# New Algebraic Properties of Biregular Functions in 2n Quaternionic Variables

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#### Abstract

In this paper we exploit some general results on the tensor product of free resolutions to deduce important new algebraic properties of biregular functions on 2n quaternionic functions. In particular we are able to construct a minimal resolution for the associated module and we are able to compute all the relevant graded Betti numbers.

### 1 Introduction

A full theory for functions of several quaternionic variables which satisfy the Cauchy-Fueter operator is described in [10]. We recall that the left Cauchy–Fueter operator  $\mathcal{D}_{\ell}$  acting on differentiable functions defined on the space  $\mathbb{H}$  of quaternions is a generalization of the Cauchy-Riemann operator and it is defined as

$$\mathcal{D}_{\ell} = \frac{\partial}{\partial x_0} + \mathbf{i} \frac{\partial}{\partial x_1} + \mathbf{j} \frac{\partial}{\partial x_2} + \mathbf{k} \frac{\partial}{\partial x_3},$$

where we are denoting by  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  the three imaginary units of the real associative algebra of the quaternions and a quaternion by  $q = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$ . Since the algebra of quaternions is non commutative, it is possible to write the imaginary units on the right and to define the right Cauchy-Fueter operator

$$\mathcal{D}_r = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1}\mathbf{i} + \frac{\partial}{\partial x_2}\mathbf{j} + \frac{\partial}{\partial x_3}\mathbf{k}.$$

It is well known that the theory of nullsolutions of the left or right Cauchy-Fueter operators are completely equivalent, both in in one or several variables. In this paper, we will study functions of an even number 2n of quaternionic variables which are simultaneously left regular in the first nvariables  $p_1, \ldots, p_n$  and right regular in the remaining variables  $q_1, \ldots, q_n$ . These functions, which are called biregular, are a non-trivial generalization of functions of one or several quaternionic variables. During the eighties, Brackx and Pincket have intensively studied biregular functions in two variables for Clifford valued functions, see [4], [5], [6], [7]. In other words, they were interested in the study of functions  $f : \mathbb{R}^k \times \mathbb{R}^q \to \mathbb{R}_m$ , 1 < k, q < m, which are left monogenic in one variable and right monogenic in the other one. They proved many results for this class of functions, for example the Cauchy Integral formulas, the existence of the Taylor expansion (in terms of suitable homogeneous polynomials) and of the Laurent series, the Hartogs' theorem on the removability of compact singularities.

Our purpose is to generalize the study to functions which are biregular with respect to several pairs of variables. Our techniques are more algebraic and will allow to solve, in particular, the following problems:

a) find an explicit expression for the compatibility conditions of the system

$$\begin{cases} \mathcal{D}_{\ell 1}(f) &= g_{\ell 1} \\ \mathcal{D}_{r 1}(f) &= g_{r 1} \\ & \cdots \\ \mathcal{D}_{\ell n}(f) &= g_{\ell n} \\ \mathcal{D}_{r n}(f) &= g_{r n} \end{cases}$$
(1)

- b) construct the free resolution for the associated module, finding the dimensions of the free modules, the degrees of the maps and the length of the corresponding complex
- c) calculate the cohomology of this complex.

As a by-product we will show that the Hartogs phenomenon holds for biregular functions in several variables or rather we will show that more general singularities can be eliminated.

*Dedication.* This paper is dedicated, with friendship and admiration, to Professor Richard Delanghe, who has done so much, as a mathematician and as a leader, to further the work on Clifford Analysis around the world.

## 2 Examples of computation of the complex associated to biregular functions in low dimension

Let us give the necessary definitions and notations.

**Definition 2.1.** Let  $f : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$  be a differentiable function. The function  $f(p_1, \ldots, p_n, q_1, \ldots, q_n)$  is said to be biregular with respect to the pairs of variables  $(p_s, q_s)$ ,  $s = 1, \ldots, n$  if and only if it satisfies the system

$$\begin{cases} \mathcal{D}_{\ell 1}(f) = 0 \\ \mathcal{D}_{r 1}(f) = 0 \\ \cdots \\ \mathcal{D}_{\ell n}(f) = 0 \\ \mathcal{D}_{r n}(f) = 0 \end{cases}$$
(2)

where  $p_s = x_{s0} + ix_{s1} + jx_{s2} + kx_{s3}$ ,  $q_s = y_{s0} + iy_{s1} + jy_{s2} + ky_{s3}$  and

$$\mathcal{D}_{\ell s} = \frac{\partial}{\partial x_{s0}} + \mathbf{i} \frac{\partial}{\partial x_{s1}} + \mathbf{j} \frac{\partial}{\partial x_{s2}} + \mathbf{k} \frac{\partial}{\partial x_{s3}},$$
$$\mathcal{D}_{rs} = \frac{\partial}{\partial y_{s0}} + \frac{\partial}{\partial y_{s1}} \mathbf{i} + \frac{\partial}{\partial y_{s2}} \mathbf{j} + \frac{\partial}{\partial y_{s3}} \mathbf{k}.$$

