Robust adaptive *hp* discontinuous Galerkin finite element methods for the Helmholtz equation

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   - Reliability
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Section 1

Introduction

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ be a bounded polygonal domain. We seek $u : \Omega \mapsto \mathbb{C}$ such that

$$
\begin{aligned}
-\Delta u - k^2 u &= f && \text{in } \Omega, \\
\nabla u \cdot \boldsymbol{n} - \mathrm{i} k u &= g && \text{on } \partial\Omega, \qquad \text{(Robin/Impedence BC)}
\end{aligned}
$$

where

$$k = \frac{\omega L}{c}$$

is the wavenumber (ω is the frequency of the wave, L is the measure of the domain, and c is the speed of sound in the material). Wavenumber is related to the wave length

$$\lambda = \frac{2\pi}{k}.$$

Multiplying by a test function and integrating by parts gives the weak formulation: Find $u \in H^1(\Omega)$ such that

$$\int_\Omega (\nabla u \cdot \nabla \bar{v} - k^2 u \bar{v})\, d\boldsymbol{x} + \int_{\Gamma_R} \mathrm{i} k u \bar{v}\, ds = \int_\Omega f \overline{v}\, d\boldsymbol{x} + \int_{\Gamma_R} g \cdot \boldsymbol{n} \bar{v}\, ds$$

for all $v \in H^1(\Omega)$. Well-posedness: [Melenk, 1995]

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We want to search for a solution in a finite dimensional subspace of $H^1(\Omega)$. To that end we.

- subdivide the domain Ω into a mesh $\mathcal{T}_h$ of non-overlapping triangles T, where each element has a size h_T and denote by $\mathcal{E}_h^I$ the union of all interior edges in the mesh
- multiple by test functions v and integrate by parts elementwise

$$\sum_{T \in \mathcal{T}_h} \left(\int_T (\nabla u \cdot \nabla \bar{v} - k^2 u \bar{v}) \, d\boldsymbol{x} - \int_{\partial T} \nabla u \cdot \boldsymbol{n}_T \bar{v} \, ds \right) = \sum_{T \in \mathcal{T}_h} \int_T f \bar{v} \, d\boldsymbol{x}.$$

- Replace continuous functions u, v by *discrete* functions u_{hp}, v_{hp} in the finite element space

$$V_{hp} = \left\{ v_{hp} \in L^2(\Omega) : \ v_{hp|_T} \in \mathbb{P}_{p_T}(T) \ \text{ for all } \ T \in \mathcal{T}_h \right\}$$

$\mathbb{P}_{p_T}(T) =$ polynomials of degree $\leq p_T$ in T

- *symmetrize* and add *jump-penalty* terms on $\mathcal{E}_h^I$ and $\partial\Omega$, without losing *consistency*

$$\{\!\{\cdot\}\!\} = \text{mean value} \qquad [\![\cdot]\!]_N = \text{jump in normal direction}$$

DG method

Find $u_{hp} \in V_{hp}$ such that $a_{hp}(u_{hp}, v_{hp}) = \ell_{hp}(v_{hp})$ for all $v_{hp} \in V_{hp}$.

$$
\begin{aligned}
a_h(u,v) := \int_\Omega (\nabla_h u \cdot \nabla_h \overline{v} - k^2 u \overline{v})\, d\boldsymbol{x} - \int_{\mathcal{E}_h^I} ([\![u]\!]_N \{\!\{\nabla_h \overline{v}\}\!\} + \{\!\{\nabla_h u\}\!\}[\![\overline{v}]\!]_N)\, ds \\
- \mathrm{i}\beta \int_{\mathcal{E}_h^I} \frac{\mathrm{h}}{\mathrm{p}} [\![\nabla_h u]\!]_N [\![\nabla_h \overline{v}]\!]_N\, ds - \mathrm{i}\alpha \int_{\mathcal{E}_h^I} \frac{\mathrm{p}^2}{\mathrm{h}} [\![u]\!]_N \cdot [\![\overline{v}]\!]_N\, ds \\
- \gamma \int_{\partial\Omega} k \frac{\mathrm{h}}{\mathrm{p}} (u \nabla_h \overline{v} \cdot \boldsymbol{n} + \nabla_h u \cdot \boldsymbol{n} \overline{v})\, ds \\
- \mathrm{i}\gamma \int_{\partial\Omega} \frac{\mathrm{h}}{\mathrm{p}} \nabla_h u \cdot \boldsymbol{n} \nabla_h \overline{v} \cdot \boldsymbol{n}\, ds - \mathrm{i} \int_{\partial\Omega} k \left(1 - \gamma k \frac{\mathrm{h}}{\mathrm{p}}\right) u \overline{v}\, ds \\
\ell_{hp}(v) := \int_\Omega f \overline{v}\, d\boldsymbol{x} + i\gamma \int_{\partial\Omega} \frac{\mathrm{h}}{\mathrm{p}} g\, \nabla_h \overline{v} \cdot \boldsymbol{n}\, ds + \int_{\partial\Omega} \left(1 - \gamma k \frac{\mathrm{h}}{\mathrm{p}}\right) g \overline{v}\, ds
\end{aligned}
$$

We use $\alpha = 10$, $\beta = 1$ and $\gamma = 1/4$.

a priori analysis proves well-posedness and quasi-optimal error estimates

[Melenk, Parsania, Sauter 2013]

Error estimation/mesh adaptation

Adaptive mesh refinement:

- Refine elements using either:
  - element subdivision (*h*-refinement)
  - increasing polynomial degree of the element (*p*-refinement)
- Need to estimate which elements need refining; therefore, need computable (*a posteriori*) local error indicators η_T for each element

A global *a posteriori* error estimate can be computed by summing local error indicators, which can be used to estimate when the actual error reaches a desired accuracy.

$$\text{error} \lesssim \text{estimate} \quad \Longrightarrow \quad \text{reliability}$$

$$\text{estimate} \lesssim \text{error} \quad \Longrightarrow \quad \text{efficiency}$$

If the constants in these inequalities are independent of h and p we have a robust estimate.

