Quaternionic hyperfunctions on 5-dimensional varieties in $\mathbb{H}^2$ - Appendix to the paper

Fabrizio Colombo
Dipartimento di Matematica
Politecnico di Milano
Via Bonardi, 9
20133 Milano, Italy
fabrizio.colombo@polimi.it

Alberto Damiano
Mathematics Department (MUUK)
Charles University
Sokolovská 83
Prague 1, 110 00 Czech Republic
damiano@karlin.mff.cuni.cz

Irene Sabadini
Dipartimento di Matematica
Politecnico di Milano
Via Bonardi, 9
20133 Milano, Italy
irene.sabadini@polimi.it

Daniele C. Struppa
Department of Mathematics and Computer Science
Chapman University
One University Drive
Orange, CA 92867, USA
struppa@chapman.edu

(A) Complete proofs of some results of Section 3

Proposition 3.6 *Let*

$$d^1 = \sum_{i,j=1}^{2} (\check{D}_{ij}\bar{\partial}_i\bar{\partial}_j + D_{ij}\partial_i\partial_j + \tilde{D}_{ij}\partial_i\bar{\partial}_j + D^*_{ij}\bar{\partial}_i\partial_j) \tag{1}$$

and let

$$\begin{aligned} d^2 = d^2_1 + d^2_2 \quad &= \check{D}_1\bar{\partial}_1 + \check{D}_2\bar{\partial}_2 + D_1\partial_1 + D_2\partial_2 \\ &+ \textstyle\sum_{i,j=1}^{2}(\check{D}_{ij}\bar{\partial}_i\bar{\partial}_j + D_{ij}\partial_i\partial_j + \tilde{D}_{ij}\partial_i\bar{\partial}_j + D^*_{ij}\bar{\partial}_i\partial_j). \end{aligned} \tag{2}$$

Then $d^2d^1 = 0$ implies, for $i, j = 1, 2,\ i \neq j$, the conditions

(1) $\check{D}_i D^*_{ii}\check{D}_j + \check{D}_i\tilde{D}_{ii}\check{D}_j = 0, \quad i, j = 1, 2, \quad i \neq j,$

(2) $\check{D}_i D^*_{jj}\check{D}_i + \check{D}_i\tilde{D}_{jj}\check{D}_i = 0, \quad i, j = 1, 2, \quad i \neq j,$

(3) $D_i D^*_{ii}\check{D}_j + D_i\tilde{D}_{ii}\check{D}_j = 0, \quad i, j = 1, 2, \quad i \neq j,$

(4) $(D_1 D^*_{22}\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1) - (D_2 D^*_{11}\check{D}_2 + D_2\tilde{D}_{11}\check{D}_2) = 0,$

(5) $\tilde{D}_{ii}D^*_{ii}\check{D}_j + \tilde{D}_{ii}\tilde{D}_{ii}\check{D}_j + D^*_{ii}D^*_{ii}\check{D}_j + D^*_{ii}\tilde{D}_{ii}\check{D}_j = 0, \quad i, j = 1, 2, \quad i \neq j,$

(6) $D_{ii}D^*_{jj}\check{D}_i + D_{ii}\tilde{D}_{jj}\check{D}_i - D_{ij}D^*_{ii}\check{D}_j - D_{ij}\tilde{D}_{ii}\check{D}_j = 0, \quad i, j = 1, 2, \quad i \neq j,$

(7) $\tilde{D}_{ii}D^*_{jj}\check{D}_i + \tilde{D}_{ii}\tilde{D}_{jj}\check{D}_i + D^*_{ii}D^*_{jj}\check{D}_i + D^*_{ii}\tilde{D}_{jj}\check{D}_i$
$- D^*_{ij}D^*_{ii}\check{D}_j - D^*_{ij}\tilde{D}_{ii}\check{D}_j = 0, \quad i, j = 1, 2, \quad i \neq j,$

(8) $\check{D}_{ij}D^*_{kk}\check{D}_\ell + \check{D}_{ij}\tilde{D}_{kk}\check{D}_\ell = 0, \quad i, j, k, \ell = 1, 2, \quad k \neq \ell,$

(9) $D_{ii}D^*_{ii}\check{D}_j + D_{ii}\tilde{D}_{ii}\check{D}_j = 0, \quad i,j,=1,2, \quad i \neq j,$

(10) $\tilde{D}_{ij}D^*_{kk}\check{D}_\ell + \tilde{D}_{ij}\tilde{D}_{kk}\check{D}_\ell = 0, \quad i,j,k,\ell = 1,2, \qquad k \neq \ell, \quad i \neq j,$

(11) $D^*_{ji}D^*_{ii}\check{D}_j + D^*_{ji}\tilde{D}_{ii}\check{D}_j = 0, \quad i,j,=1,2, \quad i \neq j,$

(12) $D_{ji}D^*_{ii}\check{D}_j + D_{ji}\tilde{D}_{ii}\check{D}_j = 0, \quad i,j,=1,2, \quad i \neq j.$

Proof. Let us consider $g \in F_1$. By the previous computations we have that $d^1 g$ is of the form

$$d^1 g = (D^*_{11}\check{D}_2 + \tilde{D}_{11}\check{D}_2)(\Delta_1 g_2 - \bar{\partial}_2\partial_1 g_1) + (D^*_{22}\check{D}_1 + \tilde{D}_{22}\check{D}_1)(\Delta_2 g_1 - \bar{\partial}_1\partial_2 g_2)$$

and

$$d^2 d^1 g = d^2_1 d^1 g + d^2_2 d^1 g$$

so

$$\begin{aligned}
d^2_1 d^1 g \; &= [\check{D}_1\bar{\partial}_1 + \check{D}_2\bar{\partial}_2 + D_1\partial_1 + D_2\partial_2][(D^*_{11}\check{D}_2 + \tilde{D}_{11}\check{D}_2)(\Delta_1 g_2 - \bar{\partial}_2\partial_1 g_1) \\
&+(D^*_{22}\check{D}_1 + \tilde{D}_{22}\check{D}_1)(\Delta_2 g_1 - \bar{\partial}_1\partial_2 g_2)] \\
&= (\check{D}_1 D^*_{11}\check{D}_2 + \check{D}_1\tilde{D}_{11}\check{D}_2)(\bar{\partial}_1\Delta_1 g_2 - \bar{\partial}_1\bar{\partial}_2\partial_1 g_1) \\
&+(\check{D}_1 D^*_{22}\check{D}_1 + \check{D}_1\tilde{D}_{22}\check{D}_1)(\bar{\partial}_1\Delta_2 g_1 - \bar{\partial}_1\bar{\partial}_1\partial_2 g_2) \\
&+(\check{D}_2 D^*_{11}\check{D}_2 + \check{D}_2\tilde{D}_{11}\check{D}_2)(\bar{\partial}_2\Delta_1 g_2 - \bar{\partial}_2\bar{\partial}_2\partial_1 g_1) \\
&+(\check{D}_2 D^*_{22}\check{D}_1 + \check{D}_2\tilde{D}_{22}\check{D}_1)(\bar{\partial}_2\Delta_2 g_1 - \bar{\partial}_2\bar{\partial}_1\partial_2 g_2) \\
&+(D_1 D^*_{11}\check{D}_2 + D_1\tilde{D}_{11}\check{D}_2)(\partial_1\Delta_1 g_2 - \partial_1\bar{\partial}_2\partial_1 g_1) \\
&+(D_1 D^*_{22}\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1)(\partial_1\Delta_2 g_1 - \Delta_1\partial_2 g_2) \\
&+(D_2 D^*_{11}\check{D}_2 + D_2\tilde{D}_{11}\check{D}_2)(\partial_2\Delta_1 g_2 - \Delta_2\partial_1 g_1) \\
&+(D_2 D^*_{22}\check{D}_1 + D_2\tilde{D}_{22}\check{D}_1)(\partial_2\Delta_2 g_1 - \partial_2\bar{\partial}_1\partial_2 g_2) = 0.
\end{aligned} \tag{3}$$

Grouping the various terms we obtain the conditions

$$\check{D}_1 D^*_{11}\check{D}_2 + \check{D}_1\tilde{D}_{11}\check{D}_2 = 0,$$

$$\check{D}_1 D^*_{22}\check{D}_1 + \check{D}_1\tilde{D}_{22}\check{D}_1 = 0,$$

$$(D_1 D^*_{22}\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1) - (D_2 D^*_{11}\check{D}_2 + D_2\tilde{D}_{11}\check{D}_2) = 0,$$

$$\check{D}_2 D^*_{11}\check{D}_2 + \check{D}_2\tilde{D}_{11}\check{D}_2 = 0,$$

$$D_1 D^*_{11}\check{D}_2 + D_1\tilde{D}_{11}\check{D}_2 = 0,$$

$$D_1 D^*_{22}\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1 = 0,$$

$$D_2 D^*_{22}\check{D}_1 + D_2\tilde{D}_{22}\check{D}_1 = 0,$$

that can be written in compact form as in (1) − −(4). By imposing $d_2^2 d^1 g = 0$ we obtain