Any quaternionic equation  $\mathcal{D}_{\ell s} f = 0$  or  $\mathcal{D}_{rs} f = 0$  translates into four real equations and can be written in matrix form as

$$\begin{bmatrix} \partial_{x_{s0}} & -\partial_{x_{s1}} & -\partial_{x_{s2}} & -\partial_{x_{s3}} \\ \partial_{x_{s1}} & \partial_{x_{s0}} & -\partial_{x_{s3}} & \partial_{x_{s2}} \\ \partial_{x_{s2}} & \partial_{x_{s3}} & \partial_{x_{s0}} & -\partial_{x_{s1}} \\ \partial_{x_{s3}} & -\partial_{x_{s2}} & \partial_{x_{s1}} & \partial_{x_{s0}} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0, \qquad \begin{bmatrix} \partial_{y_{s0}} & -\partial_{y_{s1}} & -\partial_{y_{s2}} & -\partial_{y_{s3}} \\ \partial_{y_{s1}} & \partial_{y_{s0}} & \partial_{y_{s3}} & -\partial_{y_{s2}} \\ \partial_{y_{s2}} & -\partial_{y_{s3}} & \partial_{y_{s0}} & \partial_{y_{s1}} \\ \partial_{y_{s3}} & \partial_{y_{s2}} & -\partial_{y_{s1}} & \partial_{y_{s0}} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0$$

respectively, where f has been considered as a vector  $\overline{f}$  with four real components. In the sequel, we will denote by the symbols  $D_{\ell s}$  and  $D_{rs}$  the matrices associated to the operators  $\mathcal{D}_{\ell s}$  and  $\mathcal{D}_{rs}$ . System (2) can be written in matrix form as the unique equation

$$P_n(D)\vec{f} = 0.$$

For our purposes we will be interested in the symbols of the system, i.e. in the matrix  $P_n$  obtained from  $P_n(D)$  via Fourier transform. As usual in these cases (see e.g. [10]), when we write the entries of  $P_n$  we will neglect the imaginary unit  $\sqrt{-1}$  and we will use the same variables instead of dual variables. The main objects of interest will be the cokernel of the map induced by the matrix  $P_n^t$ , i.e., the module  $M_n = R^4/\langle P_n^t \rangle$ , and its minimal free resolution (for more details we refer the reader to [10]). Here we are denoting by R the ring of polynomials in 8n variables  $R = \mathbb{C}[x_{10}, \ldots, x_{13}, \ldots, x_{n0}, \ldots, x_{n3}, y_{10}, \ldots, y_{13}, \ldots, y_{n0}, \ldots, y_{n3}]$ , and by  $\langle P_n^t \rangle$  the module generated by the columns of  $P_n^t$ .

**Remark 2.2.** We observe that the left and right operators commute with each other. This can be easily checked directly.

In this section we will use the computer package CoCoA to carry out some computations for the cases of 1, 2, and 3 variables. These results will be our motivation for the general results on section 3.

**Case** n = 1. Let us define the two  $4 \times 4$  matrices representing the symbols of the right and left operators, where we write  $p = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$  and  $q = y_0 + \mathbf{i}y_1 + \mathbf{j}y_2 + \mathbf{k}y_3$ , thus avoiding the use of double indices, and we work with matrices with entries in  $R = \mathbb{C}[x_0, \ldots, x_3, y_0, \ldots, y_3]$ .

```
Dl:=Mat[
                       -- left operator in p
  [x[0], -x[1], -x[2], -x[3]],
  [x[1], x[0], -x[3], x[2]],
  [x[2], x[3], x[0], -x[1]],
  [x[3], -x[2], x[1], x[0]]];
Dr:=Mat[
                       -- right operator in q
  [y[0], -y[1], -y[2], -y[3]],
  [y[1], y[0], y[3], -y[2]],
  [y[2], -y[3], y[0], y[1]],
  [y[3], y[2], -y[1], y[0]]];
ModB:=Module(B);
                        --resolution for biregular functions
\operatorname{Res}(\mathbb{R}^{4}/\operatorname{ModB});
0 \implies R^4(-2) \implies R^8(-1) \implies R^4
```

note that the exponents in the resolution are the Betti numbers while the numbers in parentheses indicate the degrees of the associated maps. With the command Comm(Dl,Dr) we can compute the commutator and check that it is zero. Furthermore, denoting by R the ring of coordinates, the two matrices  $D_{\ell}$  and  $D_r$  not only commute but also form a regular sequence in  $Mat_4(R)$ , since the involve two different sets of variables. We can apply the results from [9] and conclude that the syzygies associated to the module generated by the rows of the matrix  $P_1(D) = \binom{D_{\ell}}{D_r}$ are given by the matrix  $S_1(D) = (-D_r, D_{\ell})$ .

This is actually the last map of the free resolution of the module associate to biregular functions of two quaternionic variable. The complex has a Koszul-like form both in terms of Betti numbers and degrees.

$$0 \longrightarrow R^4(-2) \xrightarrow{P_1} R^8(-1) \xrightarrow{S_1} R^4 \longrightarrow 0.$$

It is possible to prove that the second cohomology module is the only nonzero one. Precisely, if we define by  $M_1$  the module associated to  $P_1$ ,

$$\operatorname{Ext}_{R}^{2}(M_{1}, R) = R^{4} / \operatorname{Im}(S_{1}).$$

On the other hand the first cohomology vanishes, as it can be easily checked since the maximal minors of the matrix  $P_1$  are coprime (see [1], Lemma 1). This situation is again totally similar to the case of holomorphic functions of two complex variables (though the real dimension in that case would be 2 instead of 4). Because the Cauchy-Riemann operators in two complex variables commute and form a regular sequence, the associated complex is again Koszul-like. We can summarize these results in a proposition (note that Hartogs' phenomenon was already proved in [6]):

**Proposition 2.3.** Let  $\mathcal{D}_{\ell}$  and  $\mathcal{D}_{r}$  be respectively the left and the right Cauchy-Fueter operators acting on functions  $f : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H}$ . Consider the non-homogeneous system

$$\begin{cases} \mathcal{D}_{\ell}(f) &= g_{\ell} \\ \mathcal{D}_{r}(f) &= g_{r} \end{cases}$$

Then the only compatibility condition on the system is given by  $\mathcal{D}_r g_\ell = \mathcal{D}_\ell g_r$  and the associated complex is Koszul-like, i.e. it has length two and its maps are constructed as in the Koszul complex. The Hartogs' phenomenon holds for the solutions of the system, while the second cohomology module of the associated complex is nonzero.

