Abstracts of the plenary lectures

Eastwood, Michael: Conjugate Functions and Semiconformal Mappings

Abstract:

Suppose that M is a Riemannian manifold and $f : M \to \mathbb{R}$ is a smooth function. We may ask whether there is a smooth function $g : M \to \mathbb{R}$ such that

$$||\nabla f|| = ||\nabla g|| \quad \text{and} \quad \langle \nabla f, \nabla g\rangle = 0. \tag{1}$$

In this case, we shall say that f and g are conjugate functions. Suppose M is 2-dimensional. If f has a conjugate, then f is harmonic. Locally, the converse is true: harmonic functions admit conjugates. What happens for manifolds of dimension ≥ 3? Is there, for example, a partial differential equation that characterises those f that admit a conjugate?

One motivation for the condition (1) comes from the theory of harmonic morphisms . A mapping $F : M \to N$ between Riemannian manifolds is said to be a harmonic morphism if and only if harmonic functions locally defined on N pull back by F to harmonic functions locally defined on M. The Hopf fibration $S^3 \to S^2$ is a harmonic morphism. If the target manifold is $\mathbb{R}^2$ and we write $F = (f, g)$, it is clear that each of f and g should be harmonic (since the coördinate functions on $\mathbb{R}^2$ are harmonic). The other condition that F be a harmonic morphism in this case was derived by Jacobi in 1848. It is (1). It is especially natural to isolate this condition when one notices that it is conformally invariant (whereas being harmonic is only conformally invariant in dimension 2). The condition (1) on $F = (f, g)$ also has a good geometric interpretation. At points where F is a submersion, it says that its derivative dF is conformal on the subspace orthogonal to its null space: F is semiconformal (some authors use (weakly) horizontally conformal).

Here are some examples of pairs $F = (f, g)$ on $\mathbb{R}^3 \setminus x_1$-axis enjoying (1):–

- $f = {x_1}^2 - {x_2}^2 - {x_3}^2 \qquad g = 2x_1\sqrt{{x_2}^2 + {x_3}^2}$,
- $f = x_2\dfrac{{x_1}^2 + {x_2}^2 + {x_3}^2}{{x_2}^2 + {x_3}^2} \qquad g = x_3\dfrac{{x_1}^2 + {x_2}^2 + {x_3}^2}{{x_2}^2 + {x_3}^2}$,
- $f = \dfrac{(1 - ||x||^2/2)x_2 + \sqrt{2}x_1x_3}{{x_2}^2 + {x_3}^2} \qquad g = \dfrac{(1 - ||x||^2/2)x_3 + \sqrt{2}x_1x_2}{{x_2}^2 + {x_3}^2}$.

This last one is the Hopf fibration conformally rearranged using stereographic coördinates: $\mathbb{R}^3 \hookrightarrow S^3 \to S^2 \hookleftarrow \mathbb{R}^2$. In none of these examples is f harmonic (whereas it is shown by Ababou, Baird, and Brossard that if f and g are polynomial and conjugate on $\mathbb{R}^n$ for any $n \geq 3$, then they must be harmonic).

This talk will show, for example, show that neither of the functions

$$f = x_1x_2x_3 \qquad \text{nor} \qquad f = x_1^3 + x_2^3 + x_3^3$$

admit a conjugate, even locally. The first of these is easily dealt with by a differential inequality that must be satisfied in case f admit a conjugate. To express this inequality, let us write

$$f_i = \nabla_i f \qquad f_{ij} = \nabla_i\nabla_j f \qquad \text{and so on,}$$

where ∇_i is the metric connection (or just coördinate derivative on $\mathbb{R}^n$). Also, let us 'raise and lower' indices with the metric in the usual fashion and write a repeated index to denote a sum over that index. Thus, $f^i{}_i = \Delta f$ is the Laplacian and $f^i g_i = \langle \nabla f, \nabla g \rangle$. Given a smooth function f on a smooth 3-manifold M, let

$$X = 2f_i{}^j f_j f^{ik} f_k - f^i f_i f^{jk} f_{jk} + f^i f_i (f^j{}_j)^2.$$

Theorem. If f admits a conjugate, then $X \leq 0$.