$$
\begin{aligned}
d_2^2 d^1 g \quad &= [(\check{D}_{11} D_{11}^* \check{D}_2 + \check{D}_{11} \tilde{D}_{11} \check{D}_2)(\bar{\partial}_1^2 \Delta_1 g_2 - \bar{\partial}_1^2 \bar{\partial}_2 \partial_1 g_1) \\
&+(\check{D}_{11} D_{22}^* \check{D}_1 + \check{D}_{11} \tilde{D}_{22} \check{D}_1)(\bar{\partial}_1^2 \Delta_2 g_1 - \bar{\partial}_1^2 \bar{\partial}_1 \partial_2 g_2)] \\
&+[(D_{11} D_{11}^* \check{D}_2 + D_{11} \tilde{D}_{11} \check{D}_2)(\partial_1^2 \Delta_1 g_2 - \partial_1^2 \bar{\partial}_2 \partial_1 g_1) \\
&+(D_{11} D_{22}^* \check{D}_1 + D_{11} \tilde{D}_{22} \check{D}_1)(\partial_1^2 \Delta_2 g_1 - \partial_1^2 \bar{\partial}_1 \partial_2 g_2)] \\
&+[(\tilde{D}_{11} D_{11}^* \check{D}_2 + \tilde{D}_{11} \tilde{D}_{11} \check{D}_2)(\Delta_1^2 g_2 - \Delta_1 \bar{\partial}_2 \partial_1 g_1) \\
&+(\tilde{D}_{11} D_{22}^* \check{D}_1 + \tilde{D}_{11} \tilde{D}_{22} \check{D}_1)(\Delta_1 \Delta_2 g_1 - \Delta_1 \bar{\partial}_1 \partial_2 g_2)] \\
&+[(D_{11}^* D_{11}^* \check{D}_2 + D_{11}^* \tilde{D}_{11} \check{D}_2)(\Delta_1^2 g_2 - \Delta_1 \bar{\partial}_2 \partial_1 g_1) \\
&+(D_{11}^* D_{22}^* \check{D}_1 + D_{11}^* \tilde{D}_{22} \check{D}_1)(\Delta_1 \Delta_2 g_1 - \Delta_1 \bar{\partial}_1 \partial_2 g_2)] \\
&+[(\check{D}_{12} D_{11}^* \check{D}_2 + \check{D}_{12} \tilde{D}_{11} \check{D}_2)(\bar{\partial}_1 \bar{\partial}_2 \Delta_1 g_2 - \bar{\partial}_1 \bar{\partial}_2^2 \partial_1 g_1) \\
&+(\check{D}_{12} D_{22}^* \check{D}_1 + \check{D}_{12} \tilde{D}_{22} \check{D}_1)(\bar{\partial}_1 \bar{\partial}_2 \Delta_2 g_1 - \bar{\partial}_1 \bar{\partial}_2 \bar{\partial}_1 \partial_2 g_2)] \\
&+[(D_{12} D_{11}^* \check{D}_2 + D_{12} \tilde{D}_{11} \check{D}_2)(\partial_1 \partial_2 \Delta_1 g_2 - \partial_1 \Delta_2 \partial_1 g_1) \\
&+(D_{12} D_{22}^* \check{D}_1 + D_{12} \tilde{D}_{22} \check{D}_1)(\partial_1 \partial_2 \Delta_2 g_1 - \partial_1 \partial_2 \bar{\partial}_1 \partial_2 g_2)] \\
&+[(\tilde{D}_{12} D_{11}^* \check{D}_2 + \tilde{D}_{12} \tilde{D}_{11} \check{D}_2)(\partial_1 \bar{\partial}_2 \Delta_1 g_2 - \partial_1 \bar{\partial}_2^2 \partial_1 g_1) \\
&+(\tilde{D}_{12} D_{22}^* \check{D}_1 + \tilde{D}_{12} \tilde{D}_{22} \check{D}_1)(\partial_1 \bar{\partial}_2 \Delta_2 g_1 - \partial_1 \bar{\partial}_2 \bar{\partial}_1 \partial_2 g_2)] \\
&+[(D_{12}^* D_{11}^* \check{D}_2 + D_{12}^* \tilde{D}_{11} \check{D}_2)(\bar{\partial}_1 \partial_2 \Delta_1 g_2 - \bar{\partial}_1 \partial_2 \bar{\partial}_2 \partial_1 g_1) \\
&+(D_{12}^* D_{22}^* \check{D}_1 + D_{12}^* \tilde{D}_{22} \check{D}_1)(\bar{\partial}_1 \partial_2 \Delta_2 g_1 - \bar{\partial}_1 \partial_2 \bar{\partial}_1 \partial_2 g_2)] \\
&+[(\check{D}_{21} D_{11}^* \check{D}_2 + \check{D}_{21} \tilde{D}_{11} \check{D}_2)(\bar{\partial}_2 \bar{\partial}_1 \Delta_1 g_2 - \bar{\partial}_2 \bar{\partial}_1 \bar{\partial}_2 \partial_1 g_1) \\
&+(\check{D}_{21} D_{22}^* \check{D}_1 + \check{D}_{21} \tilde{D}_{22} \check{D}_1)(\bar{\partial}_2 \bar{\partial}_1 \Delta_2 g_1 - \bar{\partial}_2 \bar{\partial}_1^2 \partial_2 g_2)] \\
&+[(D_{21} D_{11}^* \check{D}_2 + D_{21} \tilde{D}_{11} \check{D}_2)(\partial_2 \partial_1 \Delta_1 g_2 - \partial_2 \partial_1 \bar{\partial}_2 \partial_1 g_1) \\
&+(D_{21} D_{22}^* \check{D}_1 + D_{21} \tilde{D}_{22} \check{D}_1)(\partial_2 \partial_1 \Delta_2 g_1 - \partial_2 \Delta_1 \partial_2 g_2)] \\
&+[(\tilde{D}_{21} D_{11}^* \check{D}_2 + \tilde{D}_{21} \tilde{D}_{11} \check{D}_2)(\partial_2 \bar{\partial}_1 \Delta_1 g_2 - \partial_2 \bar{\partial}_1 \bar{\partial}_2 \partial_1 g_1) \\
&+(\tilde{D}_{21} D_{22}^* \check{D}_1 + \tilde{D}_{21} \tilde{D}_{22} \check{D}_1)(\partial_2 \bar{\partial}_1 \Delta_2 g_1 - \partial_2 \bar{\partial}_1^2 \partial_2 g_2)] \\
&+[(D_{21}^* D_{11}^* \check{D}_2 + D_{21}^* \tilde{D}_{11} \check{D}_2)(\bar{\partial}_2 \partial_1 \Delta_1 g_2 - \bar{\partial}_2 \partial_1 \bar{\partial}_2 \partial_1 g_1) \\
&+(D_{21}^* D_{22}^* \check{D}_1 + D_{21}^* \tilde{D}_{22} \check{D}_1)(\bar{\partial}_2 \partial_1 \Delta_2 g_1 - \bar{\partial}_2 \Delta_1 \partial_2 g_2)] \\
&+[(\check{D}_{22} D_{11}^* \check{D}_2 + \check{D}_{22} \tilde{D}_{11} \check{D}_2)(\bar{\partial}_2^2 \Delta_1 g_2 - \bar{\partial}_2^3 \partial_1 g_1) \\
&+(\check{D}_{22} D_{22}^* \check{D}_1 + \check{D}_{22} \tilde{D}_{22} \check{D}_1)(\bar{\partial}_2^2 \Delta_2 g_1 - \bar{\partial}_2^2 \bar{\partial}_1 \partial_2 g_2)] \\
&+[(D_{22} D_{11}^* \check{D}_2 + D_{22} \tilde{D}_{11} \check{D}_2)(\partial_2^2 \Delta_1 g_2 - \partial_2^2 \bar{\partial}_2 \partial_1 g_1) \\
&+(D_{22} D_{22}^* \check{D}_1 + D_{22} \tilde{D}_{22} \check{D}_1)(\partial_2^2 \Delta_2 g_1 - \partial_2^2 \bar{\partial}_1 \partial_2 g_2)] \\
&+[(\tilde{D}_{22} D_{11}^* \check{D}_2 + \tilde{D}_{22} \tilde{D}_{11} \check{D}_2)(\Delta_2 \Delta_1 g_2 - \Delta_2 \bar{\partial}_2 \partial_1 g_1) \\
&+(\tilde{D}_{22} D_{22}^* \check{D}_1 + \tilde{D}_{22} \tilde{D}_{22} \check{D}_1)(\Delta_2^2 g_1 - \Delta_2 \bar{\partial}_1 \partial_2 g_2)] \\
&+[(D_{22}^* D_{11}^* \check{D}_2 + D_{22}^* \tilde{D}_{11} \check{D}_2)(\Delta_2 \Delta_1 g_2 - \Delta_2 \bar{\partial}_2 \partial_1 g_1) \\
&+(D_{22}^* D_{22}^* \check{D}_1 + D_{22}^* \tilde{D}_{22} \check{D}_1)(\Delta_2^2 g_1 - \Delta_2 \bar{\partial}_1 \partial_2 g_2)] = 0.
\end{aligned} \tag{4}
$$

By grouping the various derivatives together we obtain *(5)–(12)*. □

Proposition 3.9 *Let*

$$
\begin{aligned}
d^3 = d_1^3 + d_2^3 = \quad & \check{D}_1 \bar{\partial}_1 + \check{D}_2 \bar{\partial}_2 + D_1 \partial_1 + D_2 \partial_2 \\
& + \sum_{i,j=1}^{2} (\check{D}_{ij} \bar{\partial}_i \bar{\partial}_j + D_{ij} \partial_i \partial_j + \tilde{D}_{ij} \partial_i \bar{\partial}_j + D_{ij}^* \bar{\partial}_i \partial_j).
\end{aligned} \tag{5}
$$

Then $d^3 d_1^2 = 0$ *implies*

(1) $\check{D}_i D_1 D_{22}^* \check{D}_1 + \check{D}_i D_1 \tilde{D}_{22} \check{D}_1 = 0, \quad i = 1, 2,$

(2) $D_i D_1 D_{22}^* \check{D}_1 + D_i D_1 \tilde{D}_{22} \check{D}_1 = 0, \quad i = 1, 2,$

(3) $(\tilde{D}_{ii} D_1 D_{22}^* \check{D}_1 + \tilde{D}_{ii} D_1 \tilde{D}_{22} \check{D}_1) + (D_{ii}^* D_1 D_{22}^* \check{D}_1 + D_{ii}^* D_1 \tilde{D}_{22} \check{D}_1) = 0, \quad i, j = 1, 2,\ i \neq j,$

(4) $(D_{ij}D_1D_{22}^*\check{D}_1 + D_{ij}D_1\tilde{D}_{22}\check{D}_1) = 0, \quad i,j = 1,2,$

(5) $(\check{D}_{ij}D_1D_{22}^*\check{D}_1 + \check{D}_{ij}D_1\tilde{D}_{22}\check{D}_1) = 0, \quad i,j = 1,2,$

(6) $(\tilde{D}_{ij}D_1D_{22}^*\check{D}_1 + \tilde{D}_{ij}D_1\tilde{D}_{22}\check{D}_1) = 0, \quad i,j = 1,2,\ i \neq j,$

(7) $(D_{ij}^*D_1D_{22}^*\check{D}_1 + D_{ij}^*D_1\tilde{D}_{22}\check{D}_1) = 0, \quad i,j = 1,2,\ i \neq j.$

Proof. Let $h \in F_2$, $h = (D_{11}^*\check{D}_2 + \tilde{D}_{11}\check{D}_2)h_{12} + (D_{22}^*\check{D}_1 + \tilde{D}_{22}\check{D}_1)h_{21}$. Then, using (9) (see the paper) and relations in Proposition 3.6, we can write d_1^2h as

$$d_1^2h = (D_1D_{22}^*\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1)(\partial_1h_{21} + \partial_2h_{12}).$$

Let us compute separately $d_1^3d_1^2h$ and $d_2^3d_1^2h$:

$$\begin{aligned} d_1^3d_1^2h \; &= [\check{D}_1\bar{\partial}_1 + \check{D}_2\bar{\partial}_2 + D_1\partial_1 + D_2\partial_2][(D_1D_{22}^*\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1)(\partial_1h_{21} + \partial_2h_{12})] \\ &= (\check{D}_1D_1D_{22}^*\check{D}_1 + \check{D}_1D_1\tilde{D}_{22}\check{D}_1)(\Delta_1h_{21} + \bar{\partial}_1\partial_2h_{12}) \\ &+(\check{D}_2D_1D_{22}^*\check{D}_1 + \check{D}_2D_1\tilde{D}_{22}\check{D}_1)(\bar{\partial}_2\partial_1h_{21} + \Delta_2h_{12}) \\ &+(D_1D_1D_{22}^*\check{D}_1 + D_1D_1\tilde{D}_{22}\check{D}_1)(\partial_1^2h_{21} + \partial_1\partial_2h_{12}) \\ &+(D_2D_1D_{22}^*\check{D}_1 + D_2D_1\tilde{D}_{22}\check{D}_1)(\partial_2\partial_1h_{21} + \partial_2^2h_{12}). \end{aligned} \tag{6}$$

