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Symmetry Operators for the Dirac and Hodge-deRham Equations.

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Abstract. We show how tensors satisfying generalisations of Killing's equation naturally arise in the study of symmetry operators of various differential equations. In the case of the Dirac equation, we show how the first-order symmetry operators correspond to Killing-Yano tensors. Since the Hodge-deRham operator $d-d^*$ formally resembles the spinor Dirac operator we analogously examine first-order symmetry operators for the Hodge deRham equation (the analogue of the massive Dirac equation). The results are found to be rather different.

1. Introduction.

Killing vectors have the clearest of geometrical interpretations: their flows generate isometries. That is, a vector field K is Killing if $\mathcal{L}_K g = 0$. From this it follows that K must satisfy Killing's Equation,

$$g(\nabla_X K, Y) + g(\nabla_Y K, X) = 0\,.$$

So if $K^\flat$ is the 1-form related to K by the metric, then the symmetrised covariant derivative of $K^\flat$ must vanish. Killing's equation can be generalised in an obvious way to an equation for higher degree tensors; some sort of symmetrised covariant derivative is required to vanish. One such generalisation is to totally symmetric tensors. These are often known as Killing-Stäckel tensors; we refer to them as simply Killing tensors. Another generalisation is to totally anti-symmetric tensors, known as Killing-Yano tensors. Both these tensor equations have conformally covariant generalisations. Of course it is a simple matter to generalise to equations for tensors with more complicated symmetries. However, for none of these tensor generalisations is there an immediate geometrical interpretation. Nonetheless, the existence of Killing tensors gives rise to interesting consequences. The existence of a Killing tensor in the Kerr spacetime leads to the integrability of the geodesic equations and the separability of the Hamilton-Jacobi equation [**1, 2**]. The existence of Killing-Yano tensors in relativity enable Maxwell's source-free equations to be expressed in terms of a scalar wave equation [**3, 4, 5**].

One place in which Killing tensors, of various types, arise, is in the study of symmetry operators for various important differential equations. By a symmetry operator we simply mean some operator that transforms one solution to another. This is, of course, one of the important consequences of the existence of a Killing vector; the Lie derivative with respect to a Killing vector is a symmetry operator for

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any differential equation constructed from the metric and its associated connection. Sometimes one can go further. For example, if $\triangle$ is the Laplacian on functions and L is a first-order operator then

$$[L, \triangle] = 0$$

if and only if

$$Lf = Kf + cf$$

with K Killing and c constant. That is, all the first-order symmetry operators of the Klein-Gordon equation are given by Killing vectors (up to a constant). More generally, if we look for higher-order symmetries and write

$$L = T^{ab\ldots}\nabla(X_a, X_b, \ldots) + \ldots$$

then for

$$[L, \triangle] = 0$$

we must have

$$\sum_{\mathcal{S}} \nabla_{X_a} T_{bc\ldots} = 0$$

where the notation denotes the totally symmetrised covariant derivative. Thus a Killing tensor is necessary for the existence of a higher-order symmetry [**6**]. It certainly doesn't appear sufficient, in general, although in the absence of a counter-example caution is prudent.[1]

Suppose that, more generally, $\triangle$ denotes any sort of Laplacian (for example, the Laplace-Beltrami operator on forms, or the spinor Laplacian). We may parameterise any differential operator as

$$L = T^{ab\cdots} \bullet \nabla(X_a, X_b, \ldots) + \ldots$$

where now $T^{ab\cdots}$ is a symmetric-tensor-valued tensor of some sort that acts as a linear map denoted by $\bullet$. For example, in the case of the spinor Laplacian $T^{ab\cdots}$ would be symmetric-tensor-valued forms, and the $\bullet$ would denote the Clifford action. Then in order to have

$$[L, \triangle] = 0$$

we must have

$$\sum_{\mathcal{S}} \nabla_{X_a} T_{bc\ldots} = 0\,.$$

Thus we must have some generalised Killing tensor. In general this is not sufficient. (We will see a specific example later.)