Problems with FEM:

- Number of degrees of freedom required to obtain given accuracy increases with wave number k.
- Error: best approximation + phase lag:

$$\|\nabla_h(u-u_h)\|_{L^2(\Omega)} \lesssim (kh)^p + k(kh)^{2p};$$

Problems with FEM:

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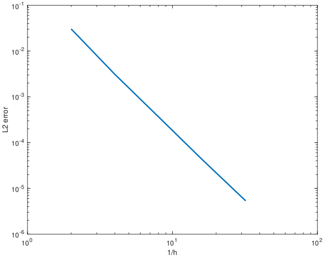
- Error: best approximation + phase lag:

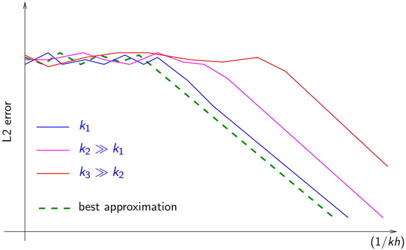
$$\|\nabla_h(u-u_h)\|_{L^2(\Omega)} \lesssim (kh)^p + k(kh)^{2p};$$

convergence like the best approximation when $k(kh)^{2p} \lesssim (kh)^p$, i.e.

$$h \lesssim k^{-1-1/p} \qquad \text{(resolution condition)}$$



Poisson



Helmholtz

Section 2

hp-robust error estimation

[C., Gedicke, Perugia, SISC 2019]

> *The observation that the pollution effect is related to the phase lead of the numerical solution leads to the definition of the shifted solution as the key auxiliary construction for the analysis of local a posteriori error estimation.*
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We consider the Helmholtz problem as a shifted Poisson problem, and use methods based on equilibrated fluxes and potential reconstructions; cf. [Braess, Pillwein, Schöberl 2009], [Ern, Vohralík 2015], [Dolejší, Ern, Vohralík 2016].

The idea of using a shifted Poisson is related to the *a posteriori* error analysis for eigenvalue problems.

[Cancès, Dusson, Maday, Stamm, Vohralík 2017 & 2018]

Helmholtz:

$$
\begin{aligned}
-\Delta u - k^2 u &= f && \text{in } \Omega,\\
\nabla u \cdot \boldsymbol{n} - \mathrm{i} k u &= g && \text{on } \partial\Omega.
\end{aligned}
$$

Shifted Poisson (Neumann):

$$
\begin{aligned}
-\Delta w &= f + k^2 u_{hp} && \text{in } \Omega,\\
\nabla w \cdot \boldsymbol{n} &= g + \mathrm{i} k u_{hp} - \gamma k (g - \nabla u_{hp} \cdot \boldsymbol{n} + \mathrm{i} k u_{hp}) && \text{on } \partial\Omega.
\end{aligned}
$$

The extra term on the right hand side is required for compatibility, which is necessary to prove that the flux reconstruction we define below exists.

Definition (Flux reconstruction)

Given $u_{hp} \in V_{hp}$, a equilibrated flux reconstruction for u_{hp} is any function $\boldsymbol{\sigma}_{hp} \in H(\mathrm{div};\Omega)$ that satisfies

$$\int_T \mathrm{div}\,\boldsymbol{\sigma}_{hp}\,d\boldsymbol{x} = \int_T (f + k^2 u_{hp})\,d\boldsymbol{x} \qquad \forall T \in \mathcal{T}_h,$$

$$\int_E \boldsymbol{\sigma}_{hp}\cdot\boldsymbol{n} = -\int_E (g + \mathrm{i}ku_{hp} - \gamma k(g - \nabla u_{hp}\cdot\boldsymbol{n} + \mathrm{i}ku_{hp}))\,ds \quad \forall E \subset \partial\Omega$$

Definition (Potential (reconstruction))

We define a potential as any function

$$s_{hp} \in H^1_*(\Omega) := \left\{ v \in H^1(\Omega) : \frac{1}{|\Omega|}\int_\Omega v\,d\boldsymbol{x} = 0 \right\}.$$

Note $\boldsymbol{\sigma}_{hp}$ and s_{hp} are not necessarily piecewise polynomial (yet).

A posteriori error estimator

For $u_{hp} \in V_{hp}$ we denote by $\mathcal{G}(u_{hp})$ its DG gradient:

$$\mathcal{G}(u_{hp}) := \nabla_h u_{hp} - \sum_{E \in \mathcal{E}_h^I} \mathcal{L}_E^0([\![u_{hp}]\!]) - \sum_{E \in \mathcal{E}_h^I} \mathcal{L}_E^1([\![\nabla u_{hp}]\!])$$

Error Estimator

$$\eta_{hp} := \sum_{T \in \mathcal{T}_h} \Big(\|\mathcal{G}(u_{hp}) + \boldsymbol{\sigma}_{hp}\|_{0,T} + \frac{h_T}{j_{1,1}} \|f + k^2 u_{hp} - \operatorname{div} \boldsymbol{\sigma}_{hp}\|_{0,T}$$
$$+ C_{tr} \sum_{E \subset \partial T \cap \partial\Omega} h_E^{1/2} \|\boldsymbol{\sigma}_{hp} \cdot \boldsymbol{n} + g + \mathrm{i} k u_{hp} - \gamma k \frac{\mathrm{h}}{\mathrm{p}} (g - \nabla_h u_{hp} \cdot \boldsymbol{n} + \mathrm{i} k u_{hp})\|_{0,E} \Big)^2$$
$$+ \sum_{T \in \mathcal{T}_h} \|\mathcal{G}(u_{hp}) - \nabla s_{hp}\|_{0,T}^2$$