Condition $d_1^3d_1^2h = 0$ implies relations (1) and (2), so that $d_1^3d_1^2h \equiv 0$. Now we compute:

$$\begin{aligned} d_2^3d_1^2h \; &= [\check{D}_{11}\bar{\partial}_1\bar{\partial}_1 + D_{11}\partial_1\partial_1 + \tilde{D}_{11}\partial_1\bar{\partial}_1 + D_{11}^*\bar{\partial}_1\partial_1 + \check{D}_{12}\bar{\partial}_1\bar{\partial}_2 \\ &+D_{12}\partial_1\partial_2 + \tilde{D}_{12}\partial_1\bar{\partial}_2 + D_{12}^*\bar{\partial}_1\partial_2 + \check{D}_{21}\bar{\partial}_2\bar{\partial}_1 \\ &+D_{21}\partial_2\partial_1 + \tilde{D}_{21}\partial_2\bar{\partial}_1 + D_{21}^*\bar{\partial}_2\partial_1 + \check{D}_{22}\bar{\partial}_2\bar{\partial}_2 + D_{22}\partial_2\partial_2 \\ &+\tilde{D}_{22}\partial_2\bar{\partial}_2 + D_{22}^*\bar{\partial}_2\partial_2][(D_1D_{22}^*\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1)(\partial_1h_{21} + \partial_2h_{12})] \\ &= [(\check{D}_{11}D_1D_{22}^*\check{D}_1 + \check{D}_{11}D_1\tilde{D}_{22}\check{D}_1)(\Delta_1\bar{\partial}_1h_{21} + \bar{\partial}_1^2\partial_2h_{12})] \\ &+[(D_{11}D_1D_{22}^*\check{D}_1 + D_{11}D_1\tilde{D}_{22}\check{D}_1)(\partial_1^3h_{21} + \partial_1^2\partial_2h_{12})] \\ &+[(\tilde{D}_{11}D_1D_{22}^*\check{D}_1 + \tilde{D}_{11}D_1\tilde{D}_{22}\check{D}_1)(\Delta_1\partial_1h_{21} + \Delta_1\partial_2h_{12})] \\ &+[(D_{11}^*D_1D_{22}^*\check{D}_1 + D_{11}^*D_1\tilde{D}_{22}\check{D}_1)(\Delta_1\partial_1h_{21} + \Delta_1\partial_2h_{12})] \\ &+[(\check{D}_{12}D_1D_{22}^*\check{D}_1 + \check{D}_{12}D_1\tilde{D}_{22}\check{D}_1)(\bar{\partial}_1\bar{\partial}_2\partial_1h_{21} + \bar{\partial}_1\Delta_2^2h_{12})] \\ &+[(D_{12}D_1D_{22}^*\check{D}_1 + D_{12}D_1\tilde{D}_{22}\check{D}_1)(\partial_1\partial_2\partial_1h_{21} + \partial_1\partial_2^2h_{12})] \\ &+[(\tilde{D}_{12}D_1D_{22}^*\check{D}_1 + \tilde{D}_{12}D_1\tilde{D}_{22}\check{D}_1)(\partial_1\bar{\partial}_2\partial_1h_{21} + \partial_1\Delta_2h_{12})] \\ &+[(D_{12}^*D_1D_{22}^*\check{D}_1 + D_{12}^*D_1\tilde{D}_{22}\check{D}_1)(\bar{\partial}_1\partial_2\partial_1h_{21} + \bar{\partial}_1\partial_2^2h_{12})] \\ &+[(\check{D}_{21}D_1D_{22}^*\check{D}_1 + \check{D}_{21}D_1\tilde{D}_{22}\check{D}_1)(\bar{\partial}_2\Delta_1h_{21} + \bar{\partial}_2\bar{\partial}_1\partial_2h_{12})] \\ &+[(D_{21}D_1D_{22}^*\check{D}_1 + D_{21}D_1\tilde{D}_{22}\check{D}_1)(\partial_2\partial_1^2h_{21} + \partial_2\partial_1\partial_2h_{12})] \\ &+[(\tilde{D}_{21}D_1D_{22}^*\check{D}_1 + \tilde{D}_{21}D_1\tilde{D}_{22}\check{D}_1)(\partial_2\Delta_1h_{21} + \partial_2\bar{\partial}_1\partial_2h_{12})] \\ &+[(D_{21}^*D_1D_{22}^*\check{D}_1 + D_{21}^*D_1\tilde{D}_{22}\check{D}_1)(\bar{\partial}_2\partial_1^2h_{21} + \bar{\partial}_2\partial_1\partial_2h_{12})] \\ &+[(\check{D}_{22}D_1D_{22}^*\check{D}_1 + \check{D}_{22}D_1\tilde{D}_{22}\check{D}_1)(\bar{\partial}_2\bar{\partial}_2\partial_1h_{21} + \bar{\partial}_2\Delta_2h_{12})] \\ &+[(D_{22}D_1D_{22}^*\check{D}_1 + D_{22}D_1\tilde{D}_{22}\check{D}_1)(\partial_2^2\partial_1h_{21} + \partial_2^3h_{12})] \\ &+[(\tilde{D}_{22}D_1D_{22}^*\check{D}_1 + \tilde{D}_{22}D_1\tilde{D}_{22}\check{D}_1)(\Delta_2\partial_1h_{21} + \Delta_2\partial_2h_{12})] \\ &+[(D_{22}^*D_1D_{22}^*\check{D}_1 + D_{22}^*D_1\tilde{D}_{22}\check{D}_1)(\Delta_2\partial_1h_{21} + \Delta_2\partial_2h_{12})]. \end{aligned} \tag{7}$$

The condition

$$d_2^3d_1^2h = 0$$

gives the relations (3) – (7) listed in the statement. □

(B) Vanishing of d^3d^2

In this Appendix we perform all the computations coming from the vanishing $d^3d^2 = 0$ (see Proposition 3.8 and Remark 3.10 in the paper).

Proposition B.1 Let, for $\ell = 2, 3$,

$$d^\ell = d_1^\ell + d_2^\ell = \check{D}_1\bar{\partial}_1 + \check{D}_2\bar{\partial}_2 + D_1\partial_1 + D_2\partial_2$$

$$+ \sum_{i,j=1}^{2} (\check{D}_{ij}\bar{\partial}_i\bar{\partial}_j + D_{ij}\partial_i\partial_j + \tilde{D}_{ij}\partial_i\bar{\partial}_j + D^*_{ij}\bar{\partial}_i\partial_j).$$

Then $d^3d^2 = 0$ implies

(1) $\check{D}_iD_1D^*_{22}\check{D}_1 + \check{D}_iD_1\tilde{D}_{22}\check{D}_1 = 0, \quad i = 1, 2,$

(2) $D_iD_1D^*_{22}\check{D}_1 + D_iD_1\tilde{D}_{22}\check{D}_1 = 0, \quad i = 1, 2,$

(3) $(\check{D}_iD_{ii}D^*_{jj}\check{D}_i + \check{D}_iD_{ii}\tilde{D}_{jj}\check{D}_i) + (D_i\tilde{D}_{ii}D^*_{jj}\check{D}_i + D_i\tilde{D}_{ii}\tilde{D}_{jj}\check{D}_i + D_iD^*_{ii}D^*_{jj}\check{D}_i + D_iD^*_{ii}\tilde{D}_{jj}\check{D}_i) + (\tilde{D}_{ii}D_1D^*_{22}\check{D}_1 + \tilde{D}_{ii}D_1\tilde{D}_{22}\check{D}_1) + (D^*_{ii}D_1D^*_{22}\check{D}_1 + D^*_{ii}D_1\tilde{D}_{22}\check{D}_1) = 0, \quad i, j = 1, 2, i \neq j$

(4) $(D_jD_{jj}D^*_{ii}\check{D}_j + D_jD_{jj}\tilde{D}_{ii}\check{D}_j) + (D_{jj}D_1D^*_{22}\check{D}_1 + D_{jj}D_1\tilde{D}_{22}\check{D}_1) = 0, \quad i, j = 1, 2,$

(5) $(\check{D}_i\tilde{D}_{ii}D^*_{jj}\check{D}_i + \check{D}_i\tilde{D}_{ii}\tilde{D}_{jj}\check{D}_i + \check{D}_iD^*_{ii}D^*_{jj}\check{D}_i + \check{D}_iD^*_{ii}\tilde{D}_{jj}\check{D}_i) + (\check{D}_{ii}D_1D^*_{22}\check{D}_1 + \check{D}_{ii}D_1\tilde{D}_{22}\check{D}_1) = 0, \quad i, j = 1, 2,$

(6) $(\check{D}_iD_{jj}D^*_{ii}\check{D}_j + \check{D}_iD_{jj}\tilde{D}_{ii}\check{D}_j) + (D^*_{ij}D_1D^*_{22}\check{D}_1 + D^*_{ij}D_1\tilde{D}_{22}\check{D}_1) = 0, \quad i, j = 1, 2,$

(7) $(\check{D}_i\tilde{D}_{jj}D^*_{ii}\check{D}_j + \check{D}_i\tilde{D}_{jj}\tilde{D}_{ii}\check{D}_j + \check{D}_iD^*_{jj}D^*_{ii}\check{D}_j + \check{D}_iD^*_{jj}\tilde{D}_{ii}\check{D}_j) + (\check{D}_{ij}D_1D^*_{22}\check{D}_1 + \check{D}_{ij}D_1\tilde{D}_{22}\check{D}_1) = 0$

(7) $(D_iD_{jj}D^*_{ii}\check{D}_j + D_iD_{jj}\tilde{D}_{ii}\check{D}_j) + (D_{ij}D_1D^*_{22}\check{D}_1 + D_{ij}D_1\tilde{D}_{22}\check{D}_1) = 0, i, j = 1, 2$ $i \neq j$

(8) $(D_i\tilde{D}_{jj}D^*_{ii}\check{D}_j + D_i\tilde{D}_{jj}\tilde{D}_{ii}\check{D}_j + D_iD^*_{jj}D^*_{ii}\check{D}_j + D_iD^*_{jj}\tilde{D}_{ii}\check{D}_j) + (\tilde{D}_{ij}D_1D^*_{22}\check{D}_1 + \tilde{D}_{ij}D_1\tilde{D}_{22}\check{D}_1) = 0, i, j = 1, 2, i \neq j$

(9) $\check{D}_{ii}D_{ii}D^*_{jj}\check{D}_i + \check{D}_{ii}D_{ii}\tilde{D}_{jj}\check{D}_i + \tilde{D}_{ii}\tilde{D}_{ii}D^*_{jj}\check{D}_i + \tilde{D}_{ii}\tilde{D}_{ii}\tilde{D}_{jj}\check{D}_i + \tilde{D}_{ii}D^*_{ii}D^*_{jj}\check{D}_i + \tilde{D}_{ii}D^*_{ii}\tilde{D}_{jj}\check{D}_i + D^*_{ii}\tilde{D}_{ii}D^*_{jj}\check{D}_i + D^*_{ii}\tilde{D}_{ii}\tilde{D}_{jj}\check{D}_i + D^*_{ii}D^*_{ii}D^*_{jj}\check{D}_i + D^*_{ii}D^*_{ii}\tilde{D}_{jj}\check{D}_i = 0, i, j = 1, 2, i \neq j$