**Case** n = 2. The situation is completely different and much more intricate when we consider functions of four quaternionic variables  $(p_1, p_2, q_1, q_2)$  and we study functions that are left regular in  $p_1, p_2$  and right regular in  $q_1, q_2$ . By Remark 2.2, every left operator commutes with every right operator, so we expect to obtain some Koszul-like syzygies associated to the pairs  $(\mathcal{D}_{\ell i}, \mathcal{D}_{rj})$ , for every  $i, j \in \{1, 2\}$ . On the other hand, the two left operators do not commute with each other, and neither do the right operators. From the two pairs  $(\mathcal{D}_{\ell 1}, \mathcal{D}_{\ell 2})$  and  $(\mathcal{D}_{r1}, \mathcal{D}_{r2})$  we then expect some syzygies of the same type appearing in the complex associated to two left (resp. right) Cauchy-Fueter operators, which are quadratic (see for example [14]).

Let  $P_2$  be the  $16 \times 4$  matrix in R representing these 4 operators. We can compute the resolution of the associated module  $M_2 = \text{Cokernel}(P_2^t)$  using CoCoA. We obtain

$$0 \implies R^{4}(-8) \implies R^{16}(-7) \implies R^{16}(-5)(+)R^{16}(-6) \implies R^{4}(-8) \implies R^{4}(-4) \implies R^{16}(-2)(+)R^{16}(-3) \implies R^{16}(-1) \implies R^{4}(-1) \implies R^{4}($$

A we can see from the Betti numbers and the degrees of the maps in the resolution, the first syzygies consists in 16 linear relations, corresponding to 4 quaternionic syzygies, and 16 quadratic relations, corresponding to 4 quaternionic syzygies. The explicit expression of such relations can be computed again with CoCoA, at least in their real counterparts, and shows that the linear conditions are exactly those coming from the commutativity and the quadratic ones are coming from the the fact that the Laplacian is real, so it commutes with any of the operators  $\mathcal{D}_{\ell i}$ ,  $\mathcal{D}_{rj}$ (see [10]). Furthermore CoCoA can also help us calculate the cohomology modules: such modules all vanish, except at the last spot of the complex where we have a nontrivial cohomology, (the cokernel of the last map). The following proposition summarizes the results for the case n = 2.

**Proposition 2.4.** Consider the non-homogeneous system

$$\begin{cases} \mathcal{D}_{\ell 1} f = g_{\ell 1} \\ \mathcal{D}_{\ell 2} f = g_{\ell 2} \\ \mathcal{D}_{r 1} f = g_{r 1} \\ \mathcal{D}_{r 2} f = g_{r 2} \end{cases}$$

The compatibility conditions of the system are given by the following four (linear) relations:

$$\mathcal{D}_{\ell i}g_{rj} = \mathcal{D}_{rj}g_{\ell i}, \quad i, j \in \{1, 2\}$$

plus the four quadratic relations

$$\mathcal{D}_{\ell i} \mathcal{D}_{\ell j} g_{\ell j} = \mathcal{D}_{\ell j}^2 g_{\ell i}, \quad i, j \in \{1, 2\}, i \neq j$$
$$\mathcal{D}_{r i} \mathcal{D}_{r j} g_{r j} = \mathcal{D}_{r j}^2 g_{r i}, \quad i, j \in \{1, 2\}, i \neq j.$$

The complex associated to the module  $M_2$  associated to the system is

$$0 \longrightarrow R^4 \xrightarrow{P_2} R^{16} \xrightarrow{S_1} R^{32} \xrightarrow{S_2} R^{40} \xrightarrow{S_3} R^{32} \xrightarrow{S_4} R^{16} \xrightarrow{S_5} R^4 \longrightarrow 0$$

where the self-duality condition holds on the maps of the resolution, i.w.  $S_5 = {}^tP_2$ ,  $S_4 = {}^tS_1$ and  $S_3 = {}^tS_2$ . The complex is exact except at the last spot where the cohomology module is the Cokernel of  $S_5$ .

*Proof.* All the statements of the theorem can be easily checked with CoCoA. It is immediate to show that the eight relations given for the system are compatibility conditions. Their sufficiency follows from a dimension argument. The vanishing of the Ext-modules  $\text{Ext}^{j}(M_{2}, R), j = 0, \ldots, 5$  can be checked directly using CoCoA (see also Corollary 3.10). The last map is the only one that gives rise to a nontrivial cohomology.

**Remark 2.5.** In the case of left regular functions of four variables, the compatibility conditions are quadratic and they include the so called *exceptional* relations that cannot be expressed in terms of the Cauchy-Fueter operator or variation of it (see [3]). In this case, on the other hand, the complex for biregular functions of four quaternionic variables has a double nature: it behaves both like the Koszul complex and the Cauchy-Fueter one. Since the number of left (or right) operators involved is only n = 2, we do not see any exceptional syzygies. The exceptional behavior can occur only with at least three Cauchy-Fueter operators of the same type (left or right). **Case** n = 3. Before we generalize the results obtained so far for the case to n pairs of quaternionic variables, let us show the computations of the complex for the case of biregular functions of 6 quaternionic variables, defined as the kernel of 3 left operators and 3 right operators. The following is the minimal free resolution of the module  $M_3$  (to get the number of quaternionic relations, it suffices to divide the Betti numbers by 4) as obtained by CoCoA:

$$0 \longrightarrow R^{16}(-12) \longrightarrow R^{144}(-11) \longrightarrow R^{564}(-10) \longrightarrow H^{1240}(-9) \longrightarrow R^{1620}(-8) \oplus R^{48}(-7) \longrightarrow (3)$$
$$R^{1200}(-7) \oplus R^{232}(-6) \longrightarrow R^{400}(-6) \oplus R^{432}(-5) \longrightarrow R^{360}(-4) \longrightarrow$$
$$R^{80}(-3) \oplus R^{9}(-2) \longrightarrow R^{24}(-1) \longrightarrow R^{4} \longrightarrow M_{3} \longrightarrow 0$$

**Remark 2.6.** We notice immediately that the nice property of self-duality does not hold in this case, since the Betti numbers are not symmetric. The length of the complex is 10, i.e. exactly the double of the length of the Cauchy-Fueter complex for 3 quaternionic variables. This was the case even in the previous examples, and indeed we will show in the next section that that the length of the complex of biregular functions in 2n variables is always equal to two times the length of the complex for (left) regular functions of n quaternionic variables.