This is deals with $f = x_1 x_2 x_3$: a computation gives $X = 6(x_1 x_2 x_3)^2$. This criterion, however, is not always sufficient:–

$$f = x_1^3 + x_2^3 + x_3^3 \quad \Rightarrow \quad X|_{(1,1,1)} = 7776 \quad \text{whilst} \quad X|_{(1,2,-2)} = -1944,$$

which does not rule out f having a conjugate near $(1, 2, -2)$. Even worse,

$$f = \log\left(\frac{\sqrt{1 + x_2{}^2 + x_3{}^2} - 1}{\sqrt{x_2{}^2 + x_3{}^2}}\right) + 2\sqrt{1 + x_2{}^2 + x_3{}^2} \tag{2}$$

yields

$$X = -2\frac{(3 + 2x_2{}^2 + 2x_3{}^2)(1 + 2x_2{}^2 + 2x_3{}^2)^2}{(x_2{}^2 + x_3{}^2)^2(1 + x_2{}^2 + x_3{}^2)^4},$$

which is everywhere negative though, in fact, f does not admit a conjugate.

Given the conformally invariant nature of the problem, it is not surprising that the quantity X is itself conformally invariant (of weight -6):–

$$g_{ij} \mapsto \widehat{g}_{ij} = \Omega^2 g_{ij} \quad \Rightarrow \quad X \mapsto \widehat{X} = \Omega^{-6} X.$$

Here are some more conformal invariants:–

$$cJ = f^i f_i \qquad Z = f^{ij} f_i f_j + J f^j{}_j \qquad Y = Z^2 - 2JXR = J f^i \nabla_i Z - 2Z f^i \nabla_i J \qquad S = J f^i \nabla_i X - 3X f^i \nabla_i J.$$

They are all even: invariant under change of orientation. Here is an odd conformal invariant:–

$$V = \epsilon^{jk\ell}(J \nabla_\ell (f^i H_{ij} f_k) - 3 f^i H_{ij} f_k \nabla_\ell J)$$

where $\epsilon_{jk\ell}$ is the volume form and

$$H_{ij} = 2J \nabla_i \nabla_j J - 3(\nabla_i J)(\nabla_j J) - 4R_{ij} J^2 \quad \text{(cf. Schwarzian)},$$

where R_{ij} is the Ricci curvature.

Theorem. If f admits a conjugate, then it satisfies, at points where $X < 0$, the following PDE:–

$$2(ZS - 2XR + 2XY)^2 + XV^2 = 0.$$

This criterion is sufficient immediately to prevent $f = x_1{}^3 + x_2{}^3 + x_3{}^3$ and the example ((2)) from having conjugates. On the other hand

$$f = \log\left(\frac{\sqrt{1 + x_2{}^2 + x_3{}^2} - 1}{\sqrt{x_2{}^2 + x_3{}^2}}\right) + \sqrt{1 + x_2{}^2 + x_3{}^2} \Rightarrow X = -\frac{2}{(x_2{}^2 + x_3{}^2)^2},$$

which is everywhere negative and f also satisfies the PDE. In fact, f has a conjugate: $g = x_1 + \arctan(x_3/x_2)$.

With more work it is possible, for functions of 3 variables, to find further partial differential equations and inequalities that completely characterise those f that admit a conjugate.

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Hall, Graham: Geometric Foundations of Classical General Relativity Theory

Abstract:

The use of geometrical techniques to solve problems in physics goes back at least as far as the ancient Greek civilisation. They were also used to great effect by Newton. Following the celebrated work of Riemann, *differential* geometric techniques were applied to Lagrangian classical mechanics and the emphasis on certain particular coordinate systems was relaxed. This "covariant" approach to Newtonian mechanics, which was inspired by the work of Riemann, Lagrange, Hamilton and others, surely made the transition to general relativity much less painful. However, the presence of Newton's absolute space and time in the classical theory meant that there was a fixed background Euclidean space and a fixed time measure which affected the physics but which itself was unaffected by it. The physical quantities (e.g. the gravitational and electromagnetic fields) simply "lived" in this unchangeable absolute space and time. This strange state of affairs, which philosophers found disturbing, is essentially corrected in Einstein's general relativity theory. This latter theory takes as its model a geometry of the Riemannian type where the gravitational field is represented by the metric tensor. Thus the gravitational field and the geometry become one and a reciprocal relationship between geometry and physics is established.

