The Four Color Theorem, graphs, knots and the vector product in $\mathbb{R}^3$

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Many times in mathematics we begin to study a problem, just to then discover that it is related to many other ones in various different fields, which we did not suspect to be related to the original one at all. The starting problem that I will present is a very well known one (to those who know it, at least...). Given a map representing countries which border each other, it asks how to color the different regions such that no two adjacent countries have the same color. Of course, with enough colors we can easily succeed! What if we have only four crayons? Somebody has shown [Appel and Haken, 1976] that this is enough. But it took more than 100 years to mathematicians to achieve such result: the many attempts that started in 1852 all revealed some fundamental flaws.

In my seminars, I will not attempt to present the proof of such Theorem (for many reasons that will be clear from the discussion) but I will use its intuitive formulation as an excuse to talk about... other things! In fact, there are at least two very nice reformulations of such problem which make use of precise mathematical objects: respectively *graphs* and the *vector product* $a \wedge b$ of two vectors in $\mathbb{R}^3$ (sometimes also called *cross product* and written as $a \times b$). This last formulation has been explored by Kauffmann in 1988. Basically it is just a statement about the non associativity of the vector product, i.e. the fact that if we pick three random vectors a, b and c in $\mathbb{R}^3$, we unfortunately have that

$$(a \wedge b) \wedge c \neq a \wedge (b \wedge c).$$

However, sometimes it happens that this is true, for a *particular* choice of a, b, c. It is fascinating that this corresponds to the possibility of coloring maps with only four colors. If someone still

Figure 1: A map of Europe with several adjacent regions and countries. How many colors do we need?

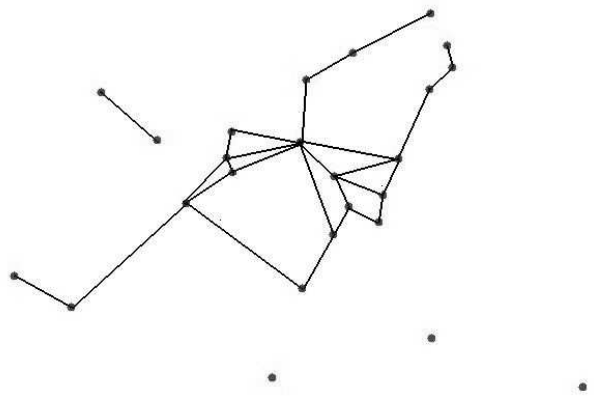

Figure 2: A planar simple undirected graph with 25 vertices. Can we label them with only three letters?

does not find it fascinating enough, still it will be hopefully interesting to see how naturally graphs come into the picture even in this description using the vector product. I will then talk a little bit about graphs and their beautiful geometric properties. The theory of graphs leads to application to computer science, for instance, but could also be presented in a purely algebraic way using quadratic monomials $x_i x_j$ and the polynomials that they generate, and forgetting about the pictures. I will only mention this briefly because it would take us too far (although recently my research interests have lead me to this fascinating problem of representing graphs and *hypergraphs* using polynomials, and making calculations with them).

Instead, I will try to keep the geometry running, leaving the cold algebra on the side. Graphs, in fact, have natural generalizations which are simply called *links* or *knots*. Knots are nothing but a manipulation of a circular rubber-band, twisted in a way that it also shows intersections when drawn on a piece of paper. Links are collection of knots. If we take a link and decide to cut each of its components we obtain basically a set of strings, some of which are "free", but some of which actually cross each other, exactly where the original knot had an intersection point. The reason why we are interested in such tied and untied strings is that they give birth to some fundamental algebraic objects (yes, algebra always comes back to us when we try to get rid of it!) called *Braid groups*. Again, rather than describing such groups formally, I prefer to talk about them using pictures, so we will learn how to make operations with the elements of this group and their basic properties only playing with the strings, crossing them, pulling them and connecting them again so to form knots and links.

As always when we study objects in mathematics, we want to associate to them some simple numbers to "label" them, namely their *invariants*. What are the invariants of maps, graphs, product of vectors and knots? in some cases, it is not so easy to answer (maybe for maps we could think of 4 being the invariant?), but for some other rich structures there is a handful of invariants we can pin on the board. Graphs have many, and I do not plan to give an exhaustive list, maybe just some examples. A course in algebraic topology will probably cover them properly. Knots are more mysterious. Their invariants come under the shape of polynomials. The *Jones polynomial*, a strange animal with a not very intuitive definition, was studied in 1983, and we will learn how to calculate it at least on some easy examples.