Adaptive Refinement for *hp*-version Trefftz Discontinuous Galerkin Methods for the Homogeneous Helmholtz Problem

Scott Congreve

Fakultät für Mathematik, Universität Wien

Joint work with
Ilaria Perugia (Universität Wien)
Paul Houston (University of Nottingham)

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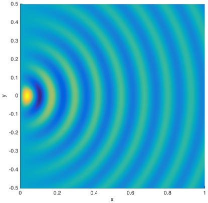
Overview

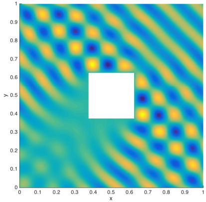
1. Trefftz DG for Helmholtz
   - Helmholtz Equation
   - Trefftz DG
   - Comparison to Polynomial DG
2. Adaptive Refinement
   - Plane Wave Direction Refinement
   - A posteriori Error Estimates
   - *hp*-adaptive Refinement

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded polygonal/polyhedral domain.

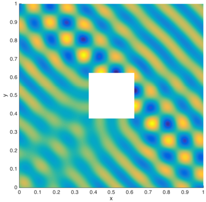
$$
\begin{aligned}
-\Delta u - k^2 u &= 0 && \text{in } \Omega, \\
u &= 0 && \text{on } \Gamma_D, && \text{(sound-soft scattering)} \\
\nabla u \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma_N, && \text{(sound-hard scattering)} \\
\nabla u \cdot \boldsymbol{n} + ik\vartheta u &= g_R && \text{on } \Gamma_R.
\end{aligned}
$$



Acoustic Wave Prop.



Sound-soft Scattering



Sound-hard Scattering

Polynomial DG Finite Element Spaces: DGFEM uses polynomial basis functions defined on a reference element $\widehat{K}$:

$$V_q^{DG}(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_K \circ F_K \in \mathcal{S}_{q_K}(\widehat{K}), K \in \mathcal{T}_h\}.$$

Trefftz FEM Spaces

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Trefftz Finite Element Space: Use basis functions defined element-wise based on general solutions to the PDE.
First define the local Trefftz spaces

$$T(K) := \{v|_K : -\Delta u - k^2 u = 0\}$$

and let

$$T(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_K \in T(K), K \in \mathcal{T}_h\}.$$

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$$T(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_K \in T(K), K \in \mathcal{T}_h\}.$$

We let $V_p(K) \subset T(K)$ be a finite dimensional local space; then, the Trefftz FE Space is given by

$$V_p(\mathcal{T}_h) := \{v \in T(\mathcal{T}_h) : v_K \in V_p(K), K \in \mathcal{T}_h\}.$$

$$V_p(K) = \left\{ v : v(\boldsymbol{x}) = \sum_{\ell=1}^{p_K} \alpha_\ell \mathrm{e}^{ik\boldsymbol{d}_\ell \cdot (\boldsymbol{x} - \boldsymbol{x}_K)}, \alpha_\ell \in \mathbb{C} \right\}$$

where p_K is the number of *degrees of freedom* for the element K, $\boldsymbol{d}_l$, $l = 1, \cdots, N_K$ are p_K (roughly) evenly spaced unit direction vectors, and $\boldsymbol{x}_K$ is the centre of the element.

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Trefftz DG has less degrees of freedom than high-order polynomials for the same accuracy.

Basis Functions	2D	3D
DG ($\mathcal{P}_q$)	$(q+1)(q+2)/2$	$(q+1)(q+2)(q+3)/6$
DG ($\mathcal{Q}_q$)	$(q+1)^2$	$(q+1)^3$
Trefftz DG	$2q+1$	$(q+1)^2$

Number of Degrees of Freedom

Plane Waves

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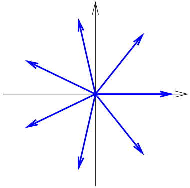
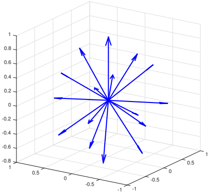
Number of Degrees of Freedom

Direction Vectors ($q = 3$):

2D



3D



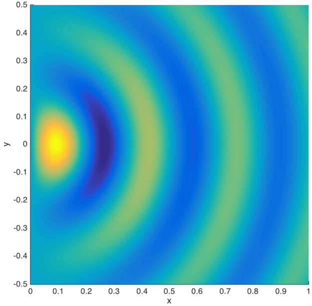
[Sloan & Womersley, 2004]

TDG/DG FEM Comparison

Consider the smooth (analytic) solution (for Acoustic Wave Propagation)

$$u(r,\theta) = \mathcal{J}_1(kr)\cos(\theta)$$

for $k = 20$ on the domain $\Omega = (0,1) \times (-1/2, 1/2)$.



Analytical Solution
(Real Part)

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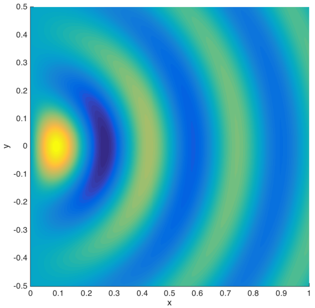
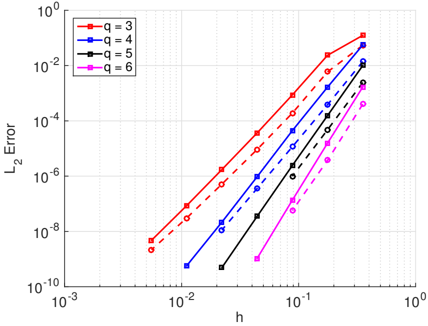
We solve using both a DGFEM (solid line) and Trefftz DGFEM (dashed).