**Remark 2.7.** Let us look now at the compatibility conditions. In the case of three left operators and three right operators we have 9 linear quaternionic syzygies and 10 quadratic ones. This makes us think that we can repeat the argument provided in the case of 4 variables in order to count the first syzygies. Indeed, the 9 pairs of operators  $(\mathcal{D}_{\ell i}, \mathcal{D}_{rj})$  commute and hence give rise to the 9 Koszul-type relations. The triple of left operators generates  $10 = 2\binom{3}{2} + 4\binom{3}{3}$  quadratic relations, of which 2 are exceptional and 8 are radial, as described for example in [14]. The same holds for the set of right operators, for a total of 20 quadratic relations. The relations described so far are obviously syzygies, and it could be shown that they are independent exactly as it has been done for the case of 3 Cauchy-Fueter operators and for the Koszul complex. Therefore, for dimension reasons, they are all the syzygies and the only ones. We will show in the next section that the complex is exact except at the last point.

### 3 Algebraic analysis of the module associated to biregular functions

We present some preliminary lemmas that will lead to the proof of our results for the general case of the module associated to n left and n right operators.

**Lemma 3.1.** Let  $A_1, \ldots, A_n$  and B be square matrices representing n+1 linear constant coefficient differential operators. Let us suppose that  $A_iB = BA_i$  for every  $i = 1 \ldots n$  and suppose that they form a left regular sequence in the ring of matrices. Let  $S = \{(S_{j1}, \ldots, S_{jn}) | j = 1 \ldots t\}$  be a set of generators for the module of left syzygies of the n-tuple  $(A_1, \ldots, A_n)$ . Then the module  $Syz(A_1, \ldots, A_n, B)$  is generated by the set  $S' = \{(S_{j1}, \ldots, S_{jn}, 0) | j = 1 \ldots t\}$  together with the set  $\mathcal{K} = \{(0, \ldots, -B, \ldots, 0, A_i) | i = 1 \ldots n\}$ .

*Proof.* It is immediate to see that the elements of S' and  $\mathcal{K}$  are syzygies. Let us now show they are sufficient to generate all of the syzygies. Let  $C_1, \ldots, C_n, D$  be n + 1 matrices such that

$$C_1A_1 + \dots + C_nA_n + DB = 0.$$

Then from  $C_1A_1 + \cdots + C_nA_n = -DB$  and the fact that  $(A_1, \ldots, A_n, B)$  is a left regular sequence it follows that  $D = T_1A_1 + \cdots + T_nA_n$  for some matrices  $T_1, \ldots, T_n$ . By substituting this expression of D and by using the commutativity we have that

$$(C_1 + T_1B)A_1 + \dots + (C_n + T_nB)A_n = 0$$

and so  $C_i + T_i B = P_1 S_{1i} + \cdots + P_t S_{ti}$  for every  $i = 1 \dots n$  and so we get that the (n + 1)-tuple  $(C_1, \dots, C_n, D)$  is of the desired form.

The following lemma exploits the computation of the Gröbner Basis of the module associated to biregular functions for small n to infer the general case. We always assume that the default term ordering on the ring of 4n variables  $\mathbb{C}[x_{i0}, \ldots, x_{i3}, y_{i0}, \ldots, y_{i3}]$  is DegRevLex.

**Lemma 3.2.** Let  $D_{\ell 1}, \ldots, D_{\ell n}$  be the symbols matrices associated to n left Cauchy-Fueter operators and let  $D_{r1}, \ldots, D_{rn}$  be the symbols of n right Cauchy–Fueter operators. Let  $\mathcal{B}_n$  be the module generated by the rows of such matrices. The reduced Gröbner Basis for  $\mathcal{B}_n$  is given by the rows of the matrices  $D_{\ell s}$  and  $D_{rs}$ ,  $i = 1, \ldots, n$  together with the rows of the matrices

$$B_{ks} = D_{\ell k} D_{\ell s} - D_{\ell s} D_{\ell k}$$
 and  $C_{ks} = D_{rk} D_{rs} - D_{rs} D_{rk}$ ,  $1 \le r < s \le n$ .

Proof. The statement can be verified directly with CoCoA for  $n \leq 4$ . For the general case, we can see that the S-polynomials generated by any two rows of  $D_{\ell i}$  give rise to the rows of  $B_{rs}$  and the S-polynomials of two rows of  $D_{ri}$  generate the rows of  $C_{rs}$ . If we pick a row of a  $D_{\ell i}$  and a row of a  $D_{rj}$ , their S-polynomial reduces to zero due to the commutativity  $D_{\ell i}D_{rj} = D_{rj}D_{\ell i}$ . Therefore, considering Buchberger's algorithms for the computation of a reduced Gröbner Basis, we have so far generated elements of the Gröbner Basis of  $\mathcal{B}_n$  by adding the rows of  $B_{rs}$  and  $C_{rs}$ . To prove that they are the only elements of the reduced Gröbner Basis, we need to show that all their S-polynomials reduce to zero. An S-polynomial generated by a row of  $D_{\ell i}$  and a row of  $B_{rs}$  is computed and reduced to zero as in the case n = 2 or n = 3, depending on the number of different indices in the triple (i, r, s). The same holds for  $D_{ri}$  and  $C_{rs}$ . An S-polynomial generated by two rows of  $B_{rs}$  is computed and reduced as in the case n = 2, n = 3 or n = 4 depending on the number of different indices. The same holds for two rows of  $C_{rs}$ . When choosing a row of  $D_{\ell i}$  and a row of  $C_{rs}$ , or a row of  $D_{ri}$  and a row of  $B_{rs}$ , or a row of  $B_{rs}$  arow of  $B_{rs}$  arow of  $B_{rs}$ .

