Adaptive Refinement for *hp*-version Trefftz Discontinuous Galerkin Methods for the Homogeneous Helmholtz Problem

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Joint work with
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16th European Finite Element Fair (Heidelberg)

Overview

1. Trefftz DG for Helmholtz
   - Helmholtz Equation
   - Trefftz DG
2. Adaptive Refinement
   - Plane Wave Direction Refinement
   - A posteriori Error Estimates
   - *hp*-adaptive Refinement

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded polygonal/polyhedral domain.

$$
\begin{aligned}
-\Delta u - k^2 u &= 0 && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma_D,\\
\nabla u \cdot \boldsymbol{n} + ik\vartheta u &= g_R && \text{on } \Gamma_R.
\end{aligned}
$$

Trefftz FEM Spaces

Polynomial DG Finite Element Spaces: DGFEM uses polynomial basis functions defined on a reference element $\widehat{K}$:

$$V_q^{DG}(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_K \circ F_K \in \mathcal{S}_{q_K}(\widehat{K}), K \in \mathcal{T}_h\}.$$

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Trefftz Finite Element Space: Use basis functions defined element-wise based on general solutions to the PDE.
First define the local Trefftz spaces

$$T(K) := \{v|_K : -\Delta u - k^2 u = 0\}$$

and let

$$T(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_K \in T(K), K \in \mathcal{T}_h\}.$$

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$$T(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_K \in T(K), K \in \mathcal{T}_h\}.$$

We let $V_p(K) \subset T(K)$ be a finite dimensional local space; then, the Trefftz FE Space is given by

$$V_p(\mathcal{T}_h) := \{v \in T(\mathcal{T}_h) : v_K \in V_p(K), K \in \mathcal{T}_h\}.$$

Plane Waves

$$V_p(K) = \left\{ v : v(\boldsymbol{x}) = \sum_{\ell=1}^{p_K} \alpha_\ell \mathrm{e}^{ik\boldsymbol{d}_\ell\cdot(\boldsymbol{x}-\boldsymbol{x}_K)}, \alpha_\ell \in \mathbb{C} \right\}$$

where p_K is the number of *degrees of freedom* for the element K, $\boldsymbol{d}_l$, $l = 1, \cdots, N_K$ are p_K (roughly) evenly spaced unit direction vectors, and $\boldsymbol{x}_K$ is the centre of the element.

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Trefftz DG has less degrees of freedom than high-order polynomials for the same accuracy.

Basis Functions	2D	3D
DG ($\mathcal{P}_q$)	$(q+1)(q+2)/2$	$(q+1)(q+2)(q+3)/6$
DG ($\mathcal{Q}_q$)	$(q+1)^2$	$(q+1)^3$
Trefftz DG	$2q+1$	$(q+1)^2$

Number of Degrees of Freedom

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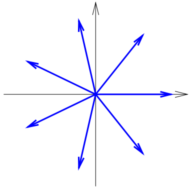
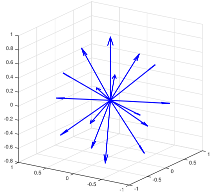
Direction Vectors ($q = 3$):

2D



3D



[Sloan & Womersley, 2004]