(10) $\check{D}_{ii}D_{jj}D^*_{ii}\check{D}_j + \check{D}_{ii}D_{jj}\tilde{D}_{ii}\check{D}_j = 0, i, j = 1, 2, i \neq j$

(11) $\check{D}_{ii}\tilde{D}_{jj}D^*_{ii}\check{D}_j + \check{D}_{ii}\tilde{D}_{jj}\tilde{D}_{ii}\check{D}_j + \check{D}_{ii}D^*_{jj}D^*_{ii}\check{D}_j + \check{D}_{ii}D^*_{jj}\tilde{D}_{ii}\check{D}_j = 0, i, j = 1, 2, i \neq j$

(12) $\check{D}_{ii}\tilde{D}_{ii}D^*_{jj}\check{D}_i + \check{D}_{ii}\tilde{D}_{ii}\tilde{D}_{jj}\check{D}_i + \check{D}_{ii}D^*_{ii}D^*_{jj}\check{D}_i + \check{D}_{ii}D^*_{ii}\tilde{D}_{jj}\check{D}_i = 0, i, j = 1, 2, i \neq j$

(13) $D_{ii}D_{ii}D^*_{jj}\check{D}_i + D_{ii}D_{ii}\tilde{D}_{jj}\check{D}_i = 0, i, j = 1, 2, i \neq j$

(14) $D_{ii}D_{jj}D^*_{ii}\check{D}_j + D_{ii}D_{jj}\tilde{D}_{ii}\check{D}_j = 0, i, j = 1, 2, i \neq j$

(15) $D_{ii}\tilde{D}_{jj}D^*_{ii}\check{D}_j + D_{ii}\tilde{D}_{jj}\tilde{D}_{ii}\check{D}_j + D_{ii}D^*_{jj}D^*_{ii}\check{D}_j + D_{ii}D^*_{jj}\tilde{D}_{ii}\check{D}_j = 0, i, j = 1, 2, i \neq j$

(16) $D_{ii}\tilde{D}_{ii}D^*_{jj}\check{D}_i + D_{ii}\tilde{D}_{ii}\tilde{D}_{jj}\check{D}_i + D_{ii}D^*_{ii}D^*_{jj}\check{D}_i + D_{ii}D^*_{ii}\tilde{D}_{jj}\check{D}_i + \tilde{D}_{ii}D_{ii}D^*_{jj}\check{D}_i + \tilde{D}_{ii}D_{ii}\tilde{D}_{jj}\check{D}_i + D^*_{ii}D_{ii}D^*_{jj}\check{D}_i + D^*_{ii}D_{ii}\tilde{D}_{jj}\check{D}_i = 0, i, j = 1, 2, i \neq j$

(17) $\tilde{D}_{ii}D_{jj}D^*_{ii}\check{D}_j + \tilde{D}_{ii}D_{jj}\tilde{D}_{ii}\check{D}_j + D^*_{ii}D_{jj}D^*_{ii}\check{D}_j + D^*_{ii}D_{jj}\tilde{D}_{ii}\check{D}_j = 0, i, j = 1, 2, i \neq j$

(18) $\tilde{D}_{ii}\tilde{D}_{jj}D^*_{ii}\check{D}_j + \tilde{D}_{ii}\tilde{D}_{jj}\tilde{D}_{ii}\check{D}_j + \tilde{D}_{ii}D^*_{jj}D^*_{ii}\check{D}_j + \tilde{D}_{ii}D^*_{jj}\tilde{D}_{ii}\check{D}_j + D^*_{ii}\tilde{D}_{jj}D^*_{ii}\check{D}_j + D^*_{ii}\tilde{D}_{jj}\tilde{D}_{ii}\check{D}_j + D^*_{ii}D^*_{jj}D^*_{ii}\check{D}_j + D^*_{ii}D^*_{jj}\tilde{D}_{ii}\check{D}_j + \check{D}_{ji}D_{ii}D^*_{jj}\check{D}_i + \check{D}_{ji}D_{ii}\tilde{D}_{jj}\check{D}_i + D^*_{ji}\tilde{D}_{ii}D^*_{jj}\check{D}_i + D^*_{ji}\tilde{D}_{ii}\tilde{D}_{jj}\check{D}_i + D^*_{ji}D^*_{ii}D^*_{jj}\check{D}_i + D^*_{ji}D^*_{ii}\tilde{D}_{jj}\check{D}_i = 0, i, j = 1, 2, i \neq j$

(19) $\check{D}_{ij}D_{ii}D^*_{jj}\check{D}_i + \check{D}_{ij}D_{ii}\tilde{D}_{jj}\check{D}_i = 0, i, j = 1, 2, i \neq j$

(20) $\check{D}_{ij}\tilde{D}_{22}D^*_{11}\check{D}_2 + \check{D}_{ij}\tilde{D}_{22}\tilde{D}_{11}\check{D}_2 + \check{D}_{ij}D^*_{22}D^*_{11}\check{D}_2 + \check{D}_{ij}D^*_{22}\tilde{D}_{11}\check{D}_2 = 0, i, j = 1, 2, i \neq j$

(21) $\check{D}_{ij}\tilde{D}_{11}D^*_{22}\check{D}_1 + \check{D}_{ij}\tilde{D}_{11}\tilde{D}_{22}\check{D}_1 + \check{D}_{ij}D^*_{11}D^*_{22}\check{D}_1 + \check{D}_{ij}D^*_{11}\tilde{D}_{22}\check{D}_1 = 0, i, j = 1, 2, i \neq j$

(22) $D_{12}D_{ii}D^*_{jj}\check{D}_i + D_{12}D_{ii}\tilde{D}_{jj}\check{D}_i = 0, i, j = 1, 2, \quad i \neq j$

(23) $D_{ij}\tilde{D}_{jj}D^*_{ii}\check{D}_j + D_{ij}\tilde{D}_{jj}\tilde{D}_{ii}\check{D}_j + D_{ij}D^*_{jj}D^*_{ii}\check{D}_j + D_{ij}D^*_{jj}\tilde{D}_{ii}\check{D}_j + \tilde{D}_{ij}D_{jj}D^*_{ii}\check{D}_j + \tilde{D}_{ij}D_{jj}\tilde{D}_{ii}\check{D}_j = 0, i, j = 1, 2, \quad i \neq j$

(24) $D_{ij}\tilde{D}_{ii}D^*_{jj}\check{D}_i + D_{ij}\tilde{D}_{ii}\tilde{D}_{jj}\check{D}_i + D_{ij}D^*_{ii}D^*_{jj}\check{D}_i + D_{ij}D^*_{ii}\tilde{D}_{jj}\check{D}_i = 0, i, j = 1, 2, \quad i \neq j$

(25) $\tilde{D}_{ij}D_{ii}D^*_{jj}\check{D}_i + \tilde{D}_{ij}D_{ii}\tilde{D}_{jj}\check{D}_i = 0, i, j = 1, 2, i \neq j$

(26) $\tilde{D}_{ij}\tilde{D}_{jj}D^*_{ii}\check{D}_j + \tilde{D}_{ij}\tilde{D}_{jj}\tilde{D}_{ii}\check{D}_j + \tilde{D}_{ij}D^*_{jj}D^*_{ii}\check{D}_j + \tilde{D}_{ij}D^*_{jj}\tilde{D}_{ii}\check{D}_j = 0, i, j = 1, 2, i \neq j$

(27) $D^*_{ij}D_{11}D^*_{22}\check{D}_1 + D^*_{ij}D_{11}\tilde{D}_{22}\check{D}_1 = 0, i, j = 1, 2, i \neq j$

(28) $D^*_{ij}D_{22}D^*_{11}\check{D}_2 + D^*_{ij}D_{22}\tilde{D}_{11}\check{D}_2, i, j = 1, 2, i \neq j = 0$

(29) $D^*_{ij}\tilde{D}_{ii}D^*_{jj}\check{D}_i + D^*_{ij}\tilde{D}_{ii}\tilde{D}_{jj}\check{D}_i + D^*_{ij}D^*_{ii}D^*_{jj}\check{D}_i + D^*_{ij}D^*_{ii}\tilde{D}_{jj}\check{D}_i = 0, i, j = 1, 2, i \neq j$

(30) $D_{21}D_{ii}D^*_{jj}\check{D}_i + D_{21}D_{ii}\tilde{D}_{jj}\check{D}_i = 0, i, j = 1, 2, i \neq j$

(31) $\tilde{D}_{21}\tilde{D}_{jj}D^*_{ii}\check{D}_j + \tilde{D}_{21}\tilde{D}_{jj}\tilde{D}_{ii}\check{D}_j + \tilde{D}_{21}D^*_{jj}D^*_{ii}\check{D}_j + \tilde{D}_{21}D^*_{jj}\tilde{D}_{ii}\check{D}_j = 0$

Proof. Let $h \in F_2$ be

$$h = (D^*_{11}\check{D}_2 + \tilde{D}_{11}\check{D}_2)h_{12} + (D^*_{22}\check{D}_1 + \tilde{D}_{22}\check{D}_1)h_{21}.$$

Then

$$d^2h = d^2_1h + d^2_2h =$$
$$(D_1D^*_{22}\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1)(\partial_1h_{21} + \partial_2h_{12}) + (D_{11}D^*_{22}\check{D}_1 + D_{11}\tilde{D}_{22}\check{D}_1)(\partial^2_1h_{21} + \partial_1\partial_2h_{12})$$
$$+(D_{22}D^*_{11}\check{D}_2 + D_{22}\tilde{D}_{11}\check{D}_2)(\partial^2_2h_{12} + \partial_2\partial_1h_{21})$$
$$+(\tilde{D}_{22}D^*_{11}\check{D}_2 + \tilde{D}_{22}\tilde{D}_{11}\check{D}_2 + D^*_{22}D^*_{11}\check{D}_2 + D^*_{22}\tilde{D}_{11}\check{D}_2)(\Delta_2h_{12} + \bar{\partial}_2\partial_1h_{21})$$
$$+(\tilde{D}_{11}D^*_{22}\check{D}_1 + \tilde{D}_{11}\tilde{D}_{22}\check{D}_1 + D^*_{11}D^*_{22}\check{D}_1 + D^*_{11}\tilde{D}_{22}\check{D}_1)(\Delta_1h_{21} + \bar{\partial}_1\partial_2h_{12})$$

Let us compute separately $d^3_1d^2_1h$, $(d^3_1d^2_2 + d^3_2d^2_1)h$ and $d^3_2d^2_2h$

$$d^3_1d^2_1h = [\check{D}_1\bar{\partial}_1 + \check{D}_2\bar{\partial}_2 + D_1\partial_1 + D_2\partial_2] \times [(D_1D^*_{22}\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1)(\partial_1h_{21} + \partial_2h_{12})]$$
$$= (\check{D}_1D_1D^*_{22}\check{D}_1 + \check{D}_1D_1\tilde{D}_{22}\check{D}_1)(\Delta_1h_{21} + \bar{\partial}_1\partial_2h_{12})$$
$$+(\check{D}_2D_1D^*_{22}\check{D}_1 + \check{D}_2D_1\tilde{D}_{22}\check{D}_1)(\bar{\partial}_2\partial_1h_{21} + \Delta_2h_{12})$$

$$+(D_1D_1D^*_{22}\check{D}_1 + D_1D_1\tilde{D}_{22}\check{D}_1)(\partial_1^2 h_{21} + \partial_1\partial_2 h_{12})$$
$$+(D_2D_1D^*_{22}\check{D}_1 + D_2D_1\tilde{D}_{22}\check{D}_1)(\partial_2\partial_1 h_{21} + \partial_2^2 h_{12})$$

The condition $d_1^3 d_1^2 h = 0$ implies the vanishing of all the relations $\check{D}_iD_1D^*_{22}\check{D}_1 + \check{D}_iD_1\tilde{D}_{22}\check{D}_1 = 0$, $D_iD_1D^*_{22}\check{D}_1 + D_iD_1\tilde{D}_{22}\check{D}_1 = 0$, i.e. relations *(1)* and *(2)*, so $d_1^3 d_1^2 h \equiv 0$.