In order to be able to describe the cohomology and the length of the resolution of the module associated to biregular function we need the Hilbert-Poincaré series. The following lemma provides the series for the general case.

**Proposition 3.3.** Let  $R = \mathbb{C}[x_{i0}, \ldots, x_{i3}, y_{i0}, \ldots, y_{i3}]$  and let  $M_n$  be the *R*-module associated to *n* left Cauchy-Fueter operators and *n* right Cauchy-Fueter operators. Then the Hilbert series of the module  $M_n$  is given by

$$\mathcal{H}_{M_n}(t) = 4 \frac{(1+(n-1)t)^2}{(1-t)^{4n+2}}.$$

Moreover, the module is Cohen-Macaulay.

*Proof.* Let us first calculate the monomial module  $LT(M_n)$ . Computations with CoCoA in the case n = 3 and n = 4 show that it is generated by the set  $\{x_{i0}e_t, y_{i0}e_t, x_{h2}x_{k1}e_t, y_{h2}y_{k1}e_t | i = 1 \dots n, t = 1 \dots 4, 1 \leq h < k \leq n\}$ , where  $e_t$  is the t-element of the canonical basis of  $R^4$ . The same argument as in Lemma 3.2 show that this is sufficient to characterize the module. Let  $I_n$  be the ideal  $I_x + I_y = \{x_{i0}, x_{h2}x_{k1} | i = 1 \dots n, 1 \leq h < k \leq n\} + \{y_{i0}, y_{h2}y_{k1} | i = 1 \dots n, 1 \leq h < k \leq n\}$ 

 $k \leq n$ }. Then obviously  $\mathcal{H}_{M_n}(t) = 4\mathcal{H}_{I_n}(t)$ . Since  $I_x$  and  $I_y$  involve a different set of variables, we can use the isomorphism

$$R/I_n \simeq R/I_x \otimes R/I_y$$

and conclude that  $\mathcal{H}_{I_n}(t) = \mathcal{H}_{I_x} \cdot \mathcal{H}_{I_y}$ . Since  $\mathcal{H}_{I_x} = \mathcal{H}_{I_y} = \frac{1+(n-1)t}{(1-t)^{2n+1}}$  as calculated in [2], we find the final form of the series. It follows that the dimension of the module  $M_n$  is 4n + 2 and to prove that it is Cohen-Macaulay we have to show that  $\operatorname{depth}(M_n) = \operatorname{dim}(M_n)$ . A maximal regular sequence for  $M_n$  can be constructed as a maximal regular sequence in  $I_n$ . The latter has obviously twice the number of elements than the one constructed in [2] for  $I_x$ , which has length 2n + 1, so the statement follows.

Being able to calculate the monomial module  $LT(M_n)$ , as in the proof of the above proposition, allows to give an explicit formula for the Betti numbers of the minimal free resolution of  $M_n$ . Note that  $LT(M_n)$  is "diagonal" in the sense specified below. The property of  $LT(M_n)$ being diagonal translates into the fact the Betti numbers of  $M_n$  are exactly the ones of the ideal  $I_n$  multiplied by 4. Moreover,  $I_n$  splits into  $I_x + I_y$ , which are the ideals in the diagonal of  $M_x$  and  $M_y$  (with obvious meaning of the symbols). Then the resolution of  $M_n$  is the "product" of the resolutions of  $M_x$  and  $M_y$ , in the following sense:

**Definition 3.4.** Let R be a ring and I and J two ideals of R. Let  $\{(F_i, \phi_i)\}_i$  and  $\{(G_j, \psi_j)\}_j$  be the two minimal free resolutions of R/I and R/J respectively. In view of the R-modules isomorphism

$$R/(I+J) \simeq R/I \otimes R/J \tag{4}$$

we can define the tensor product resolution of R/(I+J) as the complex  $\{(T_d, \tau_d)\}_d$  where

$$T_d = \bigoplus_{i+j=d} F_i \otimes G_j, \quad \tau_d = \sum_{i+j=d} \phi_i \otimes \psi_j.$$

In view of the isomorphism (4) the tensor product resolution is indeed a free resolution for the quotient R/(I + J). Its minimality follows from the definition of the maps  $\tau_d$  and form the fact that the matrices in the free resolutions of R/I and R/J do not have nonzero constant entries. Let us now state a proposition that generalizes, under suitable hypotheses that can be easily checked, the construction of a tensor product resolution to the case of modules. Given an ideal I of R, we denote by  $\Delta_s(I)$  the diagonal submodule of  $R^s$  given by  $Ie_1 + \cdots + Ie_s$  where  $e_i$  is the *i*-element of the canonical basis of  $R^s$ .