One consequence of Newton's theory of gravitation, and which is strongly supported by experimental evidence, is the so called *principle of equivalence.* According to this (and speaking classically) a spherically symmetric non-spinning chargeless "test" particle has its acceleration determined for it by the gravitational field in which it exists. In this sense the gravitational field is an *acceleration* field which is imparted indiscriminately to any such particle irrespective of its makeup. But a similar property is also possessed by the so called inertial forces which are brought into action in a frame of reference which is accelerating with respect to absolute space. Newton's theory, by its very nature, claimed a distinction between gravitational and inertial forces (and hence the ability to distinguish inertial from non-inertial frames). Einstein's theory, on the other hand, denied this distinction and was then able to take advantage of this denial, and the success of the special theory of relativity, to construct a theory of gravitation within a 4-dimensional space-time manifold admitting a metric of Lorentz signature taken as $(-1, +1, +1, +1)$. Moreover, Einstein could then avail himself of the recently discovered tensor calculus to describe his theory in a way which was independent of the observer's frame (coordinates) and thus satisfied his *principle of covariance.*

So general relativity consists of a space-time (M, g) where M is a 4-dimensional manifold and g is a Lorentz metric on M. For technical and physical reasons M is taken as being Hausdorff and connected. Einstein supplied field equations for the determination of g, given the distribution of matter and energy in M. These equations consist of a collection of ten second order partial differential equations and, not surprisingly, they are difficult to solve. In spite of this many "exact" solutions of these equations are known, arising usually after the imposition of certain simplifying assumptions. For example, one could decide that the solution required possessed certain symmetries and which were described in terms of a Lie algebra of Killing vector

fields on M. Alternatively (or in addition), one could specify the algebraic type of the Weyl tensor arising from g on M (following the classification of this latter tensor by Petrov and the physical interpretation of this classification subsequently developed). A significant number and variety of such exact solutions are now known. Amongst the successes in these exact solutions are the one body (Schwarzschild) solution which solved the long standing problem regarding the orbit of the planet Mercury and which predicted, successfully, the phenomenon of light bending, solutions (the Friedmann-Robertson-Walker metrics) which represent a smeared out, simplified expanding universe, a generalisation of the one body problem representing a rotating star (the Kerr metric) and solutions suggesting gravitational radiation.

In general relativity a signal, that is, information, is propagated along a "causal" curve whose tangent vector v satisfies $g(v,v) \leq 0$ at each point of the curve (and hence the assumption that M is connected for, otherwise, the components of M would be incapable of physical contact). Thus the physics of general relativity imposes a *causal* structure on M. One would wish such a causal structure to be, in some sense, well behaved and (at least) to exclude closed causal curves and (for reasons of stability) even "almost" closed causal curves. It turns out that any causal structure leads to a natural topology on M which may differ from the manifold topology on M but which, for causally well behaved space-times, does not. Thus the natural (manifold) topological structure of space-time follows from a physically acceptable causal structure.

A "real" gravitational field is taken as one where the Levi-Civita connection Γ arising from g is not flat. Hence one is led to ask questions such as how tightly Γ, through its geodesic structure or through its curvature (or sectional curvature) structure, determines the metric g. It turns out that, in general, the answer to each of these questions is that the determination is unique up to, at most, a constant conformal rescaling of g (that is, up to units of space-time measurement).

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Hwang, Jun-Muk: Rigidity of surjective holomorphic maps between complex projective manifolds

Abstract:

Let X and Y be two complex projective algebraic manifolds. We will discuss the following question:

Given a surjective holomorphic map $f : X \to Y$, what are the possible deformations of f?

Denote by $Hol^s(X,Y)$ the set of all surjective holomorphic maps from X to Y. It is known that $Hol^s(X,Y)$ consists of countably many irreducible components each of which is a complex variety of finite dimension. Our question is to describe the components of $Hol^s(X,Y)$ which contain the point corresponding to f.

An obvious way to deform f is to compose it with elements of $Aut_o(Y)$, the identity component of the group of biholomorphic automorphisms of Y. More generally, suppose there exists a complex manifold Z and surjective holomorphic maps $f_2 : X \to Z$ and $f_1 : Z \to Y$ such that f factors as $f = f_1 \circ f_2$. If $Aut_o(Z)$ is non-trivial, we can deform f by $f_1 \circ g \circ f_2$ with $g \in Aut_o(Z)$. Are there any other ways to deform f?