Trefftz Discontinuous Galerkin FEM for Helmholtz

Find $u_{hp} \in V_p(\mathcal{T}_h)$ such that,

$$\mathcal{A}_h(u_{hp}, v_{hp}) = \ell_h(v_{hp}),$$

for all $v_{hp} \in V_p(\mathcal{T}_h)$, where

$$\begin{aligned}
\mathcal{A}_h(u,v) = & \int_{\mathcal{F}_h^I} \{\!\{u\}\!\} [\![\nabla_h \bar{v}]\!] \, ds - \int_{\mathcal{F}_h^I} \beta (ik)^{-1} [\![\nabla_h u]\!] [\![\nabla_h \bar{v}]\!] \, ds \\
& - \int_{\mathcal{F}_h^I \cup \mathcal{F}_h^D} \{\!\{\nabla_h u\}\!\} \cdot [\![\bar{v}]\!] \, ds + \int_{\mathcal{F}_h^I \cup \mathcal{F}_h^D} \alpha ik [\![u]\!] \cdot [\![\bar{v}]\!] \, ds \\
& + \int_{\mathcal{F}_h^R} (1-\delta) u \nabla_h \bar{v} \cdot \boldsymbol{n} \, ds - \int_{\mathcal{F}_h^R} \delta (ik\vartheta)^{-1} (\nabla_h u \cdot \boldsymbol{n})(\nabla_h \bar{v} \cdot \boldsymbol{n}) \, ds \\
& - \int_{\mathcal{F}_h^R} \delta \nabla_h u \cdot \boldsymbol{n} \bar{v} \, ds + \int_{\mathcal{F}_h^R} (1-\delta) ik\vartheta u \bar{v} \, ds, \\
\ell_h(v) = & - \int_{\mathcal{F}_h^R} \delta (ik\vartheta)^{-1} g_R \nabla_h \bar{v} \cdot \boldsymbol{n} \, ds + \int_{\mathcal{F}_h^R} (1-\delta) g_R \bar{v} \, ds.
\end{aligned}$$

Plane Wave Direction Refinement

Selecting plane wave directions which align with the wave direction of the analytical solution can reduce the error.
Several existing approaches exist for selecting pane wave directions:

- Ray-tracing — requires a source term. [Betcke & Phillips, 2012]
- Approximate
$$\frac{\nabla e(\boldsymbol{x}_0)}{ike(\boldsymbol{x}_0)},$$
where e is the error. [Gittelson, 2008 (Master's Thesis)]
- Adding an extra unknown (the optimal angle of rotation) to the basis functions. [Amara, Chaudhry, Diaz, Djellouli & Fiedler, 2014]

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We propose using the Hessian of the numerical solution, based on work on anisotropic meshes for standard FE [Formaggia & Perotto, 2001, 2003].

Plane Wave Refinement Algorithm (2D)

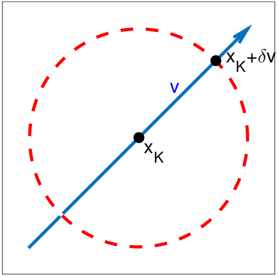
Let $(\lambda_1, \boldsymbol{v}_1), (\lambda_2, \boldsymbol{v}_2)$ be the eigenpairs of $\mathbf{H}(\mathrm{Re}(u_h(\boldsymbol{x}_K)))$, and $(\mu_1, \boldsymbol{w}_1), (\mu_2, \boldsymbol{w}_2)$ the eigenpairs of $\mathbf{H}(\mathrm{Im}(u_h(\boldsymbol{x}_K)))$ s.t. $|\lambda_1| \geq |\lambda_2|$, $|\mu_1| \geq |\mu_2|$; then, for constant $C > 1$, we can select the first plane wave direction as follows:

$\lvert\lambda_1\rvert \geq C\lvert\lambda_2\rvert$	$\lvert\mu_1\rvert \geq C\lvert\mu_2\rvert$	$\lvert\lambda_1\rvert \geq C\lvert\mu_1\rvert$	$\lvert\mu_1\rvert \geq C\lvert\lambda_1\rvert$	First PW
✓	✓	✓	✗	$\boldsymbol{v}_1$
✓	✓	✗	✓	$\boldsymbol{w}_1$
✓	✓	✗	✗	$\frac{(\boldsymbol{v}_1+\boldsymbol{w}_1)}{\lVert\boldsymbol{v}_1+\boldsymbol{w}_1\rVert}$
✓	✗	✓	✗	$\boldsymbol{v}_1$
✓	✗	✗	–	–
✗	✓	✗	✓	$\boldsymbol{w}_1$
✗	✓	–	✗	–
✗	✗	–	–	–