The case of first-order symmetries of the Dirac equation is striking: it is necessary and sufficient to have a Killing-Yano tensor in order to construct a first-order linear operator that commutes with the Dirac operator. For the weaker condition of having an operator that R-commutes (and hence corresponds to a symmetry of the massless Dirac equation) it is necessary and sufficient to have a conformal Killing-Yano tensor. Now whereas it is appealing to have this correspondence between symmetry operators and Killing-Yano tensors, without any extra curvature constraints, there are of course severe constraints for these tensors to exist. In the four-dimensional Lorentzian case the most important example of a spacetime admitting a Killing-Yano tensor is Kerr spacetime. The classification of Riemannian manifolds admitting conformal Killing-Yano tensors is the subject of the talk given at this conference by Moroianu [**7**].

[1] IMB thanks M. Eastwood for instilling this prudence.

(Note that there Killing forms and twistor forms are synonyms for Killing-Yano and conformal Killing-Yano tensors.)

Here we will start by reviewing how Dirac symmetry operators are related to Killing-Yano tensors (or Killing forms if you prefer). Although these results were all given in [**8**], the approach we give here provides insight the way KY forms are related to Dirac symmetry operators: indeed it lets us formulate the KY equation directly in terms of the Dirac operator.

Dirac's motivation that lead to the equation that bears his name was to find a find a first-order equation that 'squares' to the Klein-Gordon equation. He introduced the 'gamma matrices' that generate the Clifford algebra. The 'wave function' was required to carry an irreducible representation of this Clifford algebra: that is, it was a spinor field. Now the Laplace-Beltrami operator (on forms) has as 'square root' the Hodge deRham operator $d - d^*$, with d the exterior derivative and d^* its adjoint. The Clifford algebra can be identified with the space of forms, with the Clifford product related to the exterior and interior products. Thus one can write the Hodge deRham operator in terms of the Clifford product so that it formally resembles the Dirac operator. Given the simplicity of the result for the first-order Dirac symmetry operators, it is natural to consider in analogous fashion the case of first-order Hodge deRham symmetries. This we do in section 3. It will be shown that there are important differences.

There is another reason for looking at first-order symmetries of the Hodge deRham equation. Any such symmetry gives a second-order symmetry of the Laplace-Beltrami equation that preserves the degree of forms. We shall elaborate on this in section 4.

We conclude by summarising some outstanding problems.

2. First-Order Dirac Symmetry Operators

Normally one uses the same symbol, ∇, to denote covariant differentiation of spinors and tensors, the meaning being clear from context. Since we will be comparing and contrasting the Dirac operator on spinors with its analogue, the Hodge deRham operator acting on forms, we will use the notation S_X to denote covariant differentiation of a spinor with respect to the vector field X. We then denote the Dirac operator in the 'Feynman-like' way, $\not{S}$, where $\not{S} = e^a \not{S}_{X_a}$ where $\{e^a\}$ is dual to $\{X_a\}$ and the forms act on the spinor field with the Clifford action.

The space of spinors decomposes into even and odd spinors, with the Dirac operator being an odd operator that interchanges these spinor subspaces. Given any operator that commutes with the Dirac operator we can always arrange for one to graded commute, and it proves convenient to work with the graded commutator. Suppose that L is any first-order ($\mathbb{Z}_2$-graded) operator on spinors. Then

$$L = 2\omega^a S_{X_a} + \dots$$

for ω^a some set of (even or odd) forms. These act via the Clifford action. The dots denote terms of lower order. Then the graded commutator is given by

$$[\not{S}, L] = 2i(X^b)\omega^a \left(S^2(X_a, X_b) + S^2(X_b, X_a)\right) + \dots$$

where $i(X)$ denotes the interior derivative (or contraction operator). The second covariant derivative is denoted S^2, so that $S^2(X, Y) = S_X S_Y - S_{\nabla_X Y}$. So for the (graded) commutator to vanish it is necessary that

$$i(X^a)\omega^b + i(X^b)\omega^a = 0\,.$$

This is equivalent to

$$\omega^a = i(X^a)\omega$$

for some (possibly $\mathbb{Z}$-inhomogeneous) form ω. Thus it is necessary that L can be written as

$$L = [\,\not{S}, \omega] - \varphi\,.$$

for some pair of forms ω and φ. Hence having a first-order operator that (graded) commutes with the Dirac operator is exactly equivalent to finding a form ω that satisfies the equation

$$[\,\not{S}^2, \omega] = [\,\not{S}, \varphi] \tag{1}$$

for some form φ. Now $\not{S}^2 = S^2(X_a, X^a) - \frac{1}{4}\mathcal{R}$, with $\mathcal{R}$ the curvature scalar, so $[\,\not{S}^2, \omega] = [S^2(X_a, X^a), \omega]$ and

$$\begin{aligned} [S^2(X_a, X^a), \omega] &= 2\nabla_{X^a}\omega S_{X_a} + \nabla^2(X_a, X^a)\omega \\ [\,\not{S}, \varphi] &= 2i(X^a)\varphi S_{X_a} + \not{\nabla}\varphi\,. \end{aligned}$$