Analytical Solution (Real Part)



$\|u - u_{hp}\|_{L^2(\Omega)}$ vs. h (h-refinement)

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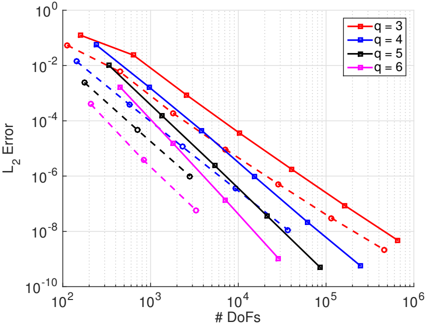

$\|u - u_{hp}\|_{L^2(\Omega)}$ vs. Degrees of Freedom
(h-refinement)

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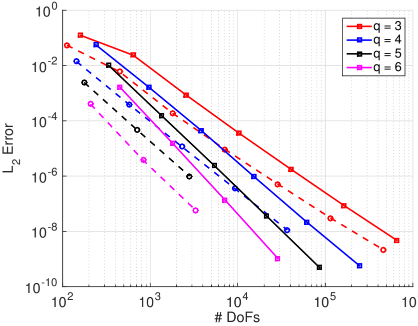
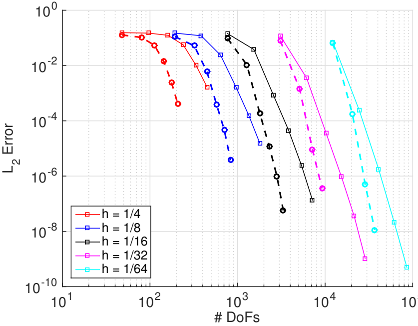


$\|u - u_{hp}\|_{L^2(\Omega)}$ vs. Degrees of Freedom (h-refinement)



$\|u - u_{hp}\|_{L^2(\Omega)}$ vs. Degrees of Freedom (p-refinement)

Trefftz Discontinuous Galerkin FEM for Helmholtz

Find $u_{hp} \in V_p(\mathcal{T}_h)$ such that,

$$\mathcal{A}_h(u_{hp}, v_{hp}) = \ell_h(v_{hp}),$$

for all $v_{hp} \in V_p(\mathcal{T}_h)$, where

$$\begin{aligned}
\mathcal{A}_h(u,v) = &\int_{\mathcal{F}_h^I \cup \mathcal{F}_h^N} \{\!\{u\}\!\} [\![\nabla_h \bar{v}]\!] \, ds - \int_{\mathcal{F}_h^I \cup \mathcal{F}_h^N} \beta (ik)^{-1} [\![\nabla_h u]\!] [\![\nabla_h \bar{v}]\!] \, ds \\
&- \int_{\mathcal{F}_h^I \cup \mathcal{F}_h^D} \{\!\{\nabla_h u\}\!\} \cdot [\![\bar{v}]\!] \, ds + \int_{\mathcal{F}_h^I \cup \mathcal{F}_h^D} \alpha ik [\![u]\!] \cdot [\![\bar{v}]\!] \, ds \\
&+ \int_{\mathcal{F}_h^R} (1-\delta) u \nabla_h \bar{v} \cdot \boldsymbol{n} \, ds - \int_{\mathcal{F}_h^R} \delta (ik\vartheta)^{-1} (\nabla_h u \cdot \boldsymbol{n})(\nabla_h \bar{v} \cdot \boldsymbol{n}) \, ds \\
&- \int_{\mathcal{F}_h^R} \delta \nabla_h u \cdot \boldsymbol{n} \bar{v} \, ds + \int_{\mathcal{F}_h^R} (1-\delta) ik\vartheta u \bar{v} \, ds, \\
\ell_h(v) = &- \int_{\mathcal{F}_h^R} \delta (ik\vartheta)^{-1} g_R \nabla_h \bar{v} \cdot \boldsymbol{n} \, ds + \int_{\mathcal{F}_h^R} (1-\delta) g_R \bar{v} \, ds.
\end{aligned}$$

Flux Parameters

Penalty Type	α	β	δ
DG-type Gittelson, Hiptmair & Perugia, 2009	$\mathtt{a}q_K^2/kh_K$	$\mathtt{b}kh_K/q_K$	$\mathtt{d}kh_K/q_K$
Constant Hiptmair, Moiola & Perugia, 2011	$\mathtt{a}$	$\mathtt{b}$	$\mathtt{d}$
UWVF Cessenat & Després, 1998	$1/2$	$1/2$	$1/2$
Non-Uniform Mesh Hiptmair, Moiola & Perugia, 2014	$\mathtt{a}h_{\max}/h_K$	$\mathtt{b}h_{\max}/h_K$	$\mathtt{d}h_{\max}/h_K$

Consider a plane wave analytical solution (for Acoustic Wave Propagation)

$$u(\boldsymbol{x}) = \mathrm{e}^{ik\boldsymbol{d}\cdot\boldsymbol{x}}$$

for $k = 20$ on the domain $\Omega = (0,1)^2$, where $\boldsymbol{d} = (1/\sqrt{2}, 1/\sqrt{2})$.

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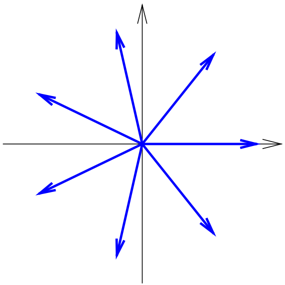
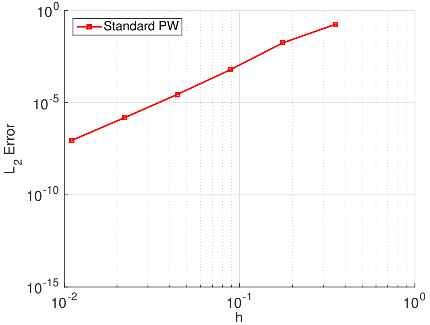
We evenly distribute directions $\boldsymbol{d}_\ell$, starting from $\boldsymbol{d}_1 = (1,0)$.



