# Symmetry solves differential equations

*or*

*What Sophus Lie taught us and we almost forgot*

*"If I only knew how I could get mathematicians interested in transformation groups and the treatment of differential equations which arises from them. I am certain, absolutely certain, that, at some point in the future, these theories will be recognized as fundamental."*

This lament was written around 1884 by a Norwegian mathematician Sophus Lie in a letter to Adolph Mayer. He was true. His almost single-handed work on what he called *transformation groups* of differential equations, a rare example of a synthesis of analysis and geometry in the 19th century mathematics, evolved into a truly fundamental field of mathematics, the group theory. But this evolution haven't really gone the straight way. Lie's most direct followers like Cartan, Weyl, Killing developed the theory of continuous (or Lie) groups from the algebraic and geometric point of view and their contributions soon found applications in the mathematical formulation of quantum mechanics. From then on, Lie groups have been one of the most useful notions of modern mathematics, at least from the physicist's point of view. What had become a little bit forgotten was the original motivation of Lie: differential equations.

An average person's first impression when he studies methods of solving differential equations may be concentrated into two words: taxonomy and trickery. This was really the state of art in the age of Sophus Lie. His results provided a concise system explaining and generalizing virtually all the known methods for solving differential equations, reducing their order, finding special solutions etc. It wasn't until the beginning of the second half of the 20th century when people turned their attention to this part of Lie's work. Since then this has been a fruitful area of mathematics with close relations with geometry, numerical mathematics and physics.

The basic idea is very simple. A first-order ordinary differential equation may be given as

$$F(x, y, y') = 0,$$

where $F$ is a smooth function on $\mathbb{R}^3$. A *point symmetry* of the equation is a transformation in the $(x, y)$ plane sending $(x, y)$ to $(\tilde{x}(x, y), \tilde{y}(x, y))$ such that

$$F(\tilde{x}, \tilde{y}, \frac{d\tilde{y}}{d\tilde{x}}) = 0.$$

If for example $F(x, y, y') = x^2 + y^2 - 1$, i.e. $F$ does not depend on $y'$, then we have an algebraic equation, which zero set is clearly the unit circle. Then any rotation $(x, y) \to (x \cos\tau - y \sin\tau, x \sin\tau + y \cos\tau)$ preserves the zero set and thus it is a symmetry of the equation. In a little bit less trivial example, $F(x, y, y') = y' = 0$, we have a one-parameter set of solutions $y = c, c \in \mathbb{R}$, i.e. horizontal lines in the $(x, y)$-plane. Setting $\tilde{x} = x + \xi$, $\tilde{y} = y + \eta$ for any $\xi, \eta \in \mathbb{R}$ we again send a solution to a solution. Both examples share a common feature: transformations preserving solutions come in families, depending on one or two real parameters, respectively. These families satisfy axioms for a group and may be described infinitesimally via vector fields. For example if we look at the group of rotations as a motion of the points of the plane in time $\tau$, this vector field corresponds to the velocity vector $(-y, x)$ of each point $(x, y)$.

If we are able to find a transformation group of an equation, we can use it to simplify the equation. For example, the equation

$$y' + y^2 = \frac{2}{x^2}$$

is invariant with respect to $(x, y) \to (xe^{\tau}, ye^{-\tau})$. This gives the "velocity" $(x, -y)$. We can find new variables $(t, u) = (t(x, y), u(x, y))$ so that the velocity in the $(t, u)$ plane will be constant. Indeed, $t = \ln x$ and $u = xy$ does the trick, since $\ln(xe^{\tau}) = \ln x + \tau$ and $xe^{\tau}ye^{-\tau} = xy$. The equation expressed in the $(t, u)$-plane hence cannot depend on $t$, since a transformation $(t, u) \to (t + \tau, u)$ would surely change it. A routine calculation shows, that we get

$$\frac{du}{dt} + u^2 - u - 2 = 0,$$

which is readily integrated. This procedure is known as the method of canonical variables.

In the seminar, we will take a brief and necesarily sketchy look at the way how to find symmetries of a differential equation. We shall also explain, how other popular methods like separation of variables or reduction of order fit into this picture and look at the beautiful geometry behind.