[C., Houston, Perugia (Submitted)]

Plane Wave Direction Refinement

If $\boldsymbol{v}$ is the eigenvector, then the direction of propagation could be either $\boldsymbol{v}$ or $-\boldsymbol{v}$ (unknown orientation). Consider the impedance on the boundary of a ball (radius δ around $\boldsymbol{x}_K$) and compare to the plane wave $u(\boldsymbol{x}) = \mathrm{e}^{ik\boldsymbol{d}\cdot(\boldsymbol{x}-\boldsymbol{x}_K)}$ for the cases when $\boldsymbol{d} = \boldsymbol{v}$ and $\boldsymbol{d} = -\boldsymbol{v}$.



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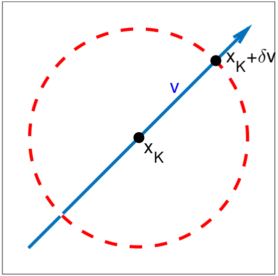

Evaluating at $\boldsymbol{x}_K + \delta\boldsymbol{v}$ we note that the normal is $\boldsymbol{v}$, so we can calculate

$$\frac{\nabla u_h(\boldsymbol{x}_K + \delta\boldsymbol{v}) \cdot \boldsymbol{v} + iku_h(\boldsymbol{x}_K + \delta\boldsymbol{v})}{iku_h(\boldsymbol{x}_K + \delta\boldsymbol{v})}.$$

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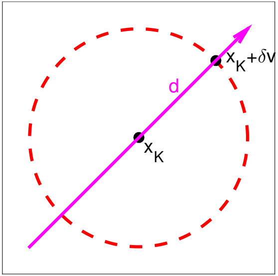

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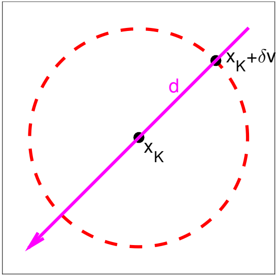
We can compare this to the impedance for u:

$$\frac{\nabla u(\boldsymbol{x}_K + \delta\boldsymbol{v}) \cdot \boldsymbol{v}}{iku(\boldsymbol{x}_K + \delta\boldsymbol{v})} + 1 = \begin{cases} 2, & \text{if } \boldsymbol{d} = \boldsymbol{v}, \\ & \end{cases}$$

If $\boldsymbol{v}$ is the eigenvector, then the direction of propagation could be either $\boldsymbol{v}$ or $-\boldsymbol{v}$ (unknown orientation). Consider the impedance on the boundary of a ball (radius δ around $\boldsymbol{x}_K$) and compare to the plane wave $u(\boldsymbol{x}) = \mathrm{e}^{ik\boldsymbol{d}\cdot(\boldsymbol{x}-\boldsymbol{x}_K)}$ for the cases when $\boldsymbol{d} = \boldsymbol{v}$ and $\boldsymbol{d} = -\boldsymbol{v}$.



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An *a posteriori* error bounds exists for the *h*-version of the method in $\mathbb{R}_2$.

A posteriori Error Bound — *h*-version Only

For the TDGFEM, with the constant flux parameters, the following error bound holds:

$$\|u - u_h\|^2_{L^2(\Omega)} \leq C(k, d_\Omega)\Bigg\{ \left\|\alpha^{1/2} h_F^s [\![u_h]\!]\right\|^2_{L^2(\mathcal{F}_h^I \cup \mathcal{F}_h^D)} + \frac{1}{k^2}\|\beta^{\frac{1}{2}} h_F^s [\![\nabla u_h]\!]\|^2_{L^2(\mathcal{F}_h^I)} + \frac{1}{k^2}\left\|\delta^{1/2} h_F^s \left(g_R - \nabla u_h \cdot \boldsymbol{n}_F + ik\vartheta u_h\right)\right\|^2_{L^2(\mathcal{F}_h^R)} \Bigg\}$$

where s depends on the regularity of the solution to the adjoint problem ($z \in H^{3/2+s}(\Omega)$).