Plane Wave Directions ($q = 3$)



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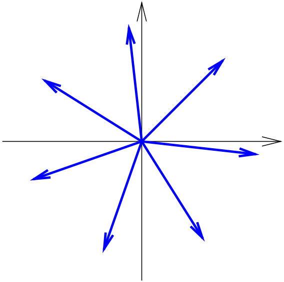
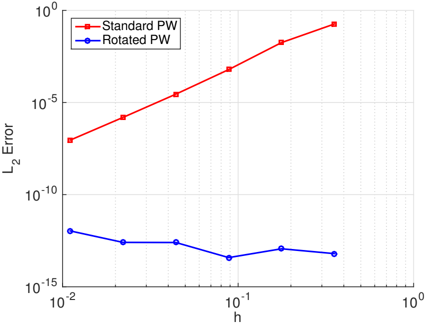
We evenly distribute directions $\boldsymbol{d}_\ell$, starting from $\boldsymbol{d}_1 = (1,0)$.
Rotating directions so that $\boldsymbol{d}_1 = \boldsymbol{d}$ gives (almost) the analytical solution.



Rotated Directions ($q = 3$)



$\|u - u_{hp}\|_{L^2(\Omega)}$ vs. h

Plane Wave Direction Refinement

Even for non-plane wave analytical solutions picking the correct main direction reduces the error.

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We need a way calculate/adapt the directions without the analytical solution. Several existing approaches exist:

- Ray-tracing — requires a source term. [Betcke & Phillips, 2012]
- Approximate
$$\frac{\nabla e(\boldsymbol{x}_0)}{ike(\boldsymbol{x}_0)},$$
where e is the error. [Gittelson, 2008 (Master's Thesis)]
- Adding an extra unknown (the optimal angle of rotation) to the basis functions. [Amara, Chaudhry, Diaz, Djellouli & Fiedler, 2014]

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Plane Wave Refinement Algorithm (2D)

Let $(\lambda_1, \boldsymbol{v}_1), (\lambda_2, \boldsymbol{v}_2)$ be the eigenpairs of $\mathbf{H}(\mathrm{Re}(u_h(\boldsymbol{x}_K)))$, and $(\mu_1, \boldsymbol{w}_1), (\mu_2, \boldsymbol{w}_2)$ the eigenpairs of $\mathbf{H}(\mathrm{Im}(u_h(\boldsymbol{x}_K)))$ s.t. $|\lambda_1| \geq |\lambda_2|$, $|\mu_1| \geq |\mu_2|$; then, for constant $C > 1$, we can select the first plane wave direction as follows:

$\|\lambda_1\| \geq C\|\lambda_2\|$	$\|\mu_1\| \geq C\|\mu_2\|$	$\|\lambda_1\| \geq C\|\mu_1\|$	$\|\mu_1\| \geq C\|\lambda_1\|$	First PW
✓	✓	✓	✗	$\boldsymbol{v}_1$
✓	✓	✗	✓	$\boldsymbol{w}_1$
✓	✓	✗	✗	$\frac{(\boldsymbol{v}_1+\boldsymbol{w}_1)}{\|\boldsymbol{v}_1+\boldsymbol{w}_1\|}$
✓	✗	✓	✗	$\boldsymbol{v}_1$
✓	✗	✗	–	–
✗	✓	✗	✓	$\boldsymbol{w}_1$
✗	✓	–	✗	–
✗	✗	–	–	–

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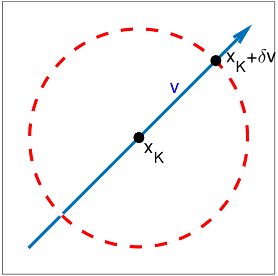
If $\boldsymbol{v}$ is the eigenvector, then the direction of propagation could be either $\boldsymbol{v}$ or $-\boldsymbol{v}$ (unknown orientation). Consider the impedance on the boundary of a ball (radius δ around $\boldsymbol{x}_K$) and compare to the plane wave $u(\boldsymbol{x}) = \mathrm{e}^{ik\boldsymbol{d}\cdot(\boldsymbol{x}-\boldsymbol{x}_K)}$ for the cases when $\boldsymbol{d} = \boldsymbol{v}$ and $\boldsymbol{d} = -\boldsymbol{v}$.



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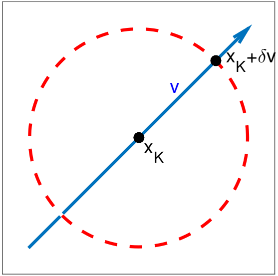

Evaluating at $\boldsymbol{x}_K + \delta\boldsymbol{v}$ we note that the normal is $\boldsymbol{v}$, so we can calculate

$$\frac{\nabla u_h(\boldsymbol{x}_K + \delta\boldsymbol{v}) \cdot \boldsymbol{v} + iku_h(\boldsymbol{x}_K + \delta\boldsymbol{v})}{iku_h(\boldsymbol{x}_K + \delta\boldsymbol{v})}.$$

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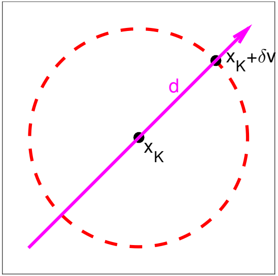

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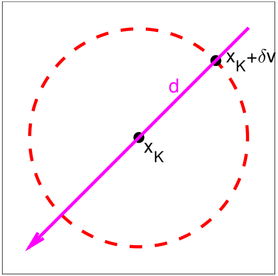
We can compare this to the impedance for u:

$$\frac{\nabla u(\boldsymbol{x}_K + \delta\boldsymbol{v}) \cdot \boldsymbol{v}}{iku(\boldsymbol{x}_K + \delta\boldsymbol{v})} + 1 = \begin{cases} 2, & \text{if } \boldsymbol{d} = \boldsymbol{v}, \\ \end{cases}$$

If $\boldsymbol{v}$ is the eigenvector, then the direction of propagation could be either $\boldsymbol{v}$ or $-\boldsymbol{v}$ (unknown orientation). Consider the impedance on the boundary of a ball (radius δ around $\boldsymbol{x}_K$) and compare to the plane wave $u(\boldsymbol{x}) = \mathrm{e}^{ik\boldsymbol{d}\cdot(\boldsymbol{x}-\boldsymbol{x}_K)}$ for the cases when $\boldsymbol{d} = \boldsymbol{v}$ and $\boldsymbol{d} = -\boldsymbol{v}$.