Any admissable flux reconstructions and potentials

$\implies$ reliability (error $\lesssim$ estimator)

Suitable localized flux and potential reconstructions

$\implies$ efficiency & robustness (estimator $\lesssim$ error)

Theorem (Reliability)

$$\|\nabla u - \mathcal{G}(u_{hp})\|_{0,\Omega} \lesssim \eta_{hp} + k^2\|u - u_{hp}\|_{0,\Omega} + k\|u - u_{hp}\|_{0,\partial\Omega} + \|\gamma k \frac{\mathrm{h}}{\mathrm{p}}(g - \nabla_h u_{hp} + \mathrm{i}k u_{hp})\|_{0,\partial\Omega}$$

The additional terms are higher order compared to the left-hand side providing that the resolution condition is met. [Sauter, Zech 2015]

Proof is similar to [Ern, Vohralík 2015].

Proof.

Let $s \in H^1_*(\Omega)$ be defined by the projection

$$\int_\Omega \nabla s \cdot \nabla \overline{v}\, d\boldsymbol{x} = \int_\Omega \mathcal{G}(u_{hp}) \cdot \nabla \overline{v}\, d\boldsymbol{x} \qquad \forall v \in H^1(\Omega).$$

Then by orthogonality, for any $s_{hp} \in H^1_*(\Omega)$,

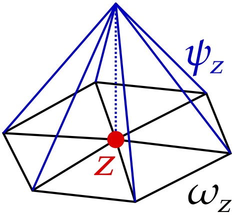
$$\begin{aligned}
\|\nabla u - \mathcal{G}(u_{hp})\|^2_{0,\Omega} &= \|\nabla(u-s)\|^2_{0,\Omega} + \|\nabla s - \mathcal{G}(u_{hp})\|^2_{0,\Omega} \\
&= \|\nabla(u-s)\|^2_{0,\Omega} + \min_{v \in H^1_*(\Omega)} \|\nabla v - \mathcal{G}(u_{hp})\|^2_{0,\Omega} \\
&\le \|\nabla(u-s)\|^2_{0,\Omega} + \|\nabla s_{hp} - \mathcal{G}(u_{hp})\|^2_{0,\Omega} \\
&= \sup_{v \in H^1_*(\Omega), \|\nabla v\|=1} \int_\Omega \nabla(u-s) \cdot \nabla \overline{v}\, d\boldsymbol{x} + \|\nabla s_{hp} - \mathcal{G}(u_{hp})\|^2_{0,\Omega} \\
&= \sup_{v \in H^1_*(\Omega), \|\nabla v\|=1} \int_\Omega \nabla(u - \mathcal{G}(u_{hp})) \cdot \nabla \overline{v}\, d\boldsymbol{x} + \|\nabla s_{hp} - \mathcal{G}(u_{hp})\|^2_{0,\Omega}
\end{aligned}$$

The first term is then boundable by definitions of weak formulation and $\boldsymbol{\sigma}_{hp}$, Poincaré and trace inequalities, interpolation results, etc. □

Suitable *localized* reconstructions

We have triangular meshes with no hanging nodes, so we can define for each node $z \in \mathcal{N}$:

- nodal patch ω_z
- nodal hat functions ψ_z (forms partition of unity)



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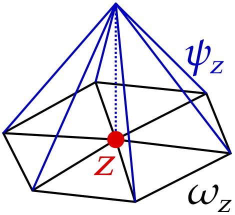
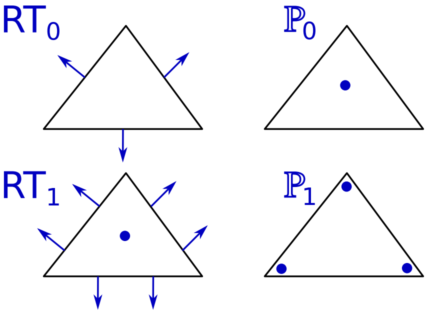


For proof of efficiency we define specific flux and potential reconstructions locally by mesh nodes:

$$\boldsymbol{\sigma}_{hp} := \sum_{z \in \mathcal{N}} \zeta_{hp}^z$$

$$s_{hp} := \widetilde{s}_{hp} - \frac{1}{|\Omega|} \int_\Omega \widetilde{s}_{hp}\, d\boldsymbol{x} \qquad \widetilde{s}_{hp} := \sum_{z \in \mathcal{N}} s_{hp}^z$$

For the local flux reconstruction we solve local patch problems using Raviart–Thomas (RT_p) elements:



For all any domain $D \subset \mathbb{R}^2$ Raviart–Thomas elements have the properties that:

- $\mathrm{RT}_p(D) \subset H(\mathrm{div}, D)$
- $\boldsymbol{v} \in \mathrm{RT}_p(D) \implies \mathrm{div}\, \boldsymbol{v} \in \mathbb{P}_p(D)$

Local flux reconstruction

We solve the following local problem in mixed form for every node $z \in \mathcal{N}$:
Find $(\zeta_{hp}^z, r_{hp}^z) \in \Sigma_{g^z,hp}^z \times Q_{hp}^z$ such that

$$\int_{\omega_z} \left(\zeta_{hp}^z \cdot \overline{\boldsymbol{\tau}}_{hp} - r_{hp}^z \operatorname{div} \overline{\boldsymbol{\tau}}_{hp}\right) \, d\boldsymbol{x} = -\int_{\omega_z} \psi_z \mathcal{G}(u_{hp}) \cdot \overline{\boldsymbol{\tau}}_{hp} \, d\boldsymbol{x} \quad \forall \boldsymbol{\tau}_{hp} \in \Sigma_{0,hp}^z$$