[Kapita, Monk & Warburton, 2015]

A posteriori Error Bound — *hp*-version

We propose the following potential *a posteriori* error bound with constants derived numerical to ensure the bound is efficient:

$$\|u - u_{hp}\|^2_{L^2(\Omega)} \leq C\Bigg\{ k\Big\|\alpha^{1/2} h_F^{1/2} q_F^{-1/2} [\![u_{hp}]\!]\Big\|^2_{L^2(\mathcal{F}_h^I \cup \mathcal{F}_h^D)} + \|\beta^{\frac{1}{2}} h_F^{3/2} q_F^{-3/2} [\![\nabla u_{hp}]\!]\|^2_{L^2(\mathcal{F}_h^I)} + \Big\|\delta^{1/2} h_F^{3/2} q_F^{-3/2} (g_R - \nabla u_{hp} \cdot \boldsymbol{n}_F + iku_{hp})\Big\|^2_{L^2(\mathcal{F}_h^R)} \Bigg\}$$

for smooth solution of the adjoint.

[C., Houston, Perugia (Submitted)]

Modified *hp*-refinement Strategy [Melenk & Wohlmuth, 2001]

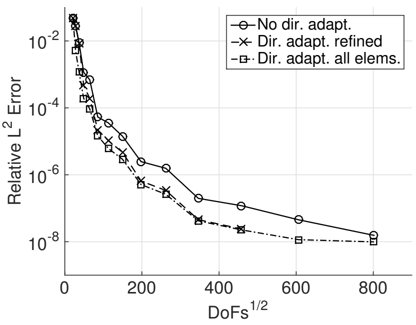
Let $\mathcal{T}_{h,0}$ be the initial mesh, $\mathcal{T}_{h,i}$ the mesh after i refinements, $\eta_{K,i}$ the error indicator for $K \in \mathcal{T}_{h,i}$, and $\eta_{K,i}^{\text{pred}}$ the predicted error for $K \in \mathcal{T}_{h,i}$.

for $K \in \mathcal{T}_{h,i}$ **do**
  if K is marked for refinement **then**
    if $\eta_{K,i}^2 > (\eta_{K,i}^{\text{pred}})^2$ **then**
      h-refinement: Subdivide K into N sons $K_s, s \in 0, \ldots, N$
      $(\eta_{K_s,i+1}^{\text{pred}})^2 \leftarrow \frac{1}{N}\gamma_h \left(\frac{1}{2}\right)^{2q_K} \eta_{K,i}^2$, $i \leq s \leq N$
    else
      p-refinement: $q_K \leftarrow q_K + 1$
      $(\eta_{K,i+1}^{\text{pred}})^2 \leftarrow \gamma_p \eta_{K,i}^2$
    end if
  else
    $(\eta_{K,i+1}^{\text{pred}})^2 \leftarrow \gamma_n (\eta_{K,i}^{\text{pred}})^2$
  end if
end for

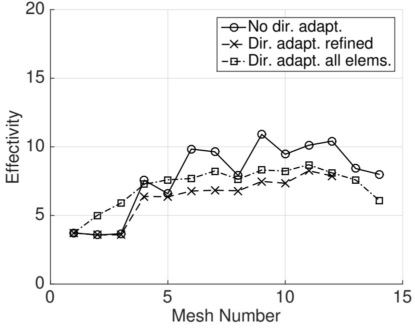
Consider the smooth (analytic) solution (for Acoustic Wave Propagation)

$$u(x,y) = \mathcal{H}_0^{(1)}(k\sqrt{(x+1/4)^2+y^2}),$$

on the domain $\Omega = (0,1)^2$ with suitable Robin BCs.
Consider h- and hp-refinement for $k = 20$.



