On Euler's function and its connection with Lie algebras

The aim of this seminar (in two parts) is to reveal a tantalizing sequence of integers which captured the attention of the English geometer Ian MacDonald, who succeeded in unravelling its mystery and hereby established a profound connection between number theory and representation theory for simple Lie algebras (see [1]). These Lie algebras, which turn out to be the main characters in most branches of differential geometry and (mathematical) physics, are nothing but a specific type of non-associative algebras (as we will see in the seminar).

In order to obtain the above-mentioned series, we will introduce *Euler's function*

$$\prod_{k=1}^{\infty}(1-x^k) \;=\; \sum_{k=-\infty}^{+\infty}(-1)^k x^{\pi_k} \ ,$$

which generates the so-called generalized pentagonal numbers π_k. This function plays an important role in elementary number theory, because its inverse can be expressed in terms of the *partition function* $p(n)$, with $n \in \mathbb{N}$. This function represents the number of partitions of a positive number n, which is the number of distinct ways of representing n as a sum of natural numbers. For example, we have that $p(4) = 5$, because obviously

$$4 \;=\; 4+0 \;=\; 3+1 \;=\; 2+2 \;=\; 2+1+1 \;=\; 1+1+1+1 \ .$$

This number $p(n)$ can be visualized graphically by means of the *Young diagrams*, which play an important role in many branches of mathematics (and physics). We will explain their importance in group representation theory, in particular for the symmetric group S_n. The latter is the group of all possible permutations of n elements, that is : all possibilities to try to mess up your father's CD-collection by putting n discs in n boxes.

We will give some identities involving the Euler function and its powers, and we will perform some explicit calculations to show that in case these powers are choosen *properly*, the resulting power series has a very simple form. Part of these calculations will be left for the student as a homework. Because that is essentially the only way to understand what is meant by '*a simple form*'...

In a second part of the seminar, we will define the so-called *affine root systems* and we will look for their explicit realisations in lower dimensions (that is dimension two, and - for those who like the game - dimension three). Root systems are encoding essential information on simple Lie algebras, and during the seminar we will introduce them on a very comprehensible level as particular configurations of vectors in a Euclidean vector space. In the second seminar we will turn our attention on one of these root systems and explain how it is related to the associated Lie algebra by means of explicit calculations. This will only require familiarity with matrices over $\mathbb{C}^3$.

We conclude with MacDonald's observation...

References

[1] MacDonald, I.G., *Affine root systems and Dedekind's η-function*, Invent. Math. **15**, 1972, pp. 91-143.

[2] Fulton, W., Harris, *Representation Theory : a first course*, Springer-Verlag, New York, 1991.

[3] Humphreys, J.E., *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York, 1987.