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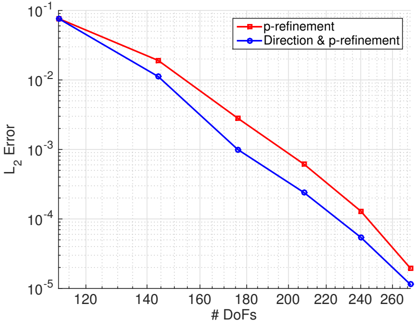
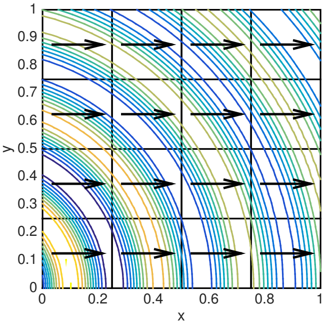
To test the direction refinement, we consider the solution

$$u(x,y) = \mathcal{H}_0^{(1)}\left(k\sqrt{(x+1/4)^2+y^2}\right),$$

with $k = 20$, on the domain $\Omega = (0,1)^2$.



$\|u - u_{hp}\|_{L^2(\Omega)}$ vs. *DoF*



First PW Direction ($p = 3$)

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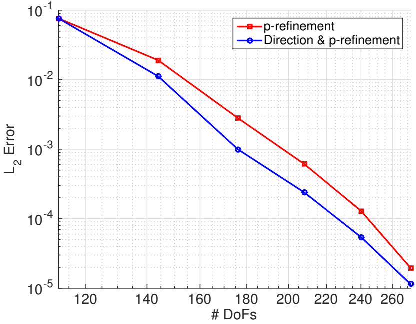

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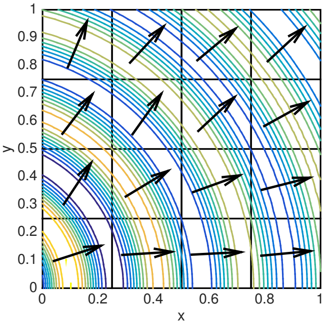

First PW Direction ($p = 4$)

Plane Wave Direction Refinement

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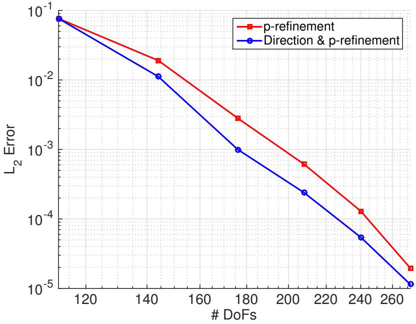
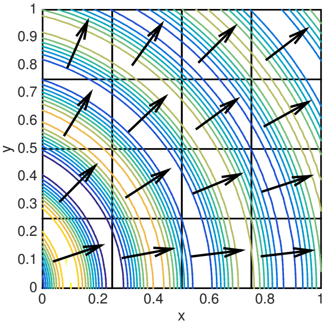


$\|u - u_{hp}\|_{L^2(\Omega)}$ vs. DoF

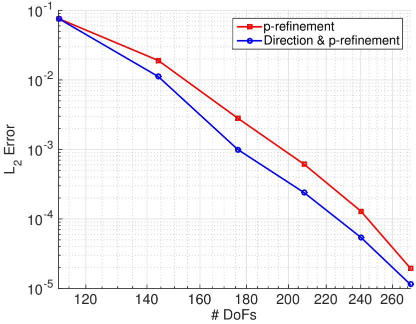


First PW Direction ($p = 5$)

To test the direction refinement, we consider the solution

$$u(x,y) = \mathcal{H}_0^{(1)}\left(k\sqrt{(x+1/4)^2+y^2}\right),$$

with $k = 20$, on the domain $\Omega = (0,1)^2$.



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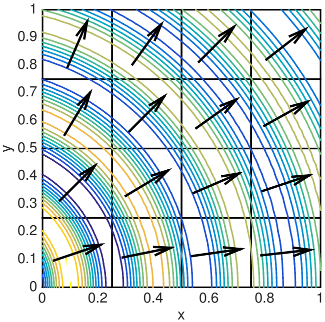

First PW Direction ($p = 6$)

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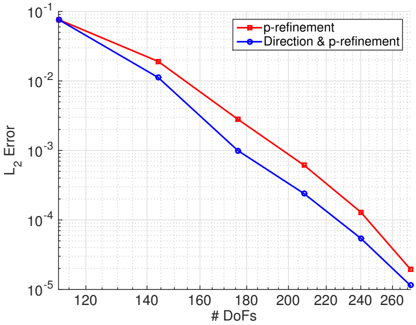

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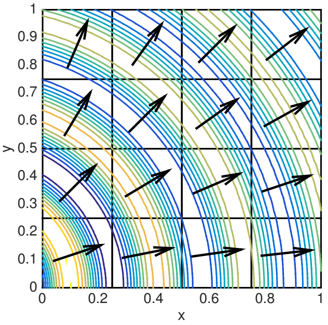

First PW Direction ($p = 7$)

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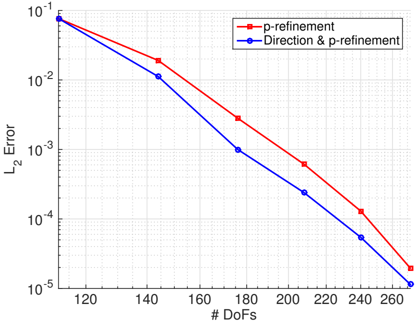

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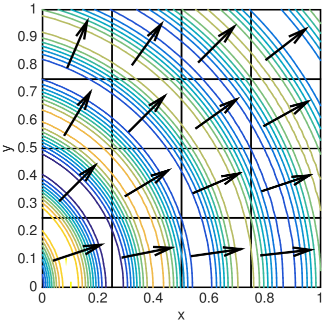

First PW Direction $(p = 8)$

Plane Wave Direction Refinement (3D)

Third eigenpair $(\lambda_3, \boldsymbol{v}_3)$ of $\mathbf{H}(\mathrm{Re}(u_{hp}(\boldsymbol{x}_K)))$, and third eigenpair $(\mu_3, \boldsymbol{w}_3)$ of $\mathbf{H}(\mathrm{Im}(u_{hp}(\boldsymbol{x}_K)))$.