Here $\not{\nabla}$ is the Hodge deRham operator, $\not{\nabla} = e^a\nabla_{X_a} = d - d^*$. Thus (1) is equivalent to the pair of equations

$$\nabla_{X_a}\omega = i(X_a)\varphi \tag{2}$$

$$\nabla^2(X_a, X^a)\omega = \not{\nabla}\varphi\,. \tag{3}$$

Now applying $e^a\wedge$ to (2) gives

$$d\omega = \Pi\varphi \tag{4}$$

where Π scales each form by its degree. So (2) requires that the p-form components of ω satisfy the Killing-Yano equation

$$\nabla_X\omega_p = \frac{1}{p+1} i(X) d\omega_p\,.$$

By differentiating (2) we get $\nabla^2\omega = -d^*\varphi$, whilst from (4) we see that all but the 0-form component of φ is exact, and hence closed. Thus (3) simply reduces to saying that any 0-form component of φ must be constant. So in fact we see that (1) just requires that ω be a Killing-Yano form. Thus the first-order Dirac symmetry operators correspond to Killing-Yano forms.

More generally we can consider the first-order symmetry operators of the massless Dirac equation, when instead of $[L, \not{S}] = 0$ we only require $\not{S}L = M\not{S}$ for some operator M. With a little more work it can be shown that [**8**] the most general first-order symmetry operators are given by

$$K(\omega) = 2i(X^a)\omega\nabla_{X_a} + \frac{p}{p+1} d\omega - \frac{n-p}{n+1-p} d^*\omega$$

where ω is a conformal Killing-Yano form,

$$\nabla_X\omega = \frac{1}{p+1} i(X) d\omega - \frac{1}{n+1-p} X^\flat \wedge d^*\omega\,.$$

3. First-Order Hodge-deRham Symmetry Operators

If we identify the space of differential forms with sections of the Clifford bundle then the Hodge deRham operator can be written as

$$\nabla\!\!\!/ = e^a \nabla_{X_a} = d - d^* \,.$$

Whereas the Hodge deRham operator formally resembles the Dirac operator on spinors, it will be important to note the difference between the spinor covariant derivative operator S_X and the pseudo-Riemannian covariant derivative ∇_X (which is why we here notationally distinguish them). Given the pleasing simplicity of the classification of first-order Dirac symmetry operators, it is natural to examine the analogous Hodge deRham case.

For simplicity we consider even dimensions where the complex Clifford algebra is a total matrix algebra. Since

$$\mathcal{M}_n \otimes \mathcal{M}_n \simeq \mathrm{End}\mathcal{M}_n$$

any linear transformation on differential forms may be given by simultaneous left and right Clifford multiplication. That is, linear transformations on the space of differential forms correspond to the tensor product of the Clifford algebra with itself. If

$$\Omega = \Omega_{IJ} e^I \otimes e^J \,,$$

where $\{e^I\}$ is a basis for the exterior algebra then we define the action on a form ψ by

$$\Omega \bullet \psi = \Omega_{IJ} e^I \psi e^J \,,$$

where the juxtaposing of symbols denotes their Clifford product. The most general first-order linear operator on differential forms is of the form

$$L = 2\Omega^a \bullet \nabla_{X_a} + \dots \,.$$

Now we will proceed analogously to the Dirac case, bearing in mind two important differences: the different connection, and the different action of Ω^a (as compared with ω^a).