We will discuss two recent results on this problem. The first one concerns the case when Y is not uniruled, namely, when Y is not covered by holomorphic images of the Riemann sphere $\mathbf{P}_1$. In this case, the recent work [HKP] shows that any surjective holomorphic map $f : X \to Y$ can be factored into $f = f_1 \circ f_2$ such that $f_1 : Z \to Y$ is an unramified finite covering and all deformations of f come from $Aut_o(Z)$ in the

manner described above. In particular, each component of $Hol^s(X,Y)$ is biholomorphic to a complex torus. Moreover if Y is simply connected, then $Hol^s(X,Y)$ is countable. The proof uses Miyaoka's semi-positivity theorem [Mi] in an essential way. Roughly speaking, if a deformation of f exists which does not come from $Aut_o(Y)$, we get a multi-valued holomorphic vector field on Y. Such a vector field induces a quotient sheaf of the cotangent bundle of Y which is semi-negative. But Miyaoka's theorem says that any quotient of the cotangent bundle of a non-uniruled manifold must be semi-positive. This implies that we get a flat vector bundle on Y, from which the unramified cover Z can be defined. This result gives a fairly satisfactory answer to our question in case Y is not uniruled.

By this result, the more interesting case of our question is when Y is uniruled. But here the answer is not so simple. One special case to note is when the target Y is a complex projective space $\mathbf{P}_n$. By definition, a complex projective algebraic manifold X is a submanifold of a complex projective space $\mathbf{P}_N$ for some large N. If we choose a projective subspace $\mathbf{P}_n \subset \mathbf{P}_N$ suitably, there exists a surjective projection $f : X \to \mathbf{P}_n$ if $n \leq \dim(X)$. Moreover, there are various ways to project. In fact, the study of $Hol^s(X, \mathbf{P}_n)$ belongs to the subject of the theory of linear systems on X, namely, the study of line bundles on X and their holomorphic sections. This is a huge subject with a long history. In general, the structure of $Hol^s(X,Y)$ for $Y = \mathbf{P}_n$ is quite complicated and depends heavily on X. This is also the case when Y is a $\mathbf{P}_n$-bundle over another manifold.

For this reason we will consider the case when Y is different from $\mathbf{P}_n$ and the second Betti number of Y is 1. The latter assumption implies that Y can not be fibered over another manifold. Under this assumption, we conjecture that all deformations of a surjective holomorphic map $f : X \to Y$ come from $Aut_o(Y)$. In [HM1], [HM2] and [HM3], this conjecture was proved for many cases which cover almost all known examples. The basic idea is to use rational curves of minimal degree on Y to define a geometric structure on Y. More precisely, let $\mathcal{C}_x \subset \mathbf{P}T_x(Y)$ be the subvariety of the projectivized tangent space of Y at a general point $x \in Y$ consisting of tangent directions to rational curves of minimal degree through x. This $\mathcal{C}_x$ is called the variety of minimal rational tangents at x and the totality of $\mathcal{C}_x$ as x varies over general points of Y defines a geometric structure on Y. For many examples, the variety of minimal rational tangents $\mathcal{C}_x$ is non-linear. In such cases, we can prove an extension theorem for biholomorphic maps preserving this geometric structure, which is analogous to Liouville's theorem for conformal structure. This extension theorem is a consequence of the fact that minimal rational curves can be regarded as 'null-geodesics' with respect to the geometric structure. This extension theorem can be used to show that the multi-valued vector field on Y induced by deformations of $f : X \to Y$ is univalent and the deformation of f come from $Aut_o(Y)$.

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Leon, Manuel: Geometric and numerical aspects of nonholonomic and vakonomic dynamics

Abstract:

Assume that $L : TQ \longrightarrow \mathbb{R}$ is a lagrangian function subjected to some constraints geometrically interpreted as a submanifold M of TQ. There are two alternative ways to describe the corresponding dynamics. The first one uses the Lagrange-D'Alembert principle and leads us to the nonholonomic equations of motion. The second one uses a variational principle and produces the so-called vakonomic dynamics. Both formulations have received much attention in the last years (see ([5],[6] for two recent surveys). Nonholonomic dynamics are useful for Mechanics; on the other side, vakonomic dynamics gives a nice description of optimal control theory.