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From the primary wave direction $\boldsymbol{d}$ we select the other directions, $\boldsymbol{d}_\ell$, $\ell = 1, \ldots, p_K - 1$, by applying a matrix $T \in \mathbb{R}^{3\times 3}$ to the 'reference' directions $\hat{\boldsymbol{d}}_\ell$, $\ell = 1, \ldots, p_K - 1$, respectively; i.e., $\boldsymbol{d}_\ell = T\hat{\boldsymbol{d}}_\ell$.

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$$\boldsymbol{d} = T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For $\boldsymbol{d} = (d_x, d_y, d_z)^\top$ we use the identity matrix if $d_x = d_y = 0$, and

$$T = \begin{pmatrix} \frac{d_x d_z}{\sqrt{d_x^2+d_y^2}} & \frac{d_y}{\sqrt{d_x^2+d_y^2}} & d_x \\ \frac{d_y d_z}{\sqrt{d_x^2+d_y^2}} & -\frac{d_x}{\sqrt{d_x^2+d_y^2}} & d_y \\ -\sqrt{d_x^2+d_y^2} & 0 & d_z \end{pmatrix} \text{ otherwise.}$$

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A posteriori Error Bound — *h*-version Only

For the TDGFEM, with the constant flux parameters, the following error bound holds:

$$\|u - u_h\|^2_{L^2(\Omega)} \leq C(k, d_\Omega)\Bigg\{ \left\| \alpha^{1/2} h_F^s [\![u_h]\!] \right\|^2_{L^2(\mathcal{F}_h^I \cup \mathcal{F}_h^D)} + \frac{1}{k^2} \|\beta^{\frac{1}{2}} h_F^s [\![\nabla u_h]\!]\|^2_{L^2(\mathcal{F}_h^I)}$$

$$+ \frac{1}{k^2} \left\| \delta^{1/2} h_F^s \left(g_R - \nabla u_h \cdot \boldsymbol{n}_F + ik\vartheta u_h \right) \right\|^2_{L^2(\mathcal{F}_h^R)} \Bigg\}$$

where s depends on the regularity of the solution to the adjoint problem ($z \in H^{3/2+s}(\Omega)$).

[Kapita, Monk & Warburton, 2015]

A posteriori Error Bound — *hp*-version

We propose the following potential *a posteriori* error bound for the *hp*-version with constant flux parameters:

$$\|u - u_{hp}\|^2_{L^2(\Omega)} \leq C\Bigg\{ k\left\|\alpha^{1/2} h_F^{1/2} q_F^{-1/2} [\![u_{hp}]\!]\right\|^2_{L^2(\mathcal{F}_h^I \cup \mathcal{F}_h^D)} + \|\beta^{\frac{1}{2}} h_F^{3/2} q_F^{-3/2} [\![\nabla u_{hp}]\!]\|^2_{L^2(\mathcal{F}_h^I)} + \left\|\delta^{1/2} h_F^{3/2} q_F^{-3/2} (g_R - \nabla u_{hp} \cdot \boldsymbol{n}_F + iku_{hp})\right\|^2_{L^2(\mathcal{F}_h^R)} \Bigg\}$$

for smooth solution of the adjoint and $d = 2$.

Consider the smooth (analytic) solution (for Acoustic Wave Propagation)

$$u(x,y) = \mathcal{H}_0^{(1)}(k\sqrt{(x+1/4)^2+y^2}),$$

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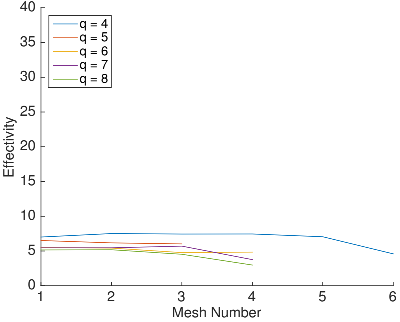
Consider uniform h-refinement for $k = 10, 20, 30, 40, 50$.



Effectivity ($k = 10$)

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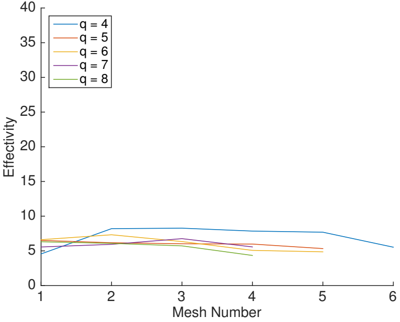

Effectivity ($k = 20$)

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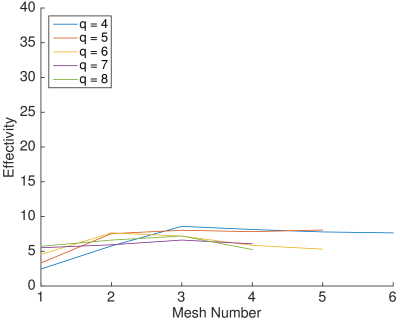

Effectivity ($k = 30$)