L^2-Error & Error Bound



Effectivity

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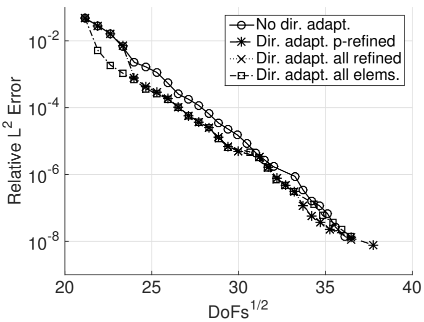

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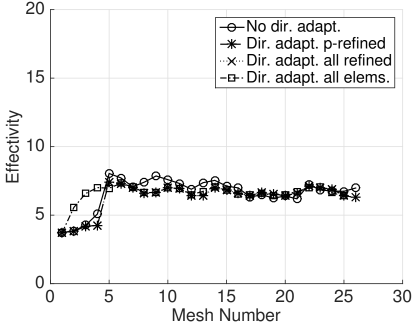

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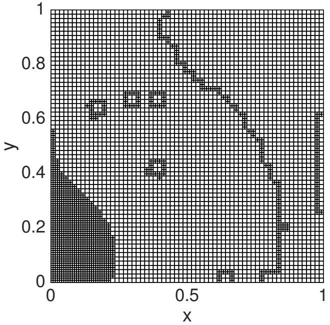
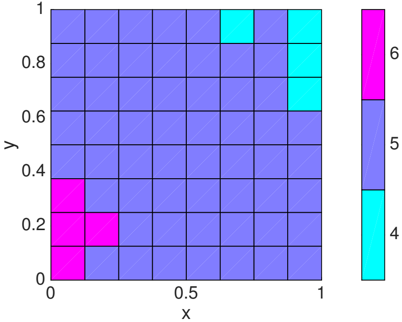


Mesh after 8 h-refinements



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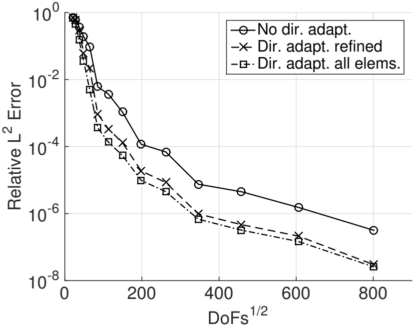

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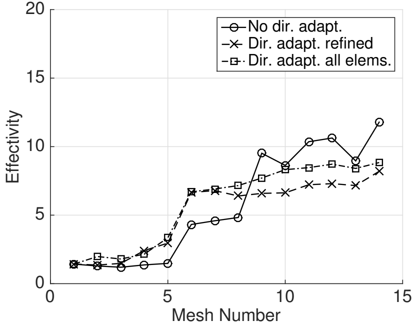

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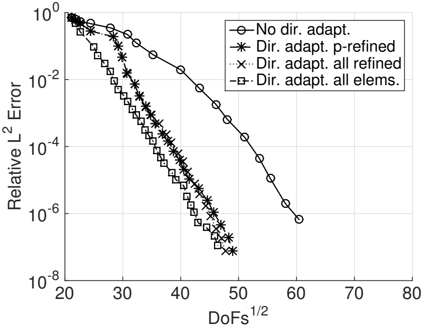

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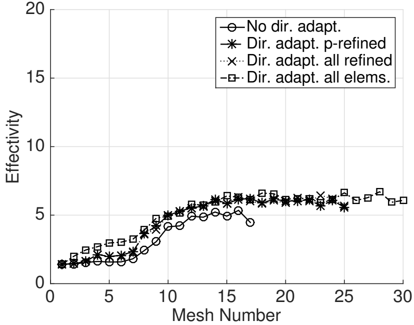