$$\int_{\omega_z} \operatorname{div} \zeta_{hp}^z \overline{q}_{hp} \, d\boldsymbol{x} = \int_{\omega_z} f_z \overline{q}_{hp} \, d\boldsymbol{x} \qquad \forall q_{hp} \in Q_{hp}^z$$

where, for $p_z \geq 1$,

$$\begin{aligned} \Sigma_{g^z,hp}^z := \big\{ \boldsymbol{\tau}_{hp} \in H(\operatorname{div}, \omega_z) : \; & \boldsymbol{\tau}_{hp}|_T \in \mathrm{RT}_{p_z}(T) \text{ for all } T \in \mathcal{T}_h(z), \\ & \boldsymbol{\tau}_{hp} \cdot \boldsymbol{n} = 0 \text{ on } \partial\omega_z \setminus \partial\Omega, \\ & \boldsymbol{\tau}_{hp} \cdot \boldsymbol{n}|_E = \Pi_E^{p_z} g^z \text{ for all } E \subset \partial\omega_z \cap \partial\Omega \big\} \end{aligned}$$

$$Q_{hp}^z := \left\{ q_{hp} \in Q_{hp}(\omega_z) : |\omega_z|^{-1} \int_{\omega_z} q_{hp} \, d\boldsymbol{x} = 0 \right\}$$

$$Q_{hp}(\omega_z) := \left\{ q_{hp} \in L^2(\omega_z) : q_{hp}|_T \in \mathbb{P}_{p_z}(T) \text{ for all } T \in \mathcal{T}_h(z) \right\}$$

$$f^z := (f + k^2 u_{hp})\psi_z - \mathcal{G}(u_{hp}) \cdot \nabla \psi_z, \qquad g_z := \dots$$

Lemma (Flux reconstruction)

$\boldsymbol{\sigma}_{hp} = \sum_{z\in\mathcal{N}} \zeta_{hp}^z$ is an equilibrated flux reconstruction in $H(\mathrm{div},\Omega)$, which satisfies for any $T \in \mathcal{T}_h$

$$\int_T \left(f + k^2 u_{hp} - \mathrm{div}\,\boldsymbol{\sigma}_{hp}\right) \overline{q}_{hp}\, d\boldsymbol{x} = 0,$$

for all $q_{hp} \in \bigcap_{z\in\mathcal{N}(T)} Q_{hp}(\omega_z)|_T$, and for any $E \subset \partial\Omega$

$$\int_E \left(\boldsymbol{\sigma}_{hp}\cdot\boldsymbol{n} + g + \mathrm{i}ku_{hp} - \gamma k\frac{\mathrm{h}}{\mathrm{p}}(g - \nabla_h u_{hp}\cdot\boldsymbol{n} + \mathrm{i}ku_{hp})\right) \overline{q}_{hp}\, ds = 0,$$

for all $q_{hp} \in \bigcap_{z\in\mathcal{N}(E)} Q_{hp}(\omega_z)|_E$.

Proof.

- $\zeta^z_{hp} \in H(\mathrm{div}, \Omega)$ by zero extension $\implies \boldsymbol{\sigma}_{hp} \in H(\mathrm{div}, \Omega)$.
- Prove just the first statement (second follows similarly):

$$
\begin{aligned}
&\int_T \left(f + k^2 u_{hp} - \mathrm{div}\,\boldsymbol{\sigma}_{hp}\right) \overline{q}_{hp}\, d\boldsymbol{x} \\
&= \sum_{z\in\mathcal{N}(T)} \int_T \left(\psi_z(f + k^2 u_{hp}) - \mathrm{div}\,\zeta^z_{hp}\right) \overline{q}_{hp}\, d\boldsymbol{x}, \\
&= \sum_{z\in\mathcal{N}(T)} \int_T \left(f^z + \mathcal{G}(u_{hp})\cdot\nabla\psi_z - \mathrm{div}\,\zeta^z_{hp}\right) \overline{q}_{hp}\, d\boldsymbol{x}, \\
&= \sum_{z\in\mathcal{N}(T)} \int_T \mathcal{G}(u_{hp})\cdot\nabla\psi_z \overline{q}_{hp}\, d\boldsymbol{x}, \qquad \left(\textstyle\sum_{z\in\mathcal{N}(T)} \nabla\psi_z = 0\right) \\
&= 0.
\end{aligned}
$$

□

Local flux reconstruction

Theorem (Flux reconstruction efficiency)

$$\begin{aligned}\|\mathcal{G}(u_{hp}) + \boldsymbol{\sigma}_{hp}\|_{0,\Omega} \lesssim \|\nabla u - \mathcal{G}(u_{hp})\|_{0,\Omega} &+ k^2\|u - u_{hp}\|_{0,\Omega} \\ &+ k\|u - u_{hp}\|_{0,\partial\Omega} + \mathrm{osc}(f) + \mathrm{osc}(g) \\ &+ \|\gamma k \frac{\mathsf{h}}{\mathsf{p}}(g - \nabla_h u_{hp} \cdot \boldsymbol{n} + \mathrm{i} k u_{hp})\|_{0,\partial\Omega} \\ &+ \|\mathrm{i}\gamma \frac{\sqrt{\mathsf{h}}}{\mathsf{p}}(g - \nabla_h u_{hp} \cdot \boldsymbol{n} + \mathrm{i} k u_{hp})\|_{0,\partial\Omega}\end{aligned}$$

Proof.