**Proposition 3.5.** Let s be an integer, let R be a polynomial ring and let  $M_1$ ,  $M_2$  and  $M = M_1 + M_2$  be finitely generated submodules of the free module  $R^s$ . Let  $I_1$  and  $I_2$  be two ideals in R such that:

- 1)  $LT(M) = LT(M_1) + LT(M_2),$
- 2)  $LT(M_i) = \Delta_s(I_i), i = 1, 2$

and let  $\alpha_i$  and  $\beta_j$  be the Betti numbers associated to  $M_1$  and  $M_2$  respectively. Then the Betti numbers of the module M are given by

$$\gamma_d = \frac{1}{s} \sum_{i+j=d} \alpha_i \beta_j.$$
(5)

*Proof.* Let  $\alpha'_i$  and  $\beta'_j$  be the Betti numbers of the resolutions associated to  $R/I_1$  and  $R/I_2$ . From condition 1) and 2) it follows that  $LT(M) = \Delta_s(I)$ , where  $I = I_1 + I_2$ , so if  $\gamma'_d$  are the Betti numbers of R/I, we have that

$$\gamma_d = s\gamma'_d, \ \alpha_i = s\alpha'_i, \ \beta_j = s\beta'_j, \ \text{for all } i, j, d.$$

By definition of tensor product resolution, we obtain the Betti numbers associated to R/I via the formula  $\gamma'_d = \sum_{i+j=d} \alpha'_i \beta'_j$  so substituting in the formula for  $\gamma_d$  we get the statement.  $\Box$ 

**Remark 3.6.** The expression for the Betti numbers given in (5) does not take into account the degrees of the maps involved in the resolution. However, one could extend the results from proposition 3.5 giving a formula for the graded Betti numbers. This requires the tensor product resolution to be endowed with the natural grading arising form the tensor products. Denoting by  $\gamma_d(\lambda)$  the *d*-th Betti number in degree  $\lambda$  of the module *M*, formula (5) becomes, with obvious meaning of symbols

$$\gamma_d(\lambda) = \frac{1}{s} \sum_{i+j=d, \, \rho+\sigma=\lambda} \alpha_i(\rho)\beta_j(\sigma).$$

We can now state all the principal results for the analysis of the module  $M_n$  in a Theorem:

**Theorem 3.7.** Let  $R = \mathbb{C}[x_{i0}, \ldots, x_{i3}, y_{i0}, \ldots, y_{i3} | i = 1 \dots n]$  and consider the system associated to n left Cauchy-Fueter operators  $\mathcal{D}_{\ell 1}, \ldots, \mathcal{D}_{\ell n}$  and n right Cauchy-Fueter operators  $\mathcal{D}_{r1}, \ldots, \mathcal{D}_{rn}$ . Let  $M_n$  be the R-module associated to map given by all the 2n operators. Then the length of the minimal free resolution of  $M_n$  is 4n - 2. The Betti numbers associated to  $M_n$  are  $\gamma_0 = 4$ ,  $\gamma_1 = 8n$  and

$$\gamma_d = 4n^2 \sum_{i+j=d} \binom{2n-1}{i} \binom{2n-1}{j} \frac{ij+1-d}{ij+1+d}, \quad d > 1.$$
(6)

Furthermore, if we consider the inhomogeneous system

$$\begin{cases}
\mathcal{D}_{\ell 1}(f) = g_{\ell 1} \\
\mathcal{D}_{r 1}(f) = g_{r 1} \\
\dots \\
\mathcal{D}_{\ell n}(f) = g_{\ell n} \\
\mathcal{D}_{r n}(f) = g_{r n}
\end{cases}$$
(7)

the compatibility conditions are given by the  $n^2$  linear relations

$$\mathcal{D}_{\ell i}g_{rj} = \mathcal{D}_{rj}g_{\ell i}, \quad i,j \in \{1,\dots,n\}$$
(8)

and the  $4\binom{n}{2} + 4\binom{n}{3}$  relations given by the following

$$\mathcal{D}_{si}\mathcal{D}_{sj}g_{sj} = \mathcal{D}_{sj}^2g_{si}, \quad i, j \in \{1, \dots, n\}, i \neq j$$
(9)

$$\mathcal{D}_{si}\mathcal{D}_{sj}g_{sk} + \mathcal{D}_{sj}\mathcal{D}_{si}g_{sk} = \mathcal{D}_{sk}\mathcal{D}_{si}g_{sk} + \mathcal{D}_{sk}\mathcal{D}_{sj}g_{si}, \quad i, j \in \{1, \dots, n\}, i \neq j$$

and finally the  $2\binom{n}{3}$  exceptional relations

$$(\mathcal{D}'_{si}\mathcal{D}_{sj} - \mathcal{D}'_{sj}\mathcal{D}_{si})g_{sk} + (\mathcal{D}'_{sj}\mathcal{D}_{sk} - \mathcal{D}'_{sk}\mathcal{D}_{sj})g_{si} + (\mathcal{D}'_{sk}\mathcal{D}_{si} - \mathcal{D}'_{si}\mathcal{D}_{sk})g_{sj} = 0, \quad 1 \le 1 < j < k \le n$$

$$(10)$$

$$(\mathcal{D}''_{si}\mathcal{D}_{sj} - \mathcal{D}''_{sj}\mathcal{D}_{si})g_{sk} + (\mathcal{D}''_{sj}\mathcal{D}_{sk} - \mathcal{D}''_{sk}\mathcal{D}_{sj})g_{si} + (\mathcal{D}''_{sk}\mathcal{D}_{si} - \mathcal{D}''_{si}\mathcal{D}_{sk})g_{sj} = 0, \quad 1 \le 1 < j < k \le n$$

where in each line  $\mathcal{D}_s$  stands for either the left operator of the right operator, and the operators  $\mathcal{D}'_s$  and  $\mathcal{D}''_s$  are Cauchy-Riemann like operators involving only two of the four real variables corresponding to the quaternionic variable given by the index (see [14] for their explicit form).