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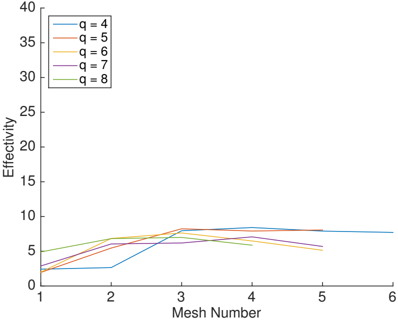

Effectivity ($k = 40$)

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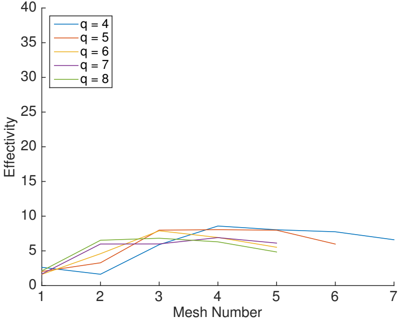

Effectivity ($k = 50$)

In order to select whether to perform *h*- or *p*-refinement at each refinement step usually involves estimates of the smoothness of the solution — several existing algorithms exist. [Mitchell, 2011]

hp-adaptive Refinement

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This method, however, will not work for TDGFEM, especially as an highly oscillatory analytical solution may be detected as non-smooth. In this case *p*-refinement could be best (our basis functions are highly oscillatory as well).

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This method, however, will not work for TDGFEM, especially as an highly oscillatory analytical solution may be detected as non-smooth. In this case *p*-refinement could be best (our basis functions are highly oscillatory as well).

Instead choose to assume *p*-refinement is the best refinement at the first step for any element, then at further refinements decide whether to perform *h*- or *p*-refinement based on whether the expected error reduction is achieved by the previous refinement. [Melenk & Wohlmuth, 2001]

Modified *hp*-refinement Strategy [Melenk & Wohlmuth, 2001]

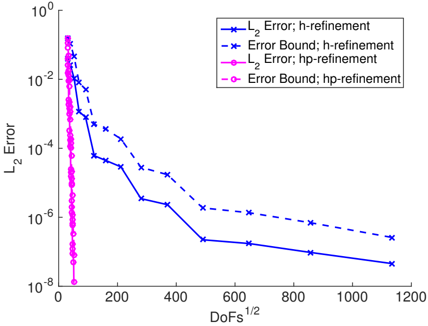
Let $\mathcal{T}_{h,0}$ be the initial mesh, $\mathcal{T}_{h,i}$ the mesh after i refinements, $\eta_{K,i}$ the error indicator for $K \in \mathcal{T}_{h,i}$, and $\eta_{K,i}^{\mathrm{pred}}$ the predicted error for $K \in \mathcal{T}_{h,i}$.

for $K \in \mathcal{T}_{h,i}$ **do**
  if K is marked for refinement **then**
    if $\eta_{K,i}^2 > (\eta_{K,i}^{\mathrm{pred}})^2$ **then**
      h-refinement: Subdivide K into N sons $K_s, s \in 0, \ldots, N$
      $(\eta_{K_s,i+1}^{\mathrm{pred}})^2 \leftarrow \frac{1}{N}\gamma_h \left(\frac{1}{2}\right)^{2q_K} \eta_{K,i}^2$, $i \leq s \leq N$
    else
      p-refinement: $q_K \leftarrow q_K + 1$
      $(\eta_{K,i+1}^{\mathrm{pred}})^2 \leftarrow \gamma_p \eta_{K,i}^2$
    end if
  else
    $(\eta_{K,i+1}^{\mathrm{pred}})^2 \leftarrow \gamma_n (\eta_{K,i}^{\mathrm{pred}})^2$
  end if
end for

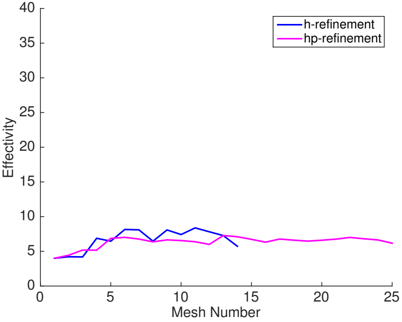
Consider the smooth (analytic) solution (for Acoustic Wave Propagation)

$$u(r,\theta) = \mathcal{J}_1(kr)\cos(\theta)$$

on the domain $\Omega = (0,1) \times (-1/2, 1/2)$.
Consider h- and hp-refinement for $k = 20$.



L^2-Error & Error Bound



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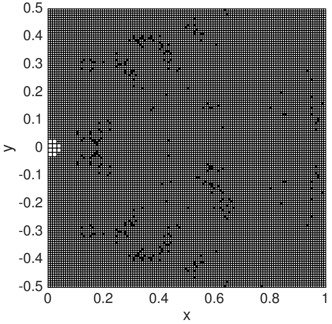
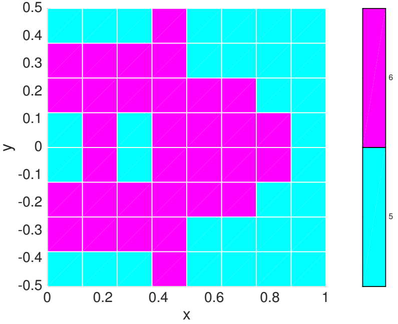


Mesh after 10 *h*-refinements



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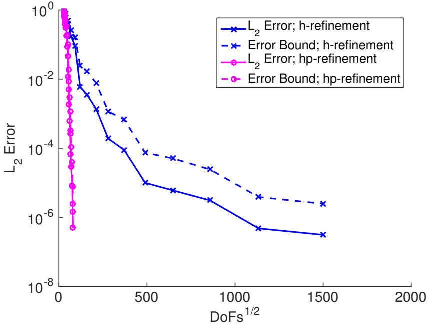

