contacts had with him, although not directly connected with the second variational derivative, are therefore here gratefully acknowledged.

This work has been partially supported by GNFM of INdAM, MIUR (PRIN 2003) and University of Torino.

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Department of Mathematics, University of Torino
via C. Alberto 10, 10123 Torino, Italy

e-mails: `francaviglia@dm.unito.it`, `palese@dm.unito.it`, `ekkehart@dm.unito.it`