Let us compute $(d_1^3 d_2^2 + d_2^3 d_1^2)h$. We have

$$d_1^3 d_2^2 h = [\check{D}_1\bar{\partial}_1 + \check{D}_2\bar{\partial}_2 + D_1\partial_1 + D_2\partial_2]$$
$$[(D_{11}D^*_{22}\check{D}_1 + D_{11}\tilde{D}_{22}\check{D}_1)(\partial_1^2 h_{21} + \partial_1\partial_2 h_{12}) + (D_{22}D^*_{11}\check{D}_2 + D_{22}\tilde{D}_{11}\check{D}_2)(\partial_2^2 h_{12} + \partial_2\partial_1 h_{21})$$
$$+(\tilde{D}_{22}D^*_{11}\check{D}_2 + \tilde{D}_{22}\tilde{D}_{11}\check{D}_2 + D^*_{22}D^*_{11}\check{D}_2 + D^*_{22}\tilde{D}_{11}\check{D}_2)(\Delta_2 h_{12} + \bar{\partial}_2\partial_1 h_{21})$$
$$+(\tilde{D}_{11}D^*_{22}\check{D}_1 + \tilde{D}_{11}\tilde{D}_{22}\check{D}_1 + D^*_{11}D^*_{22}\check{D}_1 + D^*_{11}\tilde{D}_{22}\check{D}_1)(\Delta_1 h_{21} + \bar{\partial}_1\partial_2 h_{12})]$$
$$= (\check{D}_1D_{11}D^*_{22}\check{D}_1 + \check{D}_1D_{11}\tilde{D}_{22}\check{D}_1)(\Delta_1\partial_1 h_{21} + \Delta_1\partial_2 h_{12})$$
$$+(\check{D}_1D_{22}D^*_{11}\check{D}_2 + \check{D}_1D_{22}\tilde{D}_{11}\check{D}_2)(\bar{\partial}_1\partial_2^2 h_{12} + \bar{\partial}_1\partial_2\partial_1 h_{21})$$
$$+(\check{D}_1\tilde{D}_{22}D^*_{11}\check{D}_2 + \check{D}_1\tilde{D}_{22}\tilde{D}_{11}\check{D}_2 + \check{D}_1D^*_{22}D^*_{11}\check{D}_2 + \check{D}_1D^*_{22}\tilde{D}_{11}\check{D}_2)(\bar{\partial}_1\Delta_2 h_{12} + \bar{\partial}_1\bar{\partial}_2\partial_1 h_{21})$$
$$+(\check{D}_1\tilde{D}_{11}D^*_{22}\check{D}_1 + \check{D}_1\tilde{D}_{11}\tilde{D}_{22}\check{D}_1 + \check{D}_1D^*_{11}D^*_{22}\check{D}_1 + \check{D}_1D^*_{11}\tilde{D}_{22}\check{D}_1)(\bar{\partial}_1\Delta_1 h_{21} + \bar{\partial}_1^2\partial_2 h_{12})$$
$$+(\check{D}_2D_{11}D^*_{22}\check{D}_1 + \check{D}_2D_{11}\tilde{D}_{22}\check{D}_1)(\bar{\partial}_2\partial_1^2 h_{21} + \bar{\partial}_2\partial_1\partial_2 h_{12})$$
$$+(\check{D}_2D_{22}D^*_{11}\check{D}_2 + \check{D}_2D_{22}\tilde{D}_{11}\check{D}_2)(\Delta_2\partial_2 h_{12} + \Delta_2\partial_1 h_{21})$$
$$+(\check{D}_2\tilde{D}_{22}D^*_{11}\check{D}_2 + \check{D}_2\tilde{D}_{22}\tilde{D}_{11}\check{D}_2 + \check{D}_2D^*_{22}D^*_{11}\check{D}_2 + \check{D}_2D^*_{22}\tilde{D}_{11}\check{D}_2)(\bar{\partial}_2\Delta_2 h_{12} + \bar{\partial}_2^2\partial_1 h_{21})$$
$$+(\check{D}_2\tilde{D}_{11}D^*_{22}\check{D}_1 + \check{D}_2\tilde{D}_{11}\tilde{D}_{22}\check{D}_1 + \check{D}_2D^*_{11}D^*_{22}\check{D}_1 + \check{D}_2D^*_{11}\tilde{D}_{22}\check{D}_1)(\bar{\partial}_2\Delta_1 h_{21} + \bar{\partial}_2\bar{\partial}_1\partial_2 h_{12})$$
$$+(D_1D_{11}D^*_{22}\check{D}_1 + D_1D_{11}\tilde{D}_{22}\check{D}_1)(\partial_1^3 h_{21} + \partial_1^2\partial_2 h_{12})$$
$$+(D_1D_{22}D^*_{11}\check{D}_2 + D_1D_{22}\tilde{D}_{11}\check{D}_2)(\partial_1\partial_2^2 h_{12} + \partial_1\partial_2\partial_1 h_{21})$$
$$+(D_1\tilde{D}_{22}D^*_{11}\check{D}_2 + D_1\tilde{D}_{22}\tilde{D}_{11}\check{D}_2 + D_1D^*_{22}D^*_{11}\check{D}_2 + D_1D^*_{22}\tilde{D}_{11}\check{D}_2)(\partial_1\Delta_2 h_{12} + \partial_1\bar{\partial}_2\partial_1 h_{21})$$
$$+(D_1\tilde{D}_{11}D^*_{22}\check{D}_1 + D_1\tilde{D}_{11}\tilde{D}_{22}\check{D}_1 + D_1D^*_{11}D^*_{22}\check{D}_1 + D_1D^*_{11}\tilde{D}_{22}\check{D}_1)(\partial_1\Delta_1 h_{21} + \Delta_1\partial_2 h_{12})$$
$$+(D_2D_{11}D^*_{22}\check{D}_1 + D_2D_{11}\tilde{D}_{22}\check{D}_1)(\partial_2\partial_1^2 h_{21} + \partial_2\partial_1\partial_2 h_{12})$$
$$+(D_2D_{22}D^*_{11}\check{D}_2 + D_2D_{22}\tilde{D}_{11}\check{D}_2)(\partial_2^3 h_{12} + \partial_2^2\partial_1 h_{21})$$
$$+(D_2\tilde{D}_{22}D^*_{11}\check{D}_2 + D_2\tilde{D}_{22}\tilde{D}_{11}\check{D}_2 + D_2D^*_{22}D^*_{11}\check{D}_2 + D_2D^*_{22}\tilde{D}_{11}\check{D}_2)(\partial_2\Delta_2 h_{12} + \Delta_2\partial_1 h_{21})$$
$$+(D_2\tilde{D}_{11}D^*_{22}\check{D}_1 + D_2\tilde{D}_{11}\tilde{D}_{22}\check{D}_1 + D_2D^*_{11}D^*_{22}\check{D}_1 + D_2D^*_{11}\tilde{D}_{22}\check{D}_1)(\partial_2\Delta_1 h_{21} + \partial_2\bar{\partial}_1\partial_2 h_{12})$$

and

$$d_2^3 d_1^2 h =$$
$$[\check{D}_{11}\bar{\partial}_1\bar{\partial}_1 + D_{11}\partial_1\partial_1 + \tilde{D}_{11}\partial_1\bar{\partial}_1 + D^*_{11}\bar{\partial}_1\partial_1 + \check{D}_{12}\bar{\partial}_1\bar{\partial}_2 + D_{12}\partial_1\partial_2 + \tilde{D}_{12}\partial_1\bar{\partial}_2 + D^*_{12}\bar{\partial}_1\partial_2$$
$$+\check{D}_{21}\bar{\partial}_2\bar{\partial}_1 + D_{21}\partial_2\partial_1 + \tilde{D}_{21}\partial_2\bar{\partial}_1 + D^*_{21}\bar{\partial}_2\partial_1 + \check{D}_{22}\bar{\partial}_2\bar{\partial}_2 + D_{22}\partial_2\partial_2 + \tilde{D}_{22}\partial_2\bar{\partial}_2 + D^*_{22}\bar{\partial}_2\partial_2]$$
$$[(D_1D^*_{22}\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1)(\partial_1 h_{21} + \partial_2 h_{12})]$$

$$= [(\check{D}_{11}D_1D^*_{22}\check{D}_1 + \check{D}_{11}D_1\tilde{D}_{22}\check{D}_1)(\Delta_1\bar{\partial}_1 h_{21} + \bar{\partial}_1^2\partial_2 h_{12})]$$

$$+[(D_{11}D_1D^*_{22}\check{D}_1 + D_{11}D_1\tilde{D}_{22}\check{D}_1)(\partial_1^3 h_{21} + \partial_1^2\partial_2 h_{12})]$$

$$+[(\tilde D_{11}D_1D^*_{22}\check D_1+\tilde D_{11}D_1\tilde D_{22}\check D_1)(\Delta_1\partial_1h_{21}+\Delta_1\partial_2h_{12})]$$

$$+[(D^*_{11}D_1D^*_{22}\check D_1+D^*_{11}D_1\tilde D_{22}\check D_1)(\Delta_1\partial_1h_{21}+\Delta_1\partial_2h_{12})]$$

$$+[(\check D_{12}D_1D^*_{22}\check D_1+\check D_{12}D_1\tilde D_{22}\check D_1)(\bar\partial_1\bar\partial_2\partial_1h_{21}+\bar\partial_1\Delta_2^2h_{12})]$$

$$+[(D_{12}D_1D^*_{22}\check D_1+D_{12}D_1\tilde D_{22}\check D_1)(\partial_1\partial_2\partial_1h_{21}+\partial_1\partial_2^2h_{12})]$$