The purpose of this talk is to give an introduction to both kind of dynamics. In addition, we will present some new developments in the subject:

tem We discuss the relations between nonholonomic and vakonomic dynamics ([9]).

We will introduce nonholonomic and vakonomic brackets ([1], [2], [7]).

We will study the existence of invariant measures in Chaplygin systems ([3]).

We will describe nonholonomic constraint systems with variable rank ([8]).

We will construct nonholonomic integrators ([11], [12].

We will introduce a new principle to handle mechanical systems with higher order constraints which appear in the nonholonomic motion of deformable bodies ([4]).

We will analyze singular control systems ([10]). and construct geometric integrators for control systems ([13]).

We will describe dynamics on Lie algebroids ([14],[15]).

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Moroianu, Andrei: Killing and Twistor Forms in Riemannian Geometry

Abstract:

Killing vector fields are important objects in Riemannian geometry. They are by definition infinitesimal isometries, *i.e.* their flow preserves a given metric. The existence of Killing vector fields determines the degree of symmetry of the manifold. Slightly more generally one can consider *conformal vector fields*, *i.e.* vector fields whose flows preserve a conformal class of metrics. The covariant derivative of a vector field can be seen as a section of the tensor product $\Lambda^1 M \otimes TM$ which is isomorphic to $\Lambda^1 M \otimes \Lambda^1 M$. This tensor product decomposes under the action of $O(n)$ as

$$\Lambda^1 M \otimes \Lambda^1 M \cong \mathbb{R} \oplus \Lambda^2 M \oplus S^2_0 M,$$

where $S^2_0 M$ is the space of trace–free symmetric 2–tensors, identified with the *Cartan product* of the two copies of $\Lambda^1 M$. A vector field X is a conformal vector field if and only if the projection on $S^2_0 M$ of ∇X vanishes.

More generally, the tensor product $\Lambda^1 M \otimes \Lambda^p M$ decomposes under the action of $O(n)$ as

$$\Lambda^1 M \otimes \Lambda^p M \cong \Lambda^{p-1} M \oplus \Lambda^{p+1} M \oplus \mathcal{T}^{p,1} M,$$

where again $\mathcal{T}^{p,1} M$ denotes the Cartan product. As natural generalizations of conformal vector fields, *twistor p-forms* are defined to be p–forms ψ such that the projection of $\nabla\psi$ onto $\mathcal{T}^{p,1} M$ vanishes.

Coclosed twistor p–forms are called *Killing forms*. For $p = 1$ they are dual to Killing vector fields. Note that parallel forms are trivial examples of twistor forms.

Killing forms, as generalization of Killing vector fields, were introduced by K. Yano in . Twistor forms were introduced later on by S. Tachibana , for the case of 2–forms, and by T. Kashiwada , in the general case.

The composition of the covariant derivative and the projection $\Lambda^1 M \otimes \Lambda^p M \to \mathcal{T}^{p,1} M$ defines a first order differential operator T, which was already studied in the context of Stein–Weiss operators (c.f.). As forms

in the kernel of T, twistor forms are very similar to twistor spinors in spin geometry, which were first studied in . An explicit construction relating these two objects can be found in .

The special interest for twistor forms in the physics literature stems from the fact that they can be used to define quadratic first integrals of the geodesic equation, *i.e.* functions which are constant along geodesics. Hence, they can be used to integrate the equation of motion, which was done for the first time by R. Penrose and M. Walker in . More recently Killing forms and twistor forms have been successfully applied to define symmetries of field equations (c.f. ,).

The aim of this talk is to report on recent progress which has been made towards the classification of oriented simply connected compact Riemannian manifolds with non–generic holonomy admitting twistor or Killing forms.

Let (M^n, g) be an oriented simply connected compact Riemannian manifold whose holonomy group Hol(M) is strictly included in SO_n. By the Berger–Simons holonomy theorem, we have to consider 3 cases:

M is a symmetric space of compact type

M is a Riemannian product $M = M_1 \times M_2$

M belongs to one of the following classes:

Kähler manifolds, with holonomy U_m, SU_m $(n = 2m)$ or Sp_k $(n = 4k)$

Quaternion–Kähler manifolds, with holonomy $\mathrm{Sp}_k \cdot \mathrm{Sp}_1$ $(n = 4k)$

Joyce manifolds, with holonomy G_2 $(n = 7)$ or Spin_7 $(n = 8)$

The first case is studied in , where symmetric spaces of compact type admitting non–parallel Killing p–forms for $p \geq 2$ are completely classified. The similar problem for twistor forms is still open.