Define $r^z \in H^1_*(\omega_z)$ as the solution of the continuous problem

$$\int_{\omega_z} \nabla r^z \cdot \nabla \overline{v}\, d\mathbf{x} = -\int_{\omega_z} \psi_z \mathcal{G}(u_{hp}) \cdot \nabla \overline{v}\, d\mathbf{x} + \sum_{T \in \mathcal{T}_h(z)} \int_T \Pi_T^{p_z} f^z \overline{v}\, d\mathbf{x}$$

$$- \sum_{E \subset \omega_z \cap \partial\Omega} \int_E \Pi_E^{p_z} g^z \overline{v}\, ds \qquad \forall v \in H^1(\omega_z).$$

$$\begin{aligned}
\|\mathcal{G}(u_{hp}) + \boldsymbol{\sigma}_{hp}\|_{0,\Omega} &\leq \sum_{z \in \mathcal{N}} \|\psi_z \mathcal{G}(u_{hp}) + \boldsymbol{\zeta}^z_{hp}\|_{0,\omega_z} \\
&\lesssim \sum_{z \in \mathcal{N}} \|\nabla r^z\|_{0,\omega_z} \qquad \text{[Braess, Pillwein, Schöberl 2009]} \\
&\lesssim \|\nabla w - \mathcal{G}(u_{hp})\|_{0,\Omega} + (\text{terms on RHS}) \\
&\leq \|\nabla u - \mathcal{G}(u_{hp})\|_{0,\Omega} + \sup_{v \in H^1_*(\Omega), \|v\|_{0,\Omega}=1} \int_\Omega \nabla(w-u) \cdot \nabla \overline{v}\, d\mathbf{x} + \ldots
\end{aligned}$$

For simplicity we consider only interior nodes. For $p_z \geq 1$

$$V_{hp}^z := \{v_{hp} \in C^0(\overline{\omega_z}) : v_{hp}|_T \in \mathbb{P}_{p_z+1}(T)\,\forall T \in \mathcal{T}_h(z), v_{hp} = 0 \text{ on } \partial\omega_z\}.$$

We then define

$$s_{hp}^z := \underset{v_{hp} \in V_{hp}^z}{\arg\min} \|\nabla_h(\psi_z u_{hp}) - \nabla v_{hp}\|_{0,\omega_z},$$

which is equivalent to finding $s_{hp}^z \in V_{hp}^z$ such that

$$\int_{\omega_z} \nabla s_{hp}^z \cdot \nabla \overline{v}_{hp}\, d\boldsymbol{x} = \int_{\omega_z} \nabla_h(\psi_z u_{hp}) \cdot \nabla \overline{v}_{hp}\, d\boldsymbol{x}, \qquad \text{for all } v_{hp} \in V_{hp}^z.$$

Local potential reconstruction

For simplicity we consider only interior nodes. For $p_z \geq 1$

$$V_{hp}^z := \{v_{hp} \in C^0(\overline{\omega_z}) : v_{hp}|_T \in \mathbb{P}_{p_z+1}(T) \forall T \in \mathcal{T}_h(z), v_{hp} = 0 \text{ on } \partial\omega_z\}.$$

We then define

$$s_{hp}^z := \arg\min_{v_{hp} \in V_{hp}^z} \|\nabla_h(\psi_z u_{hp}) - \nabla v_{hp}\|_{0,\omega_z},$$

which is equivalent to finding $s_{hp}^z \in V_{hp}^z$ such that

$$\int_{\omega_z} \nabla s_{hp}^z \cdot \nabla \overline{v}_{hp} \, d\mathbf{x} = \int_{\omega_z} \nabla_h(\psi_z u_{hp}) \cdot \nabla \overline{v}_{hp} \, d\mathbf{x}, \qquad \text{for all } v_{hp} \in V_{hp}^z.$$

Lemma (Potential reconstruction)

$$s_{hp} := \widetilde{s}_{hp} - \frac{1}{|\Omega|} \int_\Omega \widetilde{s}_{hp} \, d\mathbf{x}, \qquad \textit{where } \widetilde{s}_{hp} := \sum_{z \in \mathcal{N}} s_{hp}^z$$

is a potential.

Theorem (Potential reconstruction efficiency)

$$\|\mathcal{G}(u_{hp}) + \nabla s_{hp}\|_{0,\Omega}^2 \lesssim \|\nabla(u - u_{hp})\|_{0,\Omega}^2 + \sum_{E\subset\partial\Omega} h_E^{-1}\|\Pi_E^0[\![u_{hp}]\!]\|_{0,E}^2 + \sum_{E\subset\partial\Omega} \beta^2 h_E \|\mathrm{p}^{-1}\Pi_E^0[\![\nabla u_{hp}]\!]\|_{0,E}^2$$

Proof.

We can write the above minimization in mixed form and show it has an underlying continuous problem: Find $r^z \in H_*^1(\omega_z)$ such that:

$$\int_{\omega_z} \nabla r^z \cdot \nabla \overline{v}\, d\boldsymbol{x} = -\int_{\omega_z} \mathrm{rot}_h(\psi_z u_{hp}) \cdot \nabla \overline{v}\, d\boldsymbol{x} \qquad \forall v \in H^1(\omega_z).$$

Following [Ern & Vohralík, 2015] we can show that

$$\|\nabla(u_{hp} - s_{hp})\|_{0,T} \lesssim \sum_{z\in\mathcal{N}(T)} \|\nabla r^z\|_{0,\omega_z}$$

$$\|\nabla r^z\|_{0,\omega_z}^2 \lesssim \|\nabla(u - u_{hp})\|_{0,\omega_z}^2 + \sum_{E\subset\partial\omega_z\cap\partial\Omega} h_E^{-1}\|\Pi_E^0[\![u - u_{hp}]\!]\|_{0,E}^2$$

Theorem (Efficiency)

$$\eta_{hp} \lesssim \|\nabla(u-u_{hp})\|_{0,\Omega} + k^2\|u-u_{hp}\|_{0,\Omega} + k\|u-u_{hp}\|_{0,\partial\Omega}$$
$$+ \operatorname{osc}(f) + \operatorname{osc}(g) + \|\gamma k \frac{\mathrm{h}}{\mathrm{p}}(g - \nabla_h u_{hp}\cdot\boldsymbol{n} + \mathrm{i}ku_{hp})\|_{0,\partial\Omega}$$
$$+ \|\mathrm{i}\gamma\frac{\sqrt{\mathrm{h}}}{\mathrm{p}}(g - \nabla_h u_{hp}\cdot\boldsymbol{n} + \mathrm{i}ku_{hp})\|_{0,\partial\Omega}$$
$$+ \left(\sum_{E\subset\partial\Omega} h_E^{-1}\|\Pi_E^0[\![u_{hp}]\!]\|_{0,E}^2\right)^{1/2} + \left(\sum_{E\subset\partial\Omega} \beta^2 h_E\|\mathrm{p}^{-1}\Pi_E^0[\![\nabla u_{hp}]\!]\|_{0,E}^2\right)^{1/2}.$$

Proof.