*Proof.* The length of the resolution follows form the Auslander-Buchsbaum formula and the fact that the module  $M_n$  has dimension 4n + 2 by Proposition 3.3:

$$pd(M_n) = 8n - \operatorname{depth}(M_n) = 4n - 2.$$

Consider the sum  $M_n = M_x + M_y$ , where  $M_x$  is the module associated to the left operators and  $M_y$  to the right operators. From the calculation of the leading term module done in Proposition 3.3 we have that  $\Delta(I_n) = LT(M_n) = LT(M_x) + LT(M_y) = \Delta(I_x) + \Delta(I_y)$  so we can apply Proposition 3.5 to calculate the Betti numbers for  $M_n$ .  $M_x$  and  $M_y$  have the same Betti numbers, as calculated in [2]:  $\alpha_i = \beta_i = 4\binom{2n-1}{i} \frac{n(i-1)}{i+1}$  so formula (6) follows from the expression of the Betti numbers given in 3.5. Let us now focus on the computation of syzygies. Let us denote by  $D_{\ell i}$  the symbol of the operator  $\mathcal{D}_{\ell i}$  and by  $D_{ri}$  the symbol of  $\mathcal{D}_{ri}$ . Using Lemma 3.2 we know that the reduced Gröbner Basis of the module associated to the system (7) consists of vectors of degree at most two, involving two different sets of variables  $x_i$  or  $y_i$  at a time. The algorithm to compute the syzygies of the module (see [12]) consists in the computation of the S-polynomials and then a reduction step using the division algorithm. Since this last step cannot involve variables not already in the S-polynomial, each syzygy will contain at most 4 different variables, hence will involve at most four different operators. Direct computations for the case n = 4 show that in reality the number of operators involved in each syzygy is at most three. Let us then consider three operators  $(A_1, A_2, B)$  among those appearing in the system. If they are all left Cauchy-Fueter or all right Cauchy-Fueter, the syzygies arising will be of type (9) or (10). If  $A_1$  and  $A_2$  are left operators and B is a right operator (or viceversa), we can apply Lemma 3.1 because the three operators involve different variables and  $A_i$  commutes with B. We then conclude that the syzygies are either of the "Koszul type" (8) or of the "Cauchy-Fueter" types (9) or (10). 

**Remark 3.8.** It is possible to calculate the graded Betti numbers using the formula of Remark 3.6 and the graded Betti numbers of the Cauchy–Fueter complex. For the sake of brevity, we refer the reader to our webpage [11] where an explicit expression is given

The exactness of the complex associated to biregular functions in 2n quaternionic variables depends on the vanishing of the Ext-modules  $\operatorname{Ext}^{j}(M, R)$ . To show that those modules vanish for  $j = 0, 1, \ldots, 4n+1$  we use the fact that the characteristic variety associated to  $M_n$  (essentially the affine variety of points in which the rank of the matrix  $P_n$  is strictly less than 4) has dimension 4n-2 and a well known result in [13] (Proposition 2, p. 139).

#### **Theorem 3.9.** The characteristic variety $V_n$ associated to $M_n$ has dimension 4n + 2.

Proof. Let us consider the complexified algebra of quaternions  $\mathbb{H}_{\mathbb{C}} = \mathbb{H} \otimes \mathbb{C}$ . A quaternion  $\xi_i = \xi_{i0} + \xi_{i1}\mathbf{i} + \xi_{i2}\mathbf{j} + \xi_{i3}\mathbf{k}$  will also be denoted as a column vector in  $\mathbb{C}^4$ :  $\xi_i = (\xi_{i0}, \xi_{i1}, \xi_{i2}, \xi_{i3})^t$ . The conjugate of a quaternion  $\xi_i$  will be the element  $\xi_i^* = \xi_{i0} - \xi_{i1}\mathbf{i} - \xi_{i2}\mathbf{j} - \xi_{i3}\mathbf{k}$  and will be associated to the column vector  $\xi_i = (\xi_{i0}, -\xi_{i1}, -\xi_{i2}, -\xi_{i3})^t$ . We will show that the algebraic set  $V_n$  has dimension 4n + 2 in a neighborhood of an arbitrary point in  $V_n$ . An element in  $V_n$  is  $\zeta = (p_1, \ldots, p_n, q_1, \ldots, q_n)$  where  $p_i, q_i \in \mathbb{C}^{4n}, i = 1, \ldots, n$ . Note that the columns of the matrix  $P_n^t$  correspond to the quaternions (cfr. [3] and [10])

$$p_1^*, p_1^* \mathbf{i}, p_1^* \mathbf{j}, p_1^* \mathbf{k}, \dots, p_n^*, p_n^* \mathbf{i}, p_n^* \mathbf{j}, p_n^* \mathbf{k}, q_1^*, \mathbf{i}q_1^*, \mathbf{j}q_1^*, \mathbf{k}q_1^*, \dots, q_n^*, \mathbf{i}q_n^*, \mathbf{j}q_n^*, \mathbf{k}q_n^*.$$

The determinant of the *i*-th 4×4 block in  $P_n^t$  is equal to  $(p_i^*p_i)^2$  if  $i \leq n$  or to  $(q_j^*q_j)^2$  if i = n+j. The equation  $\xi_i^*\xi_i = 0$  defines a quadratic cone V of dimension three in  $\mathbb{C}^4$ . Now for  $\xi \in \mathbb{H}_{\mathbb{C}}$ , we define four complex subspaces of  $\mathbb{H}_{\mathbb{C}}$  as follows:

$$L_{\xi} = \{\xi q \mid q \in \mathbb{H}_{\mathbb{C}}\}, \qquad L_{\xi}^{\perp} = \{q \in \mathbb{H}_{\mathbb{C}} \mid : \xi q = 0\}$$

and

$$R_{\xi} = \{q\xi \mid q \in \mathbb{H}_{\mathbb{C}}\}, \qquad R_{\xi}^{\perp} = \{q \in \mathbb{H}_{\mathbb{C}} \mid q\xi = 0\}.$$

The spaces  $L_{\xi}$  and  $R_{\xi}$  are the image of left and right multiplication by  $\xi$  respectively, while the other two spaces are the kernels of these maps. By consequence

$$\dim_{\mathbb{C}} L_{\xi} + \dim_{\mathbb{C}} L_{\xi}^{\perp} = \dim_{\mathbb{C}} R_{\xi} + \dim_{\mathbb{C}} R_{\xi}^{\perp} = 4.$$