First we show that any symmetry operator can be decomposed into symmetry operators so that each factor in Ω^a is either even or odd. Let z denote the volume n-form, so that z is an invertible element of the Clifford algebra. Let $\mathcal{L}_z$ denote the operation of multiplying from the left by z, and $\mathcal{R}_z$ the operation of multiplying from the right. Then we may decompose any operator L with the projectors,

$$\begin{aligned} P_\epsilon L &= \frac{1}{2}(L + \epsilon \mathcal{L}_z L \mathcal{L}_{z^{-1}}) \\ Q_\sigma L &= \frac{1}{2}(L + \sigma \mathcal{R}_z L \mathcal{R}_{z^{-1}}) \end{aligned}$$

where $\epsilon, \sigma = \pm 1$. Clearly P_ϵ and Q_σ commute, since $\mathcal{L}_z$ and $\mathcal{R}_z$ do, and so we can decompose L into four parts, $L_{\epsilon\sigma} = P_\epsilon Q_\sigma L$. These components of L correspond to the operators where the first and second factors of Ω^a are separately even and odd. Now since $\mathcal{L}_z$ anti-commutes with $\nabla\!\!\!/$, and $\mathcal{R}_z$ commutes with $\nabla\!\!\!/$, if L commutes with $\nabla\!\!\!/$ so do the components $L_{\epsilon\sigma}$. So with no loss in generality we will start with the assumption that L is made from tensors with each factor either even or odd. Finally we note that if L has the first factor odd, and L commutes with $\nabla\!\!\!/$, then $\mathcal{L}_z L$ has the first factor odd, but it anti-commutes.

We therefore assume that both factors of Ω^a are separately even or odd, and choose to work with the graded (by the first factor) commutator. This is then expressed by

$$[\not\nabla, L] = 2i(X^b)\Omega^a\left(\nabla^2(X_a, X_b) + \nabla^2(X_b, X_a)\right) + \dots$$

where the interior derivative $i(X)$ acts on the first factor of Ω^a. So for the graded commutator to vanish we must have

$$\Omega^a = i(X^a)\Omega$$

for some exterior-algebra-valued form Ω. Thus it is necessary that L can be written as

$$L = [\not\nabla, \Omega] - \Phi\,.$$

So the existence of a symmetry operator corresponds to the existence of an Ω such that

$$[\not\nabla^2, \Omega] = [\not\nabla, \Phi] \tag{5}$$

for some exterior-algebra-valued form Φ. (For brevity we will refer to tensors such as Φ as 'double forms'.) Equation (5) resembles, of course, the expression we had in the Dirac case.

We can express the Laplace-Beltrami operator as $\not\nabla^2 = \nabla^2(X_a, X^a) - \frac{1}{4}\mathcal{R} - \frac{1}{4}R\bullet$ where $R = R_{ab}\otimes e^{ab}$, with R_{ab} the curvature 2-forms, and R acts on forms with the 'left-right' Clifford action. This extra curvature term denotes one important departure from the Dirac case. Expanding the terms gives

$$\begin{aligned} [\not\nabla^2, \Omega] &= 2\nabla_{X_a}\Omega\nabla_{X^a} + \nabla^2(X_a, X^a)\Omega - \frac{1}{4}[R, \Omega] \\ [\not\nabla, \Phi] &= 2i(X^a)\Phi\nabla_{X_a} + \not D\Phi \end{aligned}$$

where $\not D = e^a\nabla_{X_a} = D - D^*$. Although $\not D$ resembles $\not\nabla$, they act on different things. Whereas the Laplace-Beltrami operator $\not\nabla$ only acts on forms, the operator $\not D$ acts on the more general space of 'double forms'. The operator D is often called the covariant exterior derivative. Thus (5) is equivalent to the pair of equations

$$\nabla_{X_a}\Omega = i(X_a)\Phi \tag{6}$$

$$\nabla^2(X_a, X^a)\Omega - \frac{1}{4}[R, \Omega] = \not D\Phi\,. \tag{7}$$

Now (6) implies that

$$D\Omega = \Pi\Phi\,, \tag{8}$$

where Π acts on the left-hand factor in Φ. Note that (8) does not determine the 0-form component of Φ (where the grading is that of the first factor) so

$$\Phi = \Pi^{-1}D\Omega + 1\otimes T$$

for some arbitrary form T. Thus (8) enables (6) to be recognised as some sort of generalised Killing-Yano equation, where the p-form components satisfy

$$\nabla_X\Omega_p = \frac{1}{p+1}i(X)D\Omega_p\,. \tag{9}$$

Now as well as the generalised Killing-Yano equation (9) we still need to take care of (7). By differentiating (6) we get

$$\nabla^2(X_a, X^a)\Omega = -D^*\Phi$$

and this can be used to reduce (7) to

$$-\frac{1}{4}[R,\Omega_p] = \frac{1}{p+1}D^2\Omega_p + e^a \otimes \nabla_{X_a}T\,. \tag{10}$$

So there are obvious similarities with the Dirac case, but important differences. It is necessary that Ω satisfy some generalisation of the Killing-Yano equation, namely (9), but this is not, at least manifestly, sufficient: we need to satisfy (10).