The flux reconstruction efficiency and potential reconstruction efficiency bounds the first and last terms of η_{hp} respectively. The other terms are bound by the oscillation terms $\operatorname{osc}(f)$ and $\operatorname{osc}(g)$ respectively. □

Section 3

hp-adaptive mesh refinement

We can re-write the error indicator in local form as $\eta_{hp} = \sum_{T \in \mathcal{T}_h} \eta_T$.

hp-adaptive mesh refinement

Construct mesh $\mathcal{T}_h$, with uniform polynomial degree, and FE space V_{hp}.

for $j = 0, 1, 2, \dots$ **do**
  Solve $a_{hp}(u_{hp}, v_{hp}) = \ell_{hp}(v_{hp})$ on $V_{hp}^{(0)}$
  Compute ζ_{hp}^z and s_{hp}^z (in parallel) and η_T
  for $T \in \mathcal{T}_h$ **do**
    if $\eta_T \geq \theta \max_{T \in \mathcal{T}_h} \eta_T$ **then**
      Perform h- or p-refinement on T [Melenk & Wohlmuth 2001]
    end if
  end for
  Perform mesh smoothing (remove hanging nodes)
end for

To compute ζ_{hp}^z we let $p_z = \max_{T \subset \omega_z} p_T + 1$ and to compute s_{hp}^z we let $p_z = \max_{T \subset \omega_z} p_T$. We use $\theta = 0.75$ (maximum marking strategy).

hp-refinement strategy [Melenk & Wohlmuth 2001]

Decision on *h*- or *p*-refinement as performed as follows:

if T marked for refinement **then**
    if $\eta_T > \eta_T^{\text{pred}}$ **then**
        h-refinement: Divide T into 2 $(T_\pm)$ using newest vertex bisection
        $(\eta_{T_\pm}^{\text{pred}})^2 \leftarrow \frac{1}{2}\gamma_h \left(\frac{1}{2}\right)^{p_T} \eta_T^2$
    else
        p-refinement: $p_T \leftarrow p_T + 1$
        $(\eta_T^{\text{pred}})^2 \leftarrow \gamma_p \eta_T^2$
    end if
else
    $(\eta_T^{\text{pred}})^2 \leftarrow \gamma_n (\eta_T^{\text{pred}})^2$
end if

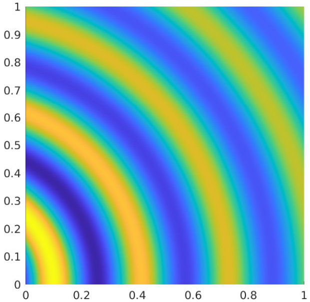
We use $\gamma_h = 4$, $\gamma_p = 0.4$, $\gamma_n = 1$, and set $\eta_T^{\text{pred}} = \infty$ initially.

Square domain example

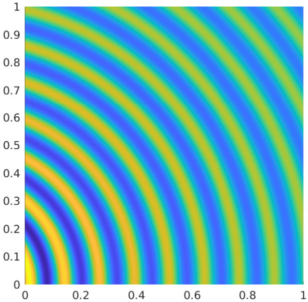
Let $\Omega = (0,1)^2$, $f = 0$, and select g such that the analytical solution is

$$u(\boldsymbol{x}) = \mathcal{H}_0^{(1)}\left(k\sqrt{(x_1 + 1/4)^2 + x_2^2}\right),$$

where $\mathcal{H}_0^{(1)}$ = Hankel function of the first kind. We consider $k = 20, 50$.



$k = 20$



$k = 50$

Let $\Omega = (0,1)^2$, $f = 0$, and select g such that the analytical solution is

$$u(\boldsymbol{x}) = \mathcal{H}_0^{(1)}\left(k\sqrt{(x_1 + 1/4)^2 + x_2^2}\right),$$

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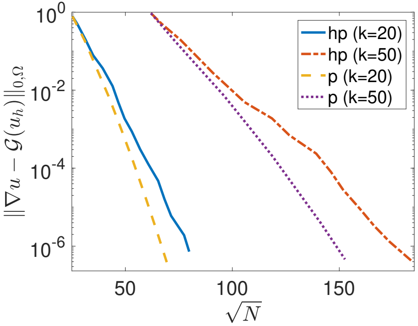
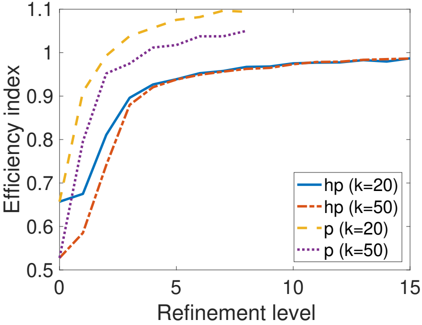


Convergence



Effectivity

Let $\Omega = (0,1)^2$, $f = 0$, and select g such that the analytical solution is

$$u(\boldsymbol{x}) = \mathcal{H}_0^{(1)}\left(k\sqrt{(x_1 + 1/4)^2 + x_2^2}\right),$$