$$+[(\tilde D_{12}D_1D^*_{22}\check D_1+\tilde D_{12}D_1\tilde D_{22}\check D_1)(\partial_1\bar\partial_2\partial_1h_{21}+\partial_1\Delta_2h_{12})]$$

$$+[(D^*_{12}D_1D^*_{22}\check D_1+D^*_{12}D_1\tilde D_{22}\check D_1)(\bar\partial_1\partial_2\partial_1h_{21}+\bar\partial_1\partial_2^2h_{12})]$$

$$+[(\check D_{21}D_1D^*_{22}\check D_1+\check D_{21}D_1\tilde D_{22}\check D_1)(\bar\partial_2\Delta_1h_{21}+\bar\partial_2\bar\partial_1\partial_2h_{12})]$$

$$+[(\tilde D_{21}D_1D^*_{22}\check D_1+\tilde D_{21}D_1\tilde D_{22}\check D_1)(\partial_2\partial_1^2h_{21}+\partial_2\partial_1\partial_2h_{12})]$$

$$+[(\tilde D_{21}D_1D^*_{22}\check D_1+\tilde D_{21}D_1\tilde D_{22}\check D_1)(\partial_2\Delta_1h_{21}+\partial_2\bar\partial_1\partial_2h_{12})]$$

$$+[(D^*_{21}D_1D^*_{22}\check D_1+D^*_{21}D_1\tilde D_{22}\check D_1)(\bar\partial_2\partial_1^2h_{21}+\bar\partial_2\partial_1\partial_2h_{12})]$$

$$+[(\check D_{22}D_1D^*_{22}\check D_1+\check D_{22}D_1\tilde D_{22}\check D_1)(\bar\partial_2\bar\partial_2\partial_1h_{21}+\bar\partial_2\Delta_2h_{12})]$$

$$+[(D_{22}D_1D^*_{22}\check D_1+D_{22}D_1\tilde D_{22}\check D_1)(\partial_2^2\partial_1h_{21}+\partial_2^3h_{12})]$$

$$+[(\tilde D_{22}D_1D^*_{22}\check D_1+\tilde D_{22}D_1\tilde D_{22}\check D_1)(\Delta_2\partial_1h_{21}+\Delta_2\partial_2h_{12})]$$

$$+[(D^*_{22}D_1D^*_{22}\check D_1+D^*_{22}D_1\tilde D_{22}\check D_1)(\Delta_2\partial_1h_{21}+\Delta_2\partial_2h_{12})]$$

The condition

$$(d_1^3d_2^2+d_2^3d_1^2)h=0$$

gives the relations *(3)-(8)* listed in the statement. Finally, the computation of $d_2^3d_2^2h$ gives

$$d_2^3d_2^2h=$$

$$[\check D_{11}\bar\partial_1^2+D_{11}\partial_1^2+\tilde D_{11}\Delta_1+D^*_{11}\Delta_1+\check D_{12}\bar\partial_1\bar\partial_2+D_{12}\partial_1\partial_2+\tilde D_{12}\partial_1\bar\partial_2+D^*_{12}\bar\partial_1\partial_2$$
$$+\check D_{21}\bar\partial_2\bar\partial_1+D_{21}\partial_2\partial_1+\tilde D_{21}\partial_2\bar\partial_1+D^*_{21}\bar\partial_2\partial_1+\check D_{22}\bar\partial_2^2+D_{22}\partial_2^2+\tilde D_{22}\Delta_2+D^*_{22}\Delta_2]$$
$$\cdot[(D_{11}D^*_{22}\check D_1+D_{11}\tilde D_{22}\check D_1)(\partial_1^2h_{21}+\partial_1\partial_2h_{12})+(D_{22}D^*_{11}\check D_2+D_{22}\tilde D_{11}\check D_2)(\partial_2^2h_{12}+\partial_2\partial_1h_{21})$$
$$+(\tilde D_{22}D^*_{11}\check D_2+\tilde D_{22}\tilde D_{11}\check D_2+D^*_{22}D^*_{11}\check D_2+D^*_{22}\tilde D_{11}\check D_2)(\Delta_2h_{12}+\bar\partial_2\partial_1h_{21})$$
$$+(\tilde D_{11}D^*_{22}\check D_1+\tilde D_{11}\tilde D_{22}\check D_1+D^*_{11}D^*_{22}\check D_1+D^*_{11}\tilde D_{22}\check D_1)(\Delta_1h_{21}+\bar\partial_1\partial_2h_{12})]$$

$$
\begin{gathered}
= (\check{D}_{11} D_{11} D^*_{22} \check{D}_1 + \check{D}_{11} D_{11} \tilde{D}_{22} \check{D}_1)(\Delta_1 \Delta_1 h_{21} + \Delta_1 \bar{\partial}_1 \partial_2 h_{12}) \\
+(\check{D}_{11} D_{22} D^*_{11} \check{D}_2 + \check{D}_{11} D_{22} \tilde{D}_{11} \check{D}_2)(\bar{\partial}_1^2 \partial_2^2 h_{12} + \bar{\partial}_1^2 \partial_2 \partial_1 h_{21}) \\
+(\check{D}_{11} \tilde{D}_{22} D^*_{11} \check{D}_2 + \check{D}_{11} \tilde{D}_{22} \tilde{D}_{11} \check{D}_2 + \check{D}_{11} D^*_{22} D^*_{11} \check{D}_2 + \check{D}_{11} D^*_{22} \tilde{D}_{11} \check{D}_2)(\bar{\partial}_1^2 \Delta_2 h_{12} + \bar{\partial}_1^2 \bar{\partial}_2 \partial_1 h_{21}) \\
+(\check{D}_{11} \tilde{D}_{11} D^*_{22} \check{D}_1 + \check{D}_{11} \tilde{D}_{11} \tilde{D}_{22} \check{D}_1 + \check{D}_{11} D^*_{11} D^*_{22} \check{D}_1 + \check{D}_{11} D^*_{11} \tilde{D}_{22} \check{D}_1)(\bar{\partial}_1^2 \Delta_1 h_{21} + \bar{\partial}_1^2 \bar{\partial}_1 \partial_2 h_{12})
\end{gathered}
$$

$$
\begin{gathered}
+(D_{11} D_{11} D^*_{22} \check{D}_1 + D_{11} D_{11} \tilde{D}_{22} \check{D}_1)(\partial_1^4 h_{21} + \partial_1^3 \partial_2 h_{12}) \\
+(D_{11} D_{22} D^*_{11} \check{D}_2 + D_{11} D_{22} \tilde{D}_{11} \check{D}_2)(\partial_1^2 \partial_2^2 h_{12} + \partial_1^2 \partial_2 \partial_1 h_{21}) \\
+(D_{11} \tilde{D}_{22} D^*_{11} \check{D}_2 + D_{11} \tilde{D}_{22} \tilde{D}_{11} \check{D}_2 + D_{11} D^*_{22} D^*_{11} \check{D}_2 + D_{11} D^*_{22} \tilde{D}_{11} \check{D}_2)(\partial_1^2 \Delta_2 h_{12} + \partial_1^2 \bar{\partial}_2 \partial_1 h_{21}) \\
+(D_{11} \tilde{D}_{11} D^*_{22} \check{D}_1 + D_{11} \tilde{D}_{11} \tilde{D}_{22} \check{D}_1 + D_{11} D^*_{11} D^*_{22} \check{D}_1 + D_{11} D^*_{11} \tilde{D}_{22} \check{D}_1)(\partial_1^2 \Delta_1 h_{21} + \partial_1 \partial_2 \Delta_1 h_{12})
\end{gathered}
$$

$$
\begin{gathered}
+(\tilde{D}_{11} D_{11} D^*_{22} \check{D}_1 + \tilde{D}_{11} D_{11} \tilde{D}_{22} \check{D}_1)(\Delta_1 \partial_1^2 h_{21} + \Delta_1 \partial_1 \partial_2 h_{12}) \\
+(\tilde{D}_{11} D_{22} D^*_{11} \check{D}_2 + \tilde{D}_{11} D_{22} \tilde{D}_{11} \check{D}_2)(\Delta_1 \partial_2^2 h_{12} + \Delta_1 \partial_2 \partial_1 h_{21}) \\
+(\tilde{D}_{11} \tilde{D}_{22} D^*_{11} \check{D}_2 + \tilde{D}_{11} \tilde{D}_{22} \tilde{D}_{11} \check{D}_2 + \tilde{D}_{11} D^*_{22} D^*_{11} \check{D}_2 + \tilde{D}_{11} D^*_{22} \tilde{D}_{11} \check{D}_2)(\Delta_1 \Delta_2 h_{12} + \Delta_1 \bar{\partial}_2 \partial_1 h_{21}) \\
+(\tilde{D}_{11} \tilde{D}_{11} D^*_{22} \check{D}_1 + \tilde{D}_{11} \tilde{D}_{11} \tilde{D}_{22} \check{D}_1 + \tilde{D}_{11} D^*_{11} D^*_{22} \check{D}_1 + \tilde{D}_{11} D^*_{11} \tilde{D}_{22} \check{D}_1)(\Delta_1 \Delta_1 h_{21} + \Delta_1 \bar{\partial}_1 \partial_2 h_{12})
\end{gathered}
$$

$$
\begin{gathered}
+(D^*_{11} D_{11} D^*_{22} \check{D}_1 + D^*_{11} D_{11} \tilde{D}_{22} \check{D}_1)(\Delta_1 \partial_1^2 h_{21} + \Delta_1 \partial_1 \partial_2 h_{12}) \\
+(D^*_{11} D_{22} D^*_{11} \check{D}_2 + D^*_{11} D_{22} \tilde{D}_{11} \check{D}_2)(\Delta_1 \partial_2^2 h_{12} + \Delta_1 \partial_2 \partial_1 h_{21}) \\
+(D^*_{11} \tilde{D}_{22} D^*_{11} \check{D}_2 + D^*_{11} \tilde{D}_{22} \tilde{D}_{11} \check{D}_2 + D^*_{11} D^*_{22} D^*_{11} \check{D}_2 + D^*_{11} D^*_{22} \tilde{D}_{11} \check{D}_2)(\Delta_1 \Delta_2 h_{12} + \Delta_1 \bar{\partial}_2 \partial_1 h_{21}) \\
+(D^*_{11} \tilde{D}_{11} D^*_{22} \check{D}_1 + D^*_{11} \tilde{D}_{11} \tilde{D}_{22} \check{D}_1 + D^*_{11} D^*_{11} D^*_{22} \check{D}_1 + D^*_{11} D^*_{11} \tilde{D}_{22} \check{D}_1)(\Delta_1 \Delta_1 h_{21} + \Delta_1 \bar{\partial}_1 \partial_2 h_{12})
\end{gathered}
$$