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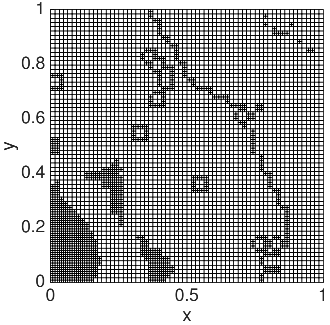

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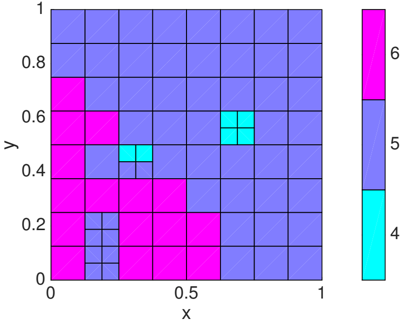

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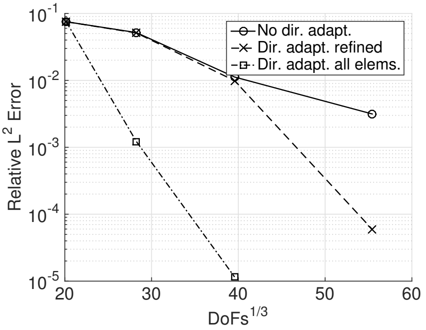

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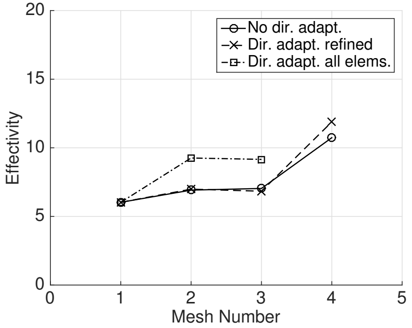

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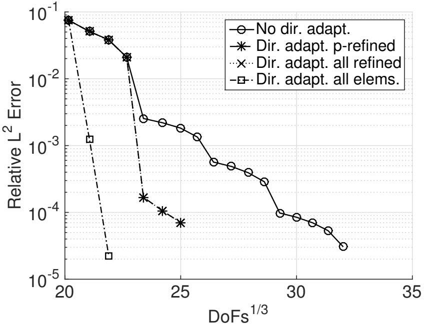

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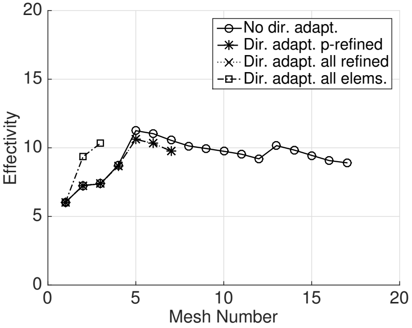

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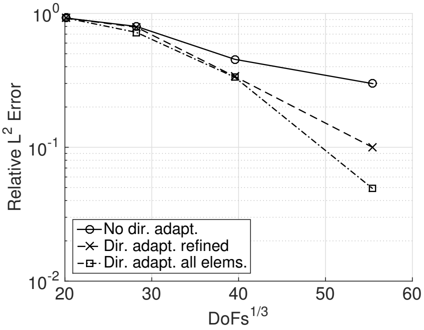
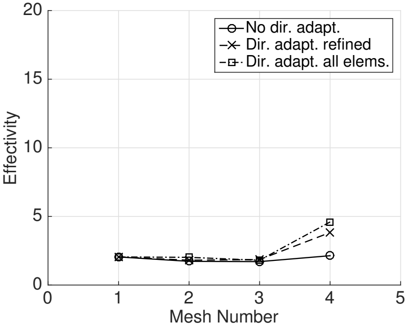