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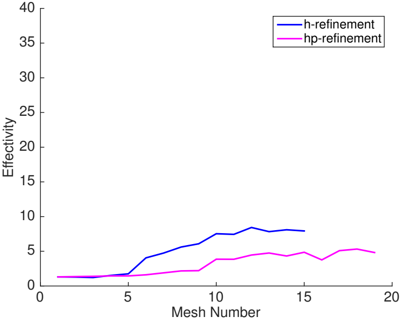

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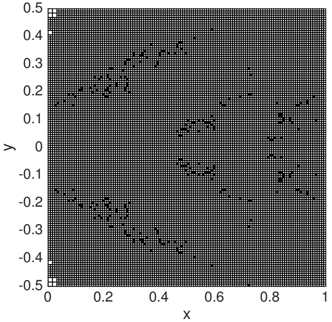

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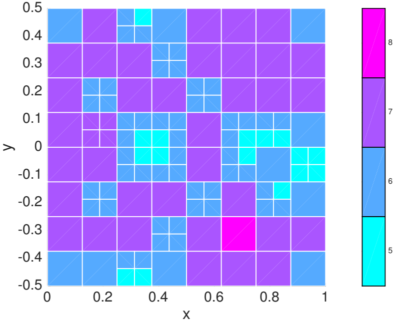

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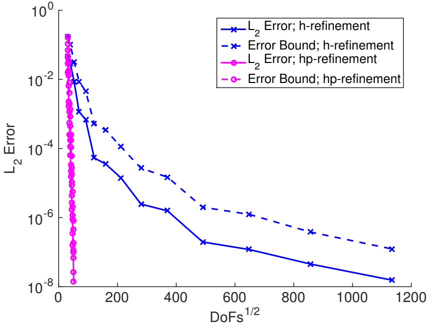

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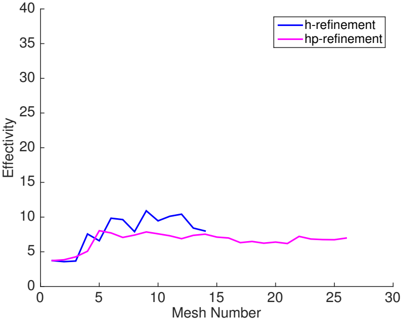

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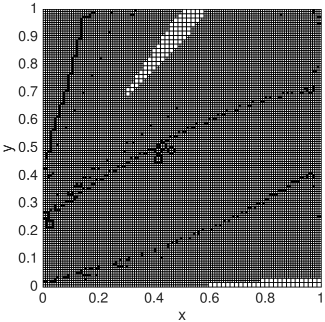
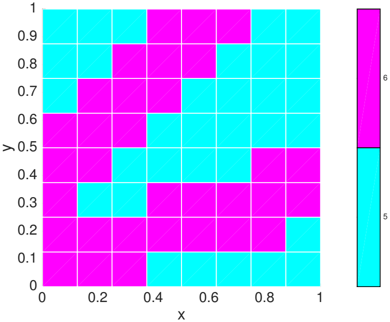


Mesh after 10 h-refinements



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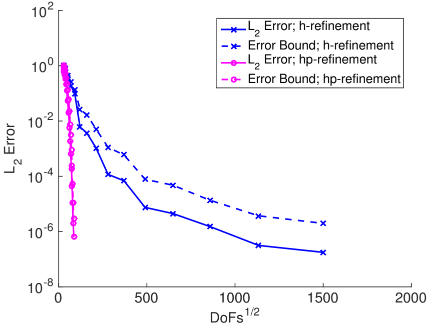

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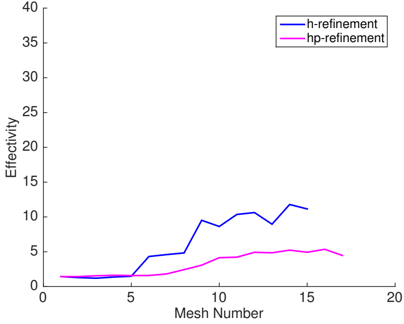

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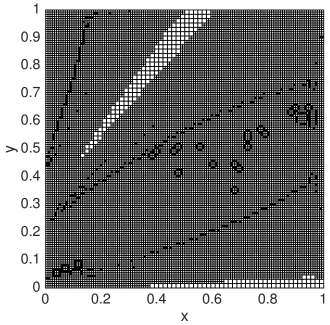

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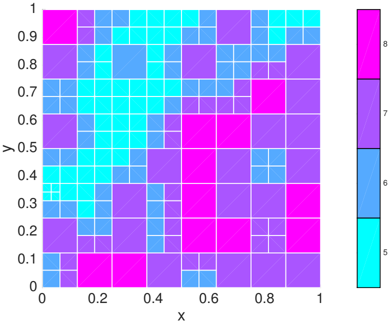

Mesh after 10 hp-refinements

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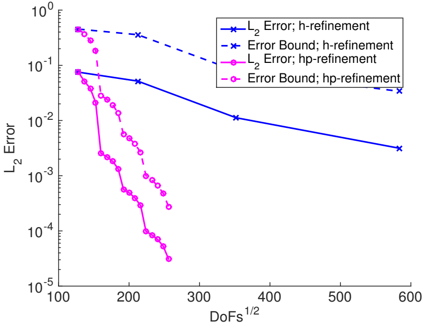

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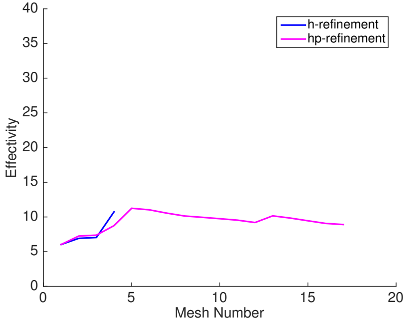

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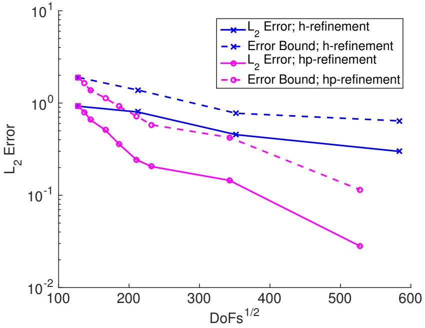