$$
\begin{gathered}
+(\check{D}_{12} D_{11} D^*_{22} \check{D}_1 + \check{D}_{12} D_{11} \tilde{D}_{22} \check{D}_1)(\bar{\partial}_1 \bar{\partial}_2 \partial_1^2 h_{21} + \bar{\partial}_1 \bar{\partial}_2 \partial_1 \partial_2 h_{12}) \\
+(\check{D}_{12} D_{22} D^*_{11} \check{D}_2 + \check{D}_{12} D_{22} \tilde{D}_{11} \check{D}_2)(\bar{\partial}_1 \partial_2 \Delta_2 h_{12} + \Delta_1 \Delta_2 h_{21}) \\
+(\check{D}_{12} \tilde{D}_{22} D^*_{11} \check{D}_2 + \check{D}_{12} \tilde{D}_{22} \tilde{D}_{11} \check{D}_2 + \check{D}_{12} D^*_{22} D^*_{11} \check{D}_2 + \check{D}_{12} D^*_{22} \tilde{D}_{11} \check{D}_2)(\bar{\partial}_1 \bar{\partial}_2 \Delta_2 h_{12} + \bar{\partial}_1 \bar{\partial}_2^2 \partial_1 h_{21}) \\
+(\check{D}_{12} \tilde{D}_{11} D^*_{22} \check{D}_1 + \check{D}_{12} \tilde{D}_{11} \tilde{D}_{22} \check{D}_1 + \check{D}_{12} D^*_{11} D^*_{22} \check{D}_1 + \check{D}_{12} D^*_{11} \tilde{D}_{22} \check{D}_1)(\bar{\partial}_1 \bar{\partial}_2 \Delta_1 h_{21} + \bar{\partial}_1 \bar{\partial}_2 \bar{\partial}_1 \partial_2 h_{12})
\end{gathered}
$$

$$
\begin{gathered}
+(D_{12} D_{11} D^*_{22} \check{D}_1 + D_{12} D_{11} \tilde{D}_{22} \check{D}_1)(\partial_1 \partial_2 \partial_1^2 h_{21} + \partial_1 \partial_2 \partial_1 \partial_2 h_{12}) \\
+(D_{12} D_{22} D^*_{11} \check{D}_2 + D_{12} D_{22} \tilde{D}_{11} \check{D}_2)(\partial_1 \partial_2^3 h_{12} + \partial_1 \partial_2^2 \partial_1 h_{21}) \\
+(D_{12} \tilde{D}_{22} D^*_{11} \check{D}_2 + D_{12} \tilde{D}_{22} \tilde{D}_{11} \check{D}_2 + D_{12} D^*_{22} D^*_{11} \check{D}_2 + D_{12} D^*_{22} \tilde{D}_{11} \check{D}_2)(\partial_1 \partial_2 \Delta_2 h_{12} + \partial_1^2 \Delta_2 h_{21}) \\
+(D_{12} \tilde{D}_{11} D^*_{22} \check{D}_1 + D_{12} \tilde{D}_{11} \tilde{D}_{22} \check{D}_1 + D_{12} D^*_{11} D^*_{22} \check{D}_1 + D_{12} D^*_{11} \tilde{D}_{22} \check{D}_1)(\partial_1 \partial_2 \Delta_1 h_{21} + \partial_1 \partial_2 \bar{\partial}_1 \partial_2 h_{12})
\end{gathered}
$$

$$
\begin{gathered}
+(\tilde{D}_{12} D_{11} D^*_{22} \check{D}_1 + \tilde{D}_{12} D_{11} \tilde{D}_{22} \check{D}_1)(\partial_1 \bar{\partial}_2 \partial_1^2 h_{21} + \partial_1 \bar{\partial}_2 \partial_1 \partial_2 h_{12}) \\
+(\tilde{D}_{12} D_{22} D^*_{11} \check{D}_2 + \tilde{D}_{12} D_{22} \tilde{D}_{11} \check{D}_2)(\partial_1 \partial_2 \Delta_2 h_{12} + \partial_1^2 \Delta_2 h_{21}) \\
+(\tilde{D}_{12} \tilde{D}_{22} D^*_{11} \check{D}_2 + \tilde{D}_{12} \tilde{D}_{22} \tilde{D}_{11} \check{D}_2 + \tilde{D}_{12} D^*_{22} D^*_{11} \check{D}_2 + \tilde{D}_{12} D^*_{22} \tilde{D}_{11} \check{D}_2)(\partial_1 \bar{\partial}_2 \Delta_2 h_{12} + \partial_1 \bar{\partial}_2^2 \partial_1 h_{21}) \\
+(\tilde{D}_{12} \tilde{D}_{11} D^*_{22} \check{D}_1 + \tilde{D}_{12} \tilde{D}_{11} \tilde{D}_{22} \check{D}_1 + \tilde{D}_{12} D^*_{11} D^*_{22} \check{D}_1 + \tilde{D}_{12} D^*_{11} \tilde{D}_{22} \check{D}_1)(\partial_1 \bar{\partial}_2 \Delta_1 h_{21} + \partial_1 \bar{\partial}_2 \bar{\partial}_1 \partial_2 h_{12})
\end{gathered}
$$

$$+(D^*_{12}D_{11}D^*_{22}\check{D}_1+D^*_{12}D_{11}\tilde{D}_{22}\check{D}_1)(\bar{\partial}_1\partial_2\partial_1^2h_{21}+\bar{\partial}_1\partial_2\partial_1\partial_2h_{12})$$
$$+(D^*_{12}D_{22}D^*_{11}\check{D}_2+D^*_{12}D_{22}\tilde{D}_{11}\check{D}_2(\bar{\partial}_1\partial_2^3h_{12}+\bar{\partial}_1\partial_2^2\partial_1h_{21})$$
$$+(D^*_{12}\tilde{D}_{22}D^*_{11}\check{D}_2+D^*_{12}\tilde{D}_{22}\tilde{D}_{11}\check{D}_2+D^*_{12}D^*_{22}D^*_{11}\check{D}_2+D^*_{12}D^*_{22}\tilde{D}_{11}\check{D}_2)(\bar{\partial}_1\partial_2\Delta_2h_{12}+\Delta_2\Delta_1h_{21})$$
$$+(D^*_{12}\tilde{D}_{11}D^*_{22}\check{D}_1+D^*_{12}\tilde{D}_{11}\tilde{D}_{22}\check{D}_1+D^*_{12}D^*_{11}D^*_{22}\check{D}_1+D^*_{12}D^*_{11}\tilde{D}_{22}\check{D}_1)(\bar{\partial}_1\partial_2\Delta_1h_{21}+\bar{\partial}_1\partial_2\bar{\partial}_1\partial_2h_{12})$$

$$+(\check{D}_{21}D_{11}D^*_{22}\check{D}_1+\check{D}_{21}D_{11}\tilde{D}_{22}\check{D}_1)(\bar{\partial}_2\partial_1\Delta_1h_{21}+\Delta_1\Delta_2h_{12})$$
$$+(\check{D}_{21}D_{22}D^*_{11}\check{D}_2+\check{D}_{21}D_{22}\tilde{D}_{11}\check{D}_2)(\bar{\partial}_2\bar{\partial}_1\partial_2^2h_{12}+\bar{\partial}_2\bar{\partial}_1\partial_2\partial_1h_{21})$$
$$+(\check{D}_{21}\tilde{D}_{22}D^*_{11}\check{D}_2+\check{D}_{21}\tilde{D}_{22}\tilde{D}_{11}\check{D}_2\check{D}_{21}D^*_{22}D^*_{11}\check{D}_2+\check{D}_{21}D^*_{22}\tilde{D}_{11}\check{D}_2)(\bar{\partial}_2\bar{\partial}_1\Delta_2h_{12}+\bar{\partial}_2\bar{\partial}_1\bar{\partial}_2\partial_1h_{21})$$
$$+(\check{D}_{21}\tilde{D}_{11}D^*_{22}\check{D}_1+\check{D}_{21}\tilde{D}_{11}\tilde{D}_{22}\check{D}_1+\check{D}_{21}D^*_{11}D^*_{22}\check{D}_1+\check{D}_{21}D^*_{11}\tilde{D}_{22}\check{D}_1)(\bar{\partial}_2\bar{\partial}_1\Delta_1h_{21}+\bar{\partial}_2\bar{\partial}_1^2\partial_2h_{12})$$

$$+(D_{21}D_{11}D^*_{22}\check{D}_1+D_{21}D_{11}\tilde{D}_{22}\check{D}_1)(\partial_2\partial_1^3h_{21}+\partial_2\partial_1^2\partial_2h_{12})$$
$$+(D_{21}D_{22}D^*_{11}\check{D}_2+D_{21}D_{22}\tilde{D}_{11}\check{D}_2)(\partial_2\partial_1\partial_2^2h_{12}+\partial_2\partial_1\partial_2\partial_1h_{21})$$
$$+(D_{21}\tilde{D}_{22}D^*_{11}\check{D}_2+D_{21}\tilde{D}_{22}\tilde{D}_{11}\check{D}_2+D_{21}D^*_{22}D^*_{11}\check{D}_2+D_{21}D^*_{22}\tilde{D}_{11}\check{D}_2)(\partial_2\partial_1\Delta_2h_{12}+\partial_2\partial_1\bar{\partial}_2\partial_1h_{21})$$
$$+(D_{21}\tilde{D}_{11}D^*_{22}\check{D}_1+D_{21}\tilde{D}_{11}\tilde{D}_{22}\check{D}_1+D_{21}D^*_{11}D^*_{22}\check{D}_1+D_{21}D^*_{11}\tilde{D}_{22}\check{D}_1)(\partial_2\partial_1\Delta_1h_{21}+\partial_2^2\Delta_1h_{12})$$

$$+(\tilde{D}_{21}D_{11}D^*_{22}\check{D}_1+\tilde{D}_{21}D_{11}\tilde{D}_{22}\check{D}_1)(\partial_2\partial_1\Delta_1h_{21}+\partial_2^2\Delta_1h_{12})$$
$$+(\tilde{D}_{21}D_{22}D^*_{11}\check{D}_2+\tilde{D}_{21}D_{22}\tilde{D}_{11}\check{D}_2)(\partial_2\bar{\partial}_1\partial_2^2h_{12}+\partial_2\bar{\partial}_1\partial_2\partial_1h_{21})$$
$$+(\tilde{D}_{21}\tilde{D}_{22}D^*_{11}\check{D}_2+\tilde{D}_{21}\tilde{D}_{22}\tilde{D}_{11}\check{D}_2+\tilde{D}_{21}D^*_{22}D^*_{11}\check{D}_2+\tilde{D}_{21}D^*_{22}\tilde{D}_{11}\check{D}_2)(\partial_2\bar{\partial}_1\Delta_2h_{12}+\partial_2\bar{\partial}_1\bar{\partial}_2\partial_1h_{21})$$
$$+(\tilde{D}_{21}\tilde{D}_{11}D^*_{22}\check{D}_1+\tilde{D}_{21}\tilde{D}_{11}\tilde{D}_{22}\check{D}_1+\tilde{D}_{21}D^*_{11}D^*_{22}\check{D}_1+\tilde{D}_{21}D^*_{11}\tilde{D}_{22}\check{D}_1)(\partial_2\bar{\partial}_1\Delta_1h_{21}+\partial_2\bar{\partial}_1^2\partial_2h_{12})$$