L^2-Error & Error Bound



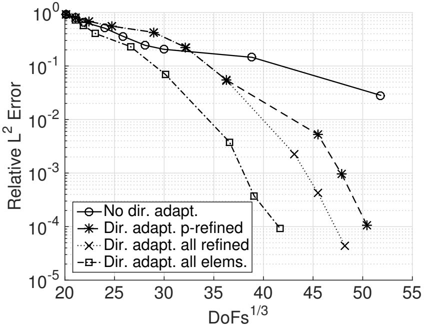
Effectivity

Consider the 3D smooth (analytic) solution (for Acoustic Wave Propagation)

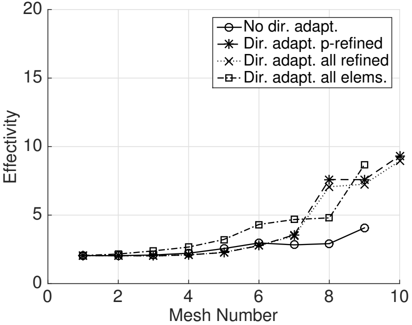
$$u(\boldsymbol{x}) = \mathrm{e}^{ik\boldsymbol{d}\cdot\boldsymbol{x}},$$

on the domain $\Omega = (0,1)^3$, where $\boldsymbol{d}_i = 1/\sqrt{3}$ for $i = 1,2,3$, with suitable Robin BCs.

Consider h- and hp-refinement for $k = 50$.



L^2-Error & Error Bound



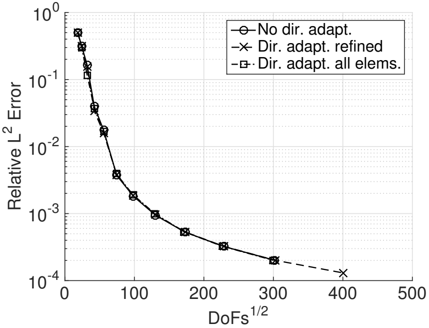
Effectivity

Consider the non-smooth solution (for Acoustic Wave Propagation)

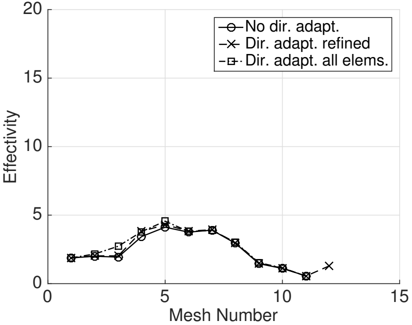
$$u(r,\theta) = \mathcal{J}_{2/3}(kr)\sin(2\theta/3),$$

on the domain L-shaped domain $\Omega = (-1,1)^2 \setminus (0,1) \times (-1,1)$, with suitable Robin BCs.

Consider h- and hp-refinement for $k = 20$.



L^2-Error & Error Bound



Effectivity

Adaptive Refinement

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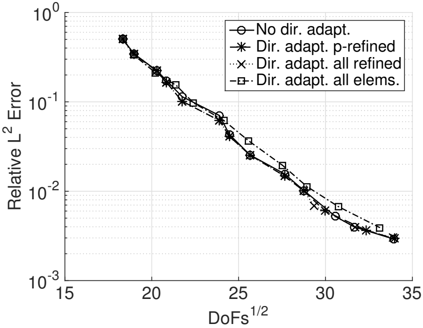

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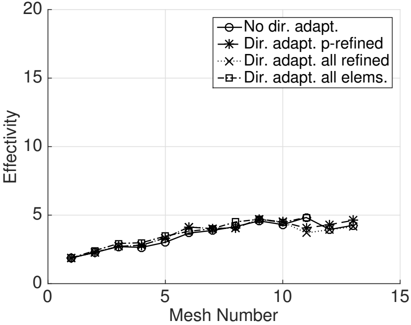