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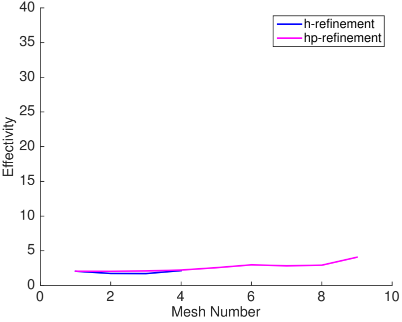

Effectivity

universität wien

Consider the non-smooth solution (for Acoustic Wave Propagation)

$$u(r,\theta) = \mathcal{J}_{2/3}(kr)\sin(2\theta/3),$$

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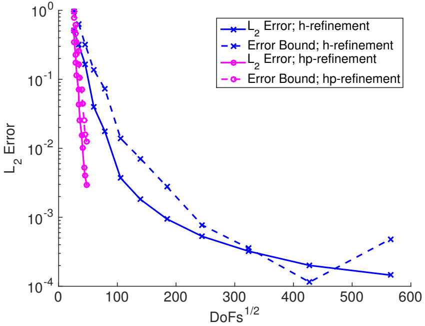
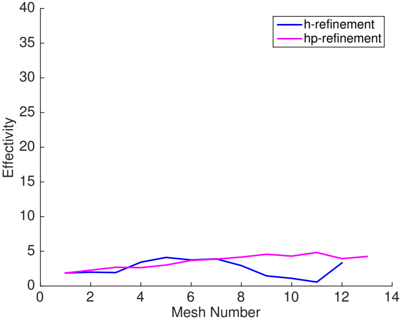
Consider h- and hp-refinement for $k = 20$.



L^2-Error & Error Bound



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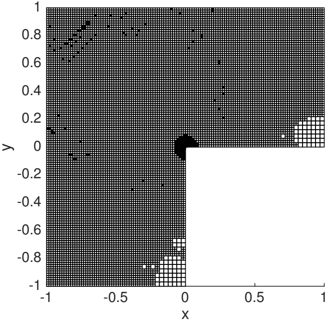
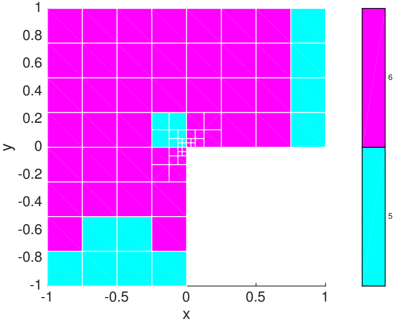
on the domain L-shaped domain $\Omega = (-1,1)^2 \setminus (0,1) \times (-1,1)$.
Consider h- and hp-refinement for $k = 20$.



Mesh after 10 h-refinements

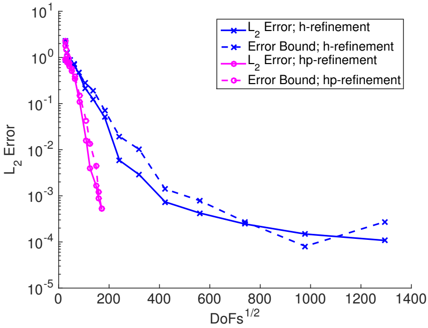


Mesh after 10 hp-refinements

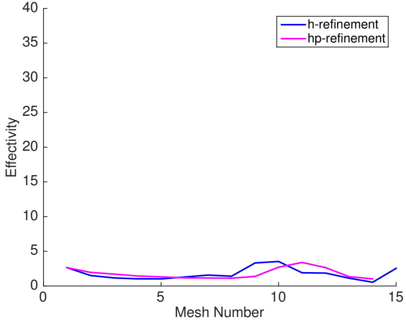
Consider the non-smooth solution (for Acoustic Wave Propagation)

$$u(r,\theta) = \mathcal{J}_{2/3}(kr)\sin(2\theta/3),$$

on the domain L-shaped domain $\Omega = (-1,1)^2 \setminus (0,1)\times(-1,1)$.
Consider h- and hp-refinement for $k = 50$.



L^2-Error & Error Bound

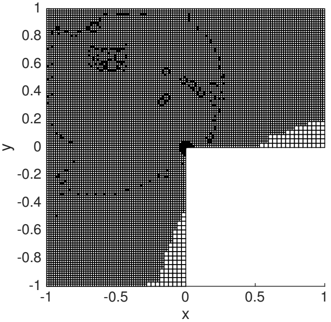


Effectivity

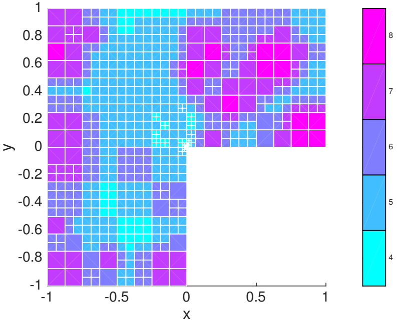
Consider the non-smooth solution (for Acoustic Wave Propagation)

$$u(r,\theta) = \mathcal{J}_{2/3}(kr)\sin(2\theta/3),$$

on the domain L-shaped domain $\Omega = (-1,1)^2 \setminus (0,1)\times(-1,1)$.
Consider h- and hp-refinement for $k = 50$.



Mesh after 10 *h*-refinements



Mesh after 10 *hp*-refinements

Conclusion

Summary:

- With plane wave basis functions it is possible to refine the wave directions.
- *hp*-adaptive refinement results in exponential convergence.

Future Aims:

- Develop an algorithm for deciding on whether to perform h or p refinement based on only the numerical solution at the current step (rather than based estimates on expected convergence).
- Use the eigenvalues/eigenvectors to develop anisotropic p-refinement (unevenly spaced plane waves).