$$+(D^*_{21}D_{11}D^*_{22}\check{D}_1+D^*_{21}D_{11}\tilde{D}_{22}\check{D}_1)(\bar{\partial}_2\partial_1^3h_{21}+\bar{\partial}_2\partial_1^2\partial_2h_{12})$$
$$+(D^*_{21}D_{22}D^*_{11}\check{D}_2+D^*_{21}D_{22}\tilde{D}_{11}\check{D}_2)(\bar{\partial}_2\partial_1\partial_2^2h_{12}+\bar{\partial}_2\partial_1\partial_2\partial_1h_{21})$$
$$+(D^*_{21}\tilde{D}_{22}D^*_{11}\check{D}_2+D^*_{21}\tilde{D}_{22}\tilde{D}_{11}\check{D}_2+D^*_{21}D^*_{22}D^*_{11}\check{D}_2+D^*_{21}D^*_{22}\tilde{D}_{11}\check{D}_2)(\bar{\partial}_2\partial_1\Delta_2h_{12}+\bar{\partial}_2\partial_1\bar{\partial}_2\partial_1h_{21})$$
$$+(D^*_{21}\tilde{D}_{11}D^*_{22}\check{D}_1+D^*_{21}\tilde{D}_{11}\tilde{D}_{22}\check{D}_1+D^*_{21}D^*_{11}D^*_{22}\check{D}_1+D^*_{21}D^*_{11}\tilde{D}_{22}\check{D}_1)(\bar{\partial}_2\partial_1\Delta_1h_{21}+\Delta_1\Delta_2h_{12})$$

$$+(\check{D}_{22}D_{11}D^*_{22}\check{D}_1+\check{D}_{22}D_{11}\tilde{D}_{22}\check{D}_1)(\bar{\partial}_2^2\partial_1^2h_{21}+\bar{\partial}_2^2\partial_1\partial_2h_{12})$$
$$+(\check{D}_{22}D_{22}D^*_{11}\check{D}_2+\check{D}_{22}D_{22}\tilde{D}_{11}\check{D}_2)(\Delta_2^2h_{12}+\bar{\partial}_2\Delta_2\partial_1h_{21})$$
$$+(\check{D}_{22}\tilde{D}_{22}D^*_{11}\check{D}_2+\check{D}_{22}\tilde{D}_{22}\tilde{D}_{11}\check{D}_2+\check{D}_{22}D^*_{22}D^*_{11}\check{D}_2+\check{D}_{22}D^*_{22}\tilde{D}_{11}\check{D}_2(\bar{\partial}_2^2\Delta_2h_{12}+\bar{\partial}_2^3\partial_1h_{21})$$
$$+(\check{D}_{22}\tilde{D}_{11}D^*_{22}\check{D}_1+\check{D}_{22}\tilde{D}_{11}\tilde{D}_{22}\check{D}_1+\check{D}_{22}D^*_{11}D^*_{22}\check{D}_1+\check{D}_{22}D^*_{11}\tilde{D}_{22}\check{D}_1)(\bar{\partial}_2^2\Delta_1h_{21}+\bar{\partial}_2^2\bar{\partial}_1\partial_2h_{12})$$

$$+(D_{22}D_{11}D^*_{22}\check{D}_1+D_{22}D_{11}\tilde{D}_{22}\check{D}_1)(\partial_2^2\partial_1^2h_{21}+\partial_2^2\partial_1\partial_2h_{12})$$
$$+(D_{22}D_{22}D^*_{11}\check{D}_2+D_{22}D_{22}\tilde{D}_{11}\check{D}_2)(\partial_2^4h_{12}+\partial_2^3\partial_1h_{21})$$
$$+(D_{22}\tilde{D}_{22}D^*_{11}\check{D}_2+D_{22}\tilde{D}_{22}\tilde{D}_{11}\check{D}_2+D_{22}D^*_{22}D^*_{11}\check{D}_2+D_{22}D^*_{22}\tilde{D}_{11}\check{D}_2)(\partial_2^2\Delta_2h_{12}+\partial_2\partial_1\Delta_2h_{21})$$
$$+(D_{22}\tilde{D}_{11}D^*_{22}\check{D}_1+D_{22}\tilde{D}_{11}\tilde{D}_{22}\check{D}_1+D_{22}D^*_{11}D^*_{22}\check{D}_1+D_{22}D^*_{11}\tilde{D}_{22}\check{D}_1)(\partial_2^2\Delta_1h_{21}+\partial_2^2\bar{\partial}_1\partial_2h_{12})$$

$$+(\tilde{D}_{22}D_{11}D^*_{22}\check{D}_1 + \tilde{D}_{22}D_{11}\tilde{D}_{22}\check{D}_1)(\Delta_2\partial_1^2 h_{21} + \Delta_2\partial_1\partial_2 h_{12})$$
$$+(\tilde{D}_{22}D_{22}D^*_{11}\check{D}_2 + \tilde{D}_{22}D_{22}\tilde{D}_{11}\check{D}_2)(\Delta_2\partial_2^2 h_{12} + \Delta_2\partial_2\partial_1 h_{21})$$
$$+(\tilde{D}_{22}\tilde{D}_{22}D^*_{11}\check{D}_2 + \tilde{D}_{22}\tilde{D}_{22}\tilde{D}_{11}\check{D}_2 + \tilde{D}_{22}D^*_{22}D^*_{11}\check{D}_2 + \tilde{D}_{22}D^*_{22}\tilde{D}_{11}\check{D}_2)(\Delta_2^2 h_{12} + \Delta_2\bar{\partial}_2\partial_1 h_{21})$$
$$+(\tilde{D}_{22}\tilde{D}_{11}D^*_{22}\check{D}_1 + \tilde{D}_{22}\tilde{D}_{11}\tilde{D}_{22}\check{D}_1 + \tilde{D}_{22}D^*_{11}D^*_{22}\check{D}_1 + \tilde{D}_{22}D^*_{11}\tilde{D}_{22}\check{D}_1)(\Delta_2\Delta_1 h_{21} + \Delta_2\bar{\partial}_1\partial_2 h_{12})$$

$$+(D^*_{22}D_{11}D^*_{22}\check{D}_1 + D^*_{22}D_{11}\tilde{D}_{22}\check{D}_1)(\Delta_2\partial_1^2 h_{21} + \Delta_2\partial_1\partial_2 h_{12})$$
$$+(D^*_{22}D_{22}D^*_{11}\check{D}_2 + D^*_{22}D_{22}\tilde{D}_{11}\check{D}_2)(\Delta_2\partial_2^2 h_{12} + \Delta_2\partial_2\partial_1 h_{21})$$
$$+(D^*_{22}\tilde{D}_{22}D^*_{11}\check{D}_2 + D^*_{22}\tilde{D}_{22}\tilde{D}_{11}\check{D}_2 + D^*_{22}D^*_{22}D^*_{11}\check{D}_2 + D^*_{22}D^*_{22}\tilde{D}_{11}\check{D}_2)(\Delta_2\Delta_2 h_{12} + \Delta_2\bar{\partial}_2\partial_1 h_{21})$$
$$+(D^*_{22}\tilde{D}_{11}D^*_{22}\check{D}_1 + D^*_{22}\tilde{D}_{11}\tilde{D}_{22}\check{D}_1 + D^*_{22}D^*_{11}D^*_{22}\check{D}_1 + D^*_{22}D^*_{11}\tilde{D}_{22}\check{D}_1)(\Delta_2\Delta_1 h_{21} + \Delta_2\bar{\partial}_1\partial_2 h_{12}).$$

Grouping the various terms we obtain the remaining relations listed in the statement. □

Proposition B.2 Let

$$k = (D_1D^*_{22}\check{D}_1 + D_1\tilde{D}_{22}\check{D}_1)k_0 + (D_{11}D^*_{22}\check{D}_1 + D_{11}\tilde{D}_{22}\check{D}_1)k_1$$
$$+(D_{22}D^*_{11}\check{D}_2 + D_{22}\tilde{D}_{11}\check{D}_2)k_2 + (\tilde{D}_{11}D^*_{22}\check{D}_1 + \tilde{D}_{11}\tilde{D}_{22}\check{D}_1 + D^*_{11}D^*_{22}\check{D}_1 + D^*_{11}\tilde{D}_{22}\check{D}_1)k_3$$
$$+(\tilde{D}_{22}D^*_{11}\check{D}_2 + \tilde{D}_{22}\tilde{D}_{11}\check{D}_2 + D^*_{22}D^*_{11}\check{D}_2 + D^*_{22}\tilde{D}_{11}\check{D}_2)k_4$$

be a generic element in F'_3. Then $d^3k = 0$ implies the following relations:

1. $\partial_i k_j - \partial_i\partial_j k_0 = 0$, $\bar{\partial}_i k_{j+2} - \partial_i\bar{\partial}_j k_0 = 0$, $\bar{\partial}_i k_i - \Delta_i k_0 = 0$, $\partial_i k_{i+2} - \bar{\partial}_i k_i$, $i, j = 1, 2$;
2. $\bar{\partial}_i k_j - \bar{\partial}_i\partial_j k_0 = 0$, $\bar{\partial}_i k_{j+2} - \partial_i\bar{\partial}_j k_0 = 0$, $i, j = 1, 2$, $i \neq j$;
3. $\Delta_i k_{i+2} - \bar{\partial}_i^2 k_i = 0$, $\partial_i\partial_j k_{j+2} - \partial_i\bar{\partial}_j k_j = 0$, $i, j = 1, 2$, $i \neq j$;
4. $\Delta_1 k_1 - \partial_1^2 k_3 = 0$, $\Delta_1 k_4 - \bar{\partial}_2\bar{\partial}_1 k_1 = 0$, $\Delta_1 k_4 - \bar{\partial}_2\partial_1 k_3 = 0$, $\Delta_2 k_3 - \bar{\partial}_1\bar{\partial}_2 k_2 = 0$, $\bar{\partial}_1\partial_2 k_4 - \bar{\partial}_1\bar{\partial}_2 k_2 = 0$.

Proof. The relations found in the previous proposition give the statement. We show just for one coefficient since the other computations are similar. □

Proposition B.3 The closure condition $d^3k = 0$ imposes no compatibility conditions on the system

$$\begin{cases} \partial_1 h_{21} + \partial_2 h_{12} = k_0 \\ \partial_1(\partial_1 h_{21} + \partial_2 h_{12}) = k_1 \\ \partial_2(\partial_1 h_{21} + \partial_2 h_{12}) = k_2 \\ \bar{\partial}_1(\partial_1 h_{21} + \partial_2 h_{12}) = k_3 \\ \bar{\partial}_2(\partial_1 h_{21} + \partial_2 h_{12}) = k_4. \end{cases}$$

Proof. It suffices to use the constraints $\partial_i k_0 = k_i$, $\bar{\partial}_i k_0 = k_{i+2}$, $i = 1, 2$ in the relations 1,2,3,4 of the above proposition. □