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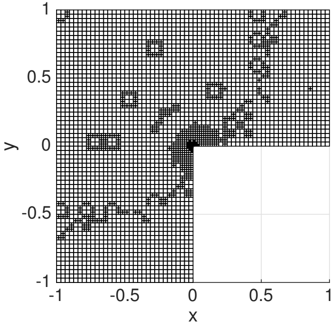
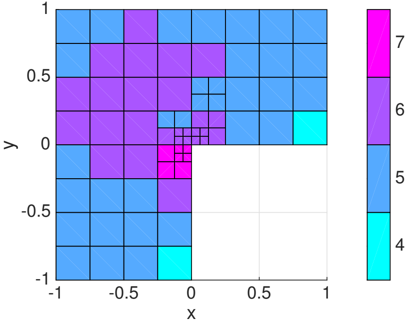


Mesh after 8 h-refinements



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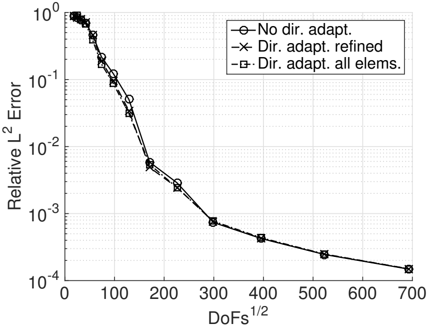
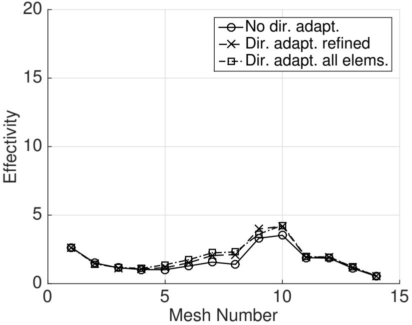


L^2-Error & Error Bound



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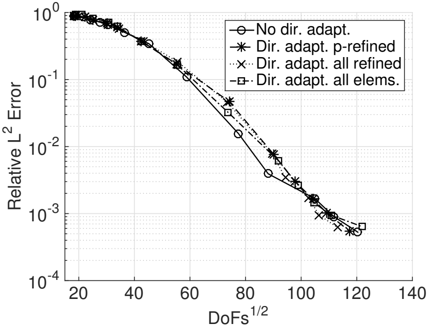
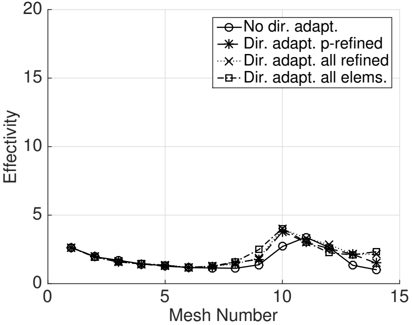


L^2-Error & Error Bound



Effectivity

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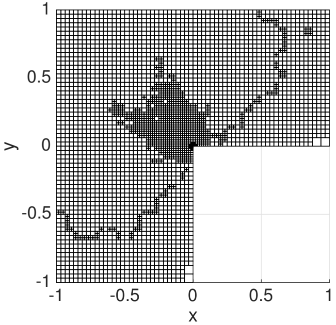
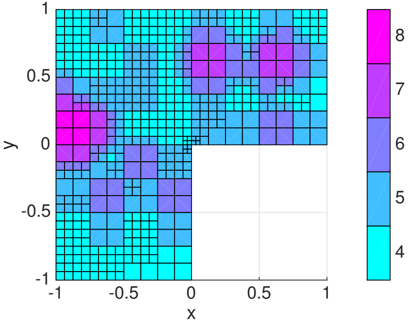


Mesh after 8 h-refinements



Mesh after 8 hp-refinements

Conclusion

Summary:

- With plane wave basis functions it is possible to refine the wave directions.
- *hp*-adaptive refinement results in exponential convergence.
- Combining plane wave direction adaptivity with *hp*-adaptive refinement often leads to reduced error compared to standard refinement.

Future Aims:

- Develop robust *hp*-version *a posteriori* error bounds..
- Use the eigenvalues/eigenvectors to develop anisotropic *p*-refinement (unevenly spaced plane waves).