It is known (see [3] and [10]) that if  $\xi \in V$  and  $\xi \neq 0$  then  $\xi^* \xi = 0$  and  $\dim_{\mathbb{C}} L_{\xi} + \dim_{\mathbb{C}} L_{\xi}^{\perp} = 2$ , (in fact the map of the left multiplication by  $\xi$  corresponds to the first four columns of  $P_n^t$  with  $\xi^*$  substituted in it). It is easy to verify, for example using CoCoA, that the 3 × 3 minors of this matrix are multiples of  $\xi^* \xi$  and the fact that  $\xi \neq 0$  implies that not all the 2 × 2 minors are zero. Since  $L_{\xi} \subseteq L_{\xi^*}^{\perp}$ , as a consequence of the dimension we get  $L_{\xi} = L_{\xi^*}^{\perp}$  and similarly  $\dim_{\mathbb{C}} R_{\xi} = 2$  and  $R_{\xi} = R_{\xi^*}^{\perp}$ .

We now prove the following:

$$\zeta \in V(M_n) \iff p_1, q_1 \in V \text{ and } p_j \in R_{p_1}, q_j \in L_{q_1}, j = 2, \dots n$$

Let us prove the implication  $\Leftarrow$ . If  $p_1 \in V$  and  $p_j \in R_{p_1}$  for  $j = 2, \ldots n$  then  $p_j = p'_j p_1$  for a suitable  $p'_j \in \mathbb{H}_{\mathbb{C}}$ , therefore  $p^*_j e \in L_{p^*_1}$  where e = 1, i, j, k so that the space generated by the first n columns is contained in the two dimensional space  $L_{p^*_1}$ . In an analogue way, if  $q_1 \in V$  and  $q_j \in L_{q_1}$  then the space generated by the last n columns is contained in the two dimensional space  $R_{q^*_1}$ . The rank of  $P^t_n$  is not maximum and so  $\zeta \in V_n$ .

Let us prove the converse. Let us suppose that  $\zeta \in V_n$ . Then the matrix  $P_n^t$  is not of maximal rank and, by consequence the determinant of the first  $4 \times 4$  block is zero. This corresponds to  $p_1 \in V$  and since dim<sub>C</sub>  $L_{p_1^*} = 2$  we may assume that  $p_1^*$  and  $p_1^*\mathbf{i}$  form a basis for  $L_{p_1^*}$ . Moreover, the hypothesis on the rank implies that, for any fixed  $\ell$ , the elements  $p_1^*, p_1^*\mathbf{i}, p_\ell^*, p_\ell^*\mathbf{i}$  are linearly dependent. Reasoning as in [3], we deduce that

$$p_{\ell}^* \in L_{p_1}^{\perp} = L_{p_1^*}.$$

We conclude that  $p_{\ell}^* = p_1^* p'$  for some  $p' \in \mathbb{H}_{\mathbb{C}}$  thus  $p_{\ell} \in R_{p_1}$ . We now look at the last *n* blocks of the matrix  $P_n^t$ . It is obvious that  $q_1^* \in V$ . Note that by adding the columns corresponding to the quaternions  $q_j$  to columns corresponding to quaternions  $p_i$  it is not anymore true that the space of the columns has dimension two. For example, if we consider the elements  $p_1^*, p_1^*\mathbf{i}, q_1^*, \mathbf{i}q_1^*$ we have that they are linearly dependent, but  $\mathbf{i}q_1^*$  does not belong to the subspace generated by  $p_1^*, p_1^*\mathbf{i}$  and so the rank of the  $4 \times 4$  matrix they form is three. Neverthless, we have that if  $q_1 \in V$ then dim<sub>C</sub>  $R_{q_1^*} = 2$  and we may assume that  $q_1^*$  and  $\mathbf{i}q_1^*$  form a basis for  $R_{q_1^*}$ . Moreover, for any fixed  $\ell$ , the elements  $q_1^*, \mathbf{i}q_1^*, q_\ell^*, \mathbf{i}q_\ell^*$  are linearly dependent and, arguing as before, we deduce that

$$q_\ell^* \in R_{q_1}^\perp = R_{q_1^*}$$

We conclude that, for any choice of  $\ell$ , we still have that the columns of the last n blocks form a two dimensional subspace  $R_{q_1^*}$ . It follows immediately that

dim 
$$(V(M_n)) = 2 \dim_{\mathbb{C}} V + (n-1) \dim_{\mathbb{C}} R_{p_1} + (n-1) \dim_{\mathbb{C}} L_{q_1}$$
  
= 6 + 4(n-1) = 4n + 2.

As a consequence of the theorem we obtain

**Corollary 3.10.** If  $M_n$  is as above, then we have

 $Ext^{i}(M_{n}, R) = 0, \quad for \ all \quad i = 0, \dots, 4n - 3$ 

and

$$Ext^{4n-2}(M_n, R) \neq 0.$$

The complex associated to the biregular functions in 2n variables is exact, except at the last spot.

*Proof.* Since the characteristic variety  $V(M_n)$  has dimension 4n+2, we immediately obtain that  $\operatorname{Ext}^i(M_n, R) = 0$ , for all  $i = 0, \ldots, 8n - (4n+2) - 1$ .

**Remark 3.11.** Note that all the examples in section 2 are an immediate consequence of Theorem 3.7 and Corollary 3.10.

**Remark 3.12.** As we have widely discussed in [10], Theorem 3.7 and Corollary 3.10 have a significant number of analytical consequences. In particular they allow an immediate proof of the Hartogs' phenomenon for biregular functions in 2n quaternionic variables (the case n = 1 had already been proved in [6]). They also allow the construction of a hyperfunction like theory for boundary values of such functions. We may return to some of these questions in a subsequent paper.

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