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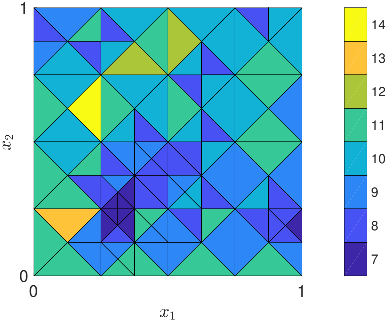
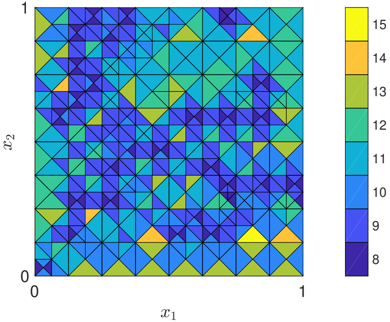


Final mesh ($k = 20$)



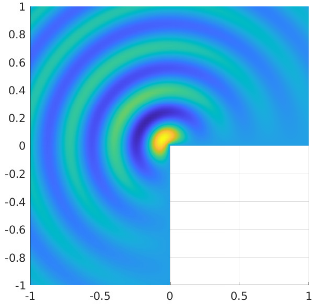
Final mesh ($k = 50$)

L-shaped domain example

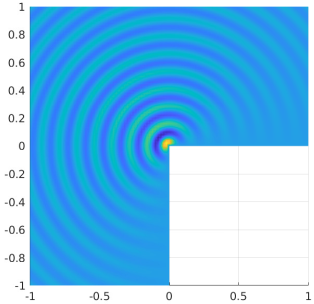
Let $\Omega = (-1,1)^2 \setminus ((0,1) \times (-1,0))$, $f = 0$, and select g such that

$$u(r,\varphi) = \mathcal{J}_{2/3}(kr)\sin(2\varphi/3),$$

where $\mathcal{J}_{2/3}$ denotes the Bessel function of first kind. We consider $k = 20, 50$.



$k = 20$



$k = 50$

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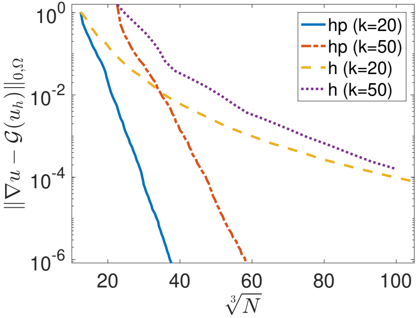
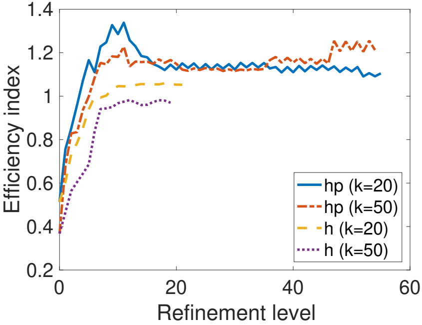


Convergence



Effectivity

Let $\Omega = (-1,1)^2 \backslash ((0,1) \times (-1,0))$, $f = 0$, and select g such that

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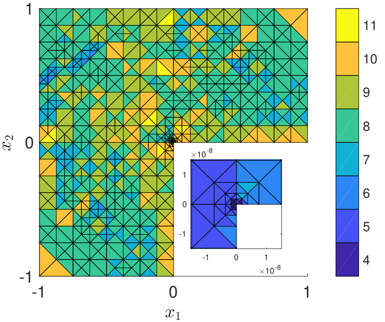
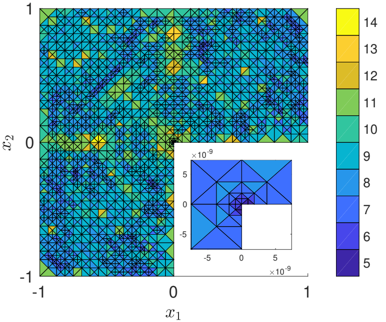


Final mesh ($k = 20$)



Final mesh ($k = 50$)

L-shaped domain example

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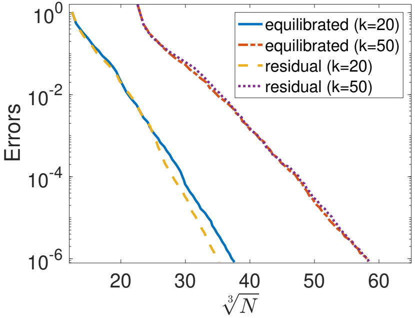
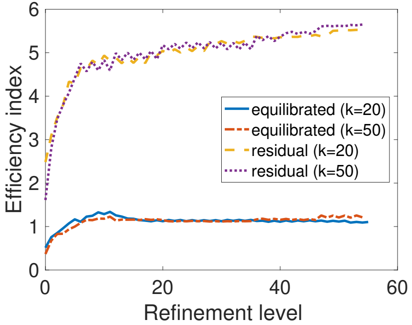
Let $\Omega = (-1,1)^2 \backslash ((0,1)\times(-1,0))$, $f = 0$, and select g such that

$$u(r,\varphi) = \mathcal{J}_{2/3}(kr)\sin(2\varphi/3),$$

where $\mathcal{J}_{2/3}$ denotes the Bessel function of first kind. We consider $k = 20, 50$.