There is of course one trivial case in which (10) is automatically satisfied; the case of flat space. Since D^2 involves curvature terms we can, in the flat case, satisfy (10) by simply choosing $T = 0$. In this case we may write $\Omega = \Omega^I \otimes e_I$ with $\{e_I\}$ a *parallel* basis for the exterior algebra. Then (9) requires that each of the forms Ω^I is a Killing-Yano form. So we see that in the case of flat space, the first-order symmetries of the Hodge deRham operator are made from a Killing-Yano form, as in the Dirac case, composed with right multiplication by a parallel form.

There is one more case in which we had better be able to satisfy (10), that being the case in which we have a Killing vector. Clearly the Lie derivative *is* a first-order linear symmetry operator in this case. Certainly any Killing-Yano form satisfies (9). For the special case in which Ω is just a form (rather than a 'double-form') $D\Omega = d\Omega$ and so $D^2\Omega = 0$. For the case of $\Omega = K^\flat$ we may write the curvature term as

$$-\frac{1}{4}[R,K^\flat] = e^c \otimes e^a \wedge R(X_c,X_a)K^\flat\,.$$

Killing's equation has the integrability condition

$$e^a \wedge R(Y,X_a)K^\flat = \frac{1}{2}\nabla_Y dK^\flat$$

and so we may satisfy (10) by choosing

$$T = \frac{1}{2}dK^\flat$$

so that

$$\Phi = \frac{1}{2}\left(dK^\flat \otimes 1 + 1 \otimes dK^\flat\right)\,.$$

This gives us the symmetry operator

$$L = 2\nabla_K + \frac{1}{2}\left(dK^\flat \otimes 1 - 1 \otimes dK^\flat\right)$$

which we may recognise [**9**] as a Lie derivative

$$L = 2\mathcal{L}_K\,.$$

As we've already noted, any Killing-Yano form certainly satisfies (9). Suppose now that ω is a Killing-Yano p-form. Then is (10) satisfied for some choice of T? In this case (10) reduces to

$$-\frac{1}{2}R(X_a,X_b)\omega \otimes e^{ab} = e^a \otimes \nabla_{X_a}T\,.$$

On the left-hand side we have the tensor product of a p-form and a 2-form. So there is no way we can choose T to equate these terms, unless of course $R(X,Y)\omega = 0$. This can only be the case if $\nabla_X d\omega = 0$, or equivalently $\nabla^2(X,Y)\omega = 0$. There certainly are Killing-Yano forms for which this is not true. (The Killing-Yano 2-form in Schwartzchild, for example.)

In fact it is hard to see how (10) can be satisfied, save for the case of a Killing vector, without imposing extra curvature conditions. Certainly we can say that (10) is not automatically satisfied whenever we have a solution to (9).

4. Second-order operators from first order ones

Given any first-order symmetry operator there's an easy way to make a second-order one: apply it twice! For example, for the case of the scalar Laplacian we have first-order symmetry operators given by Killing vectors. So we have the second-order symmetry operator $\mathcal{L}_{K_1}\mathcal{L}_{K_2} + \mathcal{L}_{K_2}\mathcal{L}_{K_1}$. This can be written in terms of the trivial Killing tensor $K_1 \otimes K_2 + K_2 \otimes K_1$. If we have a second-order symmetry operator given by a non-trivial Killing tensor (and recall that we can't be sure that all such Killing tensors do give symmetry operators) then that symmetry operator cannot be made out of two first-order symmetry operators *of that scalar equation.* However, suppose that we had a symmetry operator of the equation $\triangle\varphi = m^2\varphi$, with $\triangle$ the Laplace-Beltrami operator, and φ a form of arbitrary degree. Such a symmetry operator need not preserve the degree of a form. For example, we could map 0-form solutions to p-form solutions, and then maybe map p-form solutions back to 0-form solutions. In this way we could have a second-order symmetry operator for the scalar equation, that was obtained from first-order operators for a broader class of solutions (forms of arbitrary degree, not just 0-forms). This is one motivation for looking at first-order symmetries of the Hodge deRham equation, it gives a way of constructing second-order symmetries of the Laplace-Beltrami equation. This motivation was alluded to in the introduction. Now we've examined the first-order Hodge deRham symmetry operators, and the results were perhaps a little disappointing. Nonetheless, we use those results to illustrate how the ideas come together.