The existence of twistor forms on Riemannian products is completely understood. I will explain in my talk that they are, roughly speaking, generated by twistor forms on one of the factors.

Finally, suppose that the holonomy group of M belongs to the Berger list. If M is Kähler, twistor forms on M are in a certain sense characterized by Hamiltonian 2–forms, and there are no non–parallel Killing p–forms for $2 \leq p < n$ (cf.). If M is quaternion–Kähler or has exceptional holonomy, then M carries no non–parallel Killing p–forms for $2 \leq p < n$ (cf. ,). On the other hand, it is not yet known whether such spaces could carry non–trivial twistor forms, although there is some evidence that the answer should be negative (cf.).

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Uhlmann, Gunther: Boundary rigidity and the Dirichlet-to-Neumann map

Abstract:

Let (M, g) be a compact Riemannian manifold with boundary ∂M. Let $d_g(x, y)$ denote the geodesic distance between x and y. The inverse problem we address in this talk is whether we can determine the Riemannian metric g knowing $d_g(x, y)$ for any $x \in \partial M$, $y \in \partial M$. This problem arose in rigidity questions in Riemannian geometry [M], [C], [Gr]. For the case in which M is a bounded domain of Euclidean space and the metric is conformal to the Euclidean one, this problem is known as the inverse kinematic problem which arose in Geophysics and has a long history (see for instance [R] and the references cited there).

The metric g cannot be determined from this information alone. We have $d_{\psi^* g} = d_g$ for any diffeomorphism $\psi : M \to M$ that leaves the boundary pointwise fixed, i.e., $\psi|_{\partial M} = \mathbb{1}$, where $\mathbb{1}$ denotes the identity map and $\psi^* g$ is the pull-back of the metric g. The natural question is whether this is the only obstruction to unique identifiability of the metric. It is easy to see that this is not the case. Namely one can construct a metric g and find a point x_0 in M so that $d_g(x_0, \partial M) > \sup_{x,y\in\partial M} d_g(x, y)$. For such a metric, d_g is independent of a change of g in a neighborhood of x_0. The hemisphere of the round sphere is another example.

Therefore it is necessary to impose some a-priori restrictions on the metric. One such restriction is to assume that the Riemannian manifold is **simple**, i.e., given two points there is a unique geodesic joining the points and ∂M is strictly convex. ∂M is strictly convex if the second fundamental form of the boundary is positive definite in every boundary point.

R. Michel conjectured in [M] that simple manifolds are boundary distance rigid that is d_g determines g uniquely up to an isometry which is the identity on the boundary. This is known for simple subspaces of Euclidean space (see [Gr]), simple subspaces of an open hemisphere in two dimensions (see [M]), simple

subspaces of symmetric spaces of constant negative curvature [BCG], simple two dimensional spaces of negative curvature (see [C1] or [O]). We remark that simplicity of a compact manifold with boundary can be determined from the boundary distance function.

Recently we have shown the conjecture to be valid in two dimensions [PU].

Theorem. Let $(M, g_i), i = 1, 2$, be two Riemannian metrics on a compact, simple Riemannian manifold with boundary. Assume

$$d_{g_1}(x,y) = d_{g_2}(x,y) \quad \forall (x,y) \in \partial M \times \partial M$$

then there exists a diffeomorphism $\psi : M \to M$, $\psi|_{\partial M} = Id$, so that

$$g_2 = \psi^* g_1.$$

As was pointed out in [I] Theorem 1 together with the results of [I] implies the following

Theorem. Let (M, g_1) be a compact simple Riemannian manifold and g_2 another metric on M such that $d_{g_1}(x,y) \geq d_{g_2}(x,y)$ for all x and y in the boundary. Then Area $(g_1) \geq$ Area (g_2) with equality in area implying the isometry of g_1 and g_2.

The proof of Theorem 1 involves a connection between the boundary distance function the scattering relation and the Dirichlet-to-Neumann map.

References.

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