Convergence vs. Sauter & Zech 2015



Effectivity vs. Sauter & Zech 2015

Internal reflection/refraction

We now consider a wavenumber k given by the piecewise constant function

$$k(x,y) = \begin{cases} k_1 := \omega n_1 & \text{if } y \leq 0, \\ k_2 := \omega n_2 & \text{if } y > 0, \end{cases}$$

where, we let $\omega = 20$, $n_1 = 2$, and $n_2 = 1$, with appropriate boundary conditions, such that , for a constant $0 \leq \theta_i \leq \pi/2$,

$$u(x,y) = \begin{cases} T\mathrm{e}^{i(K_1 x + K_2 y)} & \text{if } y > 0, \\ \mathrm{e}^{ik_1(x\cos(\theta_i) + y\sin(\theta_i))} + R\mathrm{e}^{ik_1(x\cos(\theta_i) - y\sin(\theta_i))} & \text{if } y < 0, \end{cases}$$

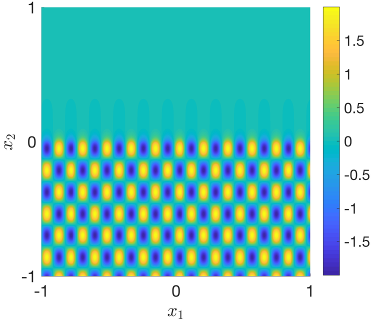
where $K_1 = k_1 \cos(\theta_i)$, $K_2 = \sqrt{k_2^2 - k_1^2\cos^2(\theta_i)}$,

$$R = -\frac{K_2 - k_1\sin(\theta_i)}{K_2 + k_1\sin(\theta_i)},$$

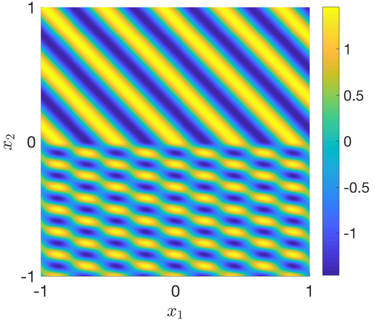
and $T = 1 + R$.

Internal reflection/refraction

There exists a critical angle θ_{crit}, such that when $\theta_i > \theta_{crit}$ the wave is refracted, while $\theta_i < \theta_{crit}$ results in internal reflection.

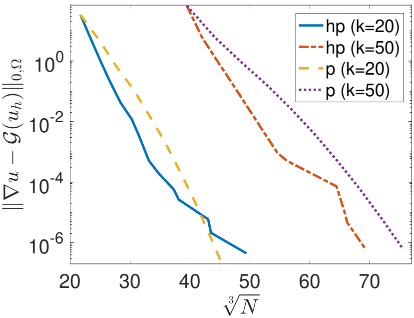


$\theta_i = 29^\circ$ — Analytical Soln.

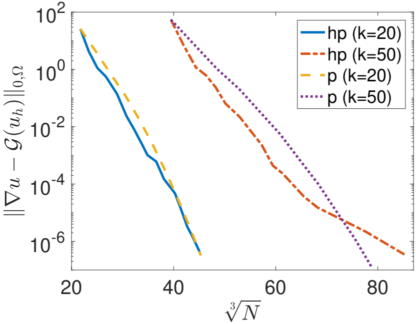


$\theta_i = 69^\circ$ — Analytical Soln.

There exists a critical angle θ_{crit}, such that when $\theta_i > \theta_{crit}$ the wave is refracted, while $\theta_i < \theta_{crit}$ results in internal reflection.



$\theta_i = 29°$ — Convergence



$\theta_i = 69°$ — Convergence

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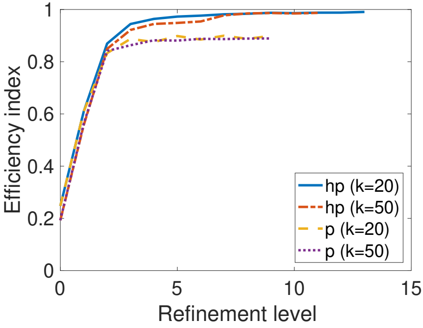
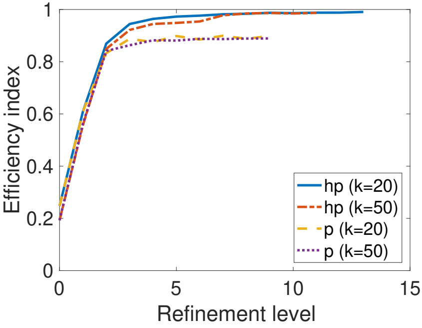
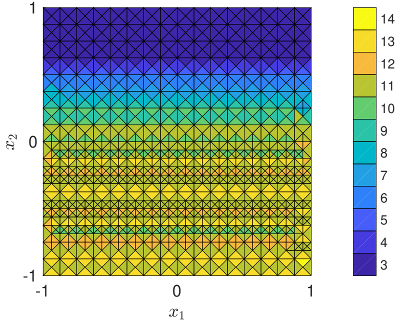


$\theta_i = 29^\circ$ — Effectivity (h)

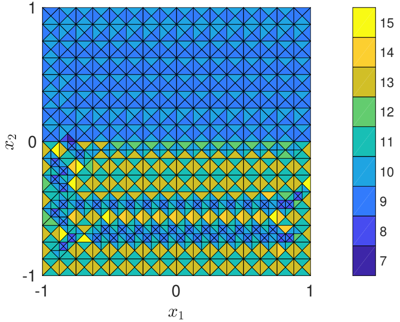


$\theta_i = 69^\circ$ — Effectivity (h)

There exists a critical angle θ_{crit}, such that when $\theta_i > \theta_{crit}$ the wave is refracted, while $\theta_i < \theta_{crit}$ results in internal reflection.



$\theta_i = 29^\circ$ — Final mesh



$\theta_i = 69^\circ$ — Final mesh

Conclusion

Summary:

- *a posteriori* error estimator based for Helmholtz
- Shown reliabilty and efficiency providing resolution condition met
- Demonstrated robust in polynomial degree

Further work:

- The analysis of the potential reconstruction is 2D only.
  [Ern, Vohralík 2017] has argument for 3D.
- Trefftz discontinuous Galerkin FEM