Suppose that L is any operator such that $[L, \triangle] = 0$. Then since the Laplace-Beltrami operator is homogeneous of degree 0 (it preserves the degree of a form), the homogeneous components of L are separately symmetry operators. Suppose that L is homogeneous of degree p, that is, it maps q-forms to $p+q$-forms. Since $\triangle$ is self-adjoint, then if L is a symmetry operator so is L^*. If L is homogeneous of degree p then L^* is homogeneous of degree $-p$. Thus (for example) L^*L is a second-order symmetry operator of degree 0.

For a Killing-Yano p-form ω with $\nabla d\omega = 0$,

$$K(\omega) = 2i(X^a)\omega\nabla_{X_a} + \frac{p}{p+1}d\omega$$

is a Hodge deRham symmetry operator. Hence $[K(\omega), \triangle] = 0$. Since we're here assuming that $d\omega$ is parallel, both terms in $K(\omega)$ separately commute with $\triangle$. As we've noted, the homogeneous components of a symmetry operator must be symmetry operators. Taking the component of highest degree gives the symmetry operator L, where

$$L = i(X^a)\omega \wedge \nabla_{X_a}\,. \tag{11}$$

So we have the symmetry operator that commutes with the scalar Laplacian

$$-L^*(\omega)L(\omega) = T^{ab}\nabla^2(X_a, X_b) + \nabla_{\mathrm{div}T}$$

where

$$2T^{ab} = i(X^a)\omega \otimes i(X^b)\omega + i(X^b)\omega \otimes i(X^a)\omega\,.$$

We make one more observation about the symmetry operator L given in (11). One of the key properties of the Lie derivative is that it commutes with the exterior derivative. This property enables the Lie derivative to be extended to an operator that acts as a graded derivation on the exterior algebra, such as to commute with the exterior derivative [**10**]. Such generalised Lie derivatives are defined by vector-valued forms. Whenever we have a metric we can map any p-form ω into a vector-valued $(p-1)$-form $\hat{\omega}$,

$$\hat{\omega} = i(X^a)\omega \otimes X_a \,.$$

In general, whenever we have a metric, and associated connection, we can relate the (generalised) Lie derivative to the covariant derivative. For the case of a Killing-Yano p-form ω we have

$$\mathcal{L}(\hat{\omega}) = i(X^a)\omega \wedge \nabla_{X_a} + (-1)^p \frac{p}{p+1} i(X^a)d\omega \wedge I(X_a) \,. \tag{12}$$

Now the first term on the right-hand side is our operator, given in (11), that commutes with $\triangle$ in the case in which $d\omega$ is parallel. In that case the second term on the right-hand side of (12) certainly commutes with $\triangle$. So we see that, in this special case in which $d\omega$ is parallel, we can construct Lie derivatives from Killing-Yano forms, such that

$$[\mathcal{L}(\hat{\omega}), \triangle] = 0 \,.$$

Now of course, having $d\omega$ parallel is, as we've pointed out, very special. Nonetheless, the above result is worthy of mention.

5. Conclusions

By looking for symmetry operators for various equations, related to some sort of Laplacian, we are naturally lead to various sorts of Killing tensors. The only case in which we are able to say for sure that, in general, the existence of such a tensor guarantees a corresponding symmetry operator, is that of the first-order symmetry operators for the Dirac equation. Although the corresponding case of the Hodge deRham operator appears disappointing, it serves as an important example, showing that in general having a generalised Killing tensor, so as to make the highest-order terms in the commutator vanish, is not enough in order to have a symmetry operator.

Here, to keep things simple, we only presented the case of first-order symmetries of the massive Dirac equation, although we refer to the massless result. Proceeding by analogy we consider symmetry operators that commute with the Hodge deRham operator. Of course we can consider the more general case in which we have a symmetry of the massless equations (of which Maxwell's are the prime example). This can be done by extending what we have presented here.

The section on second-order symmetry operators warrants further investigation. In the four-dimensional Lorentzian case at least, we can construct a second-order symmetry operator for the conformally-covariant scalar Laplace equation from a conformal Killing-Yano 2-form [**11**]. It remains to be seen whether this result generalises to higher dimensions, but certainly suggests that we don't yet have the whole truth.

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