Adaptive Refinement for the *hp*-Version Trefftz Discontinuous Galerkin Finite Element Method

Scott Congreve

Fakultät für Mathematik, Universität Wien

Joint work with
Ilaria Perugia (Universität Wien)
Paul Houston (University of Nottingham)

2nd Workshop on CENTRAL Trends
in Analysis and Numerics for PDEs

Overview

1. Trefftz DG for Helmholtz
   - Helmholtz Equation
   - Trefftz DG
   - Comparison to Polynomial DG
   - Comparison of Flux Parameters

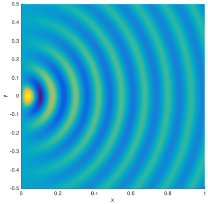
2. Adaptive Refinement
   - Plane Wave Direction Refinement
   - A posteriori Error Estimates
   - *hp*-adaptive Refinement

Section 1

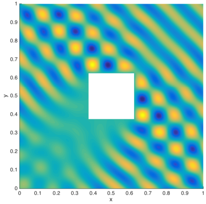
Trefftz DG for Helmholtz

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded polygonal/polyhedral domain.

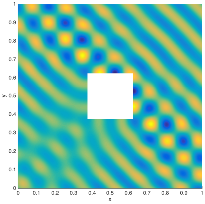
$$\begin{aligned} -\Delta u - k^2 u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, && \text{(sound-soft scattering)} \\ \nabla u \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma_N, && \text{(sound-hard scattering)} \\ \nabla u \cdot \boldsymbol{n} + ik\vartheta u &= g_R && \text{on } \Gamma_R. \end{aligned}$$



Acoustic Wave Prop.



Sound-soft Scattering



Sound-hard Scattering

Trefftz FEM Spaces

Polynomial DG Finite Element Spaces: DGFEM uses polynomial basis functions defined on a reference element $\widehat{K}$:

$$V_q^{DG}(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_K \circ F_K \in \mathcal{S}_{q_K}(\widehat{K}), K \in \mathcal{T}_h\}.$$

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Trefftz Finite Element Space: Use basis functions defined element-wise based on general solutions to the PDE.
First define the local Trefftz spaces

$$T(K) := \{v|_K : -\Delta u - k^2 u = 0\}$$

and let

$$T(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_K \in T(K), K \in \mathcal{T}_h\}.$$

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$$T(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_K \in T(K), K \in \mathcal{T}_h\}.$$

We let $V_p(K) \subset T(K)$ be a finite dimensional local space; then, the Trefftz FE Space is given by

$$V_p(\mathcal{T}_h) := \{v \in T(\mathcal{T}_h) : v_K \in V_p(K), K \in \mathcal{T}_h\}.$$

Plane Waves

$$V_p(K) = \left\{ v : v(\boldsymbol{x}) = \sum_{\ell=1}^{p_K} \alpha_\ell \mathrm{e}^{ik\boldsymbol{d}_\ell \cdot (\boldsymbol{x}-\boldsymbol{x}_K)}, \alpha_\ell \in \mathbb{C} \right\}$$

where p_K is the number of *degrees of freedom* for the element K, $\boldsymbol{d}_l$, $l = 1, \cdots, N_K$ are p_K (roughly) evenly spaced unit direction vectors, and $\boldsymbol{x}_K$ is the centre of the element.

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Trefftz DG has less degrees of freedom than high-order polynomials for the same accuracy.

Basis Functions	2D	3D
DG ($\mathcal{P}_q$)	$(q+1)(q+2)/2$	$(q+1)(q+2)(q+3)/6$
DG ($\mathcal{Q}_q$)	$(q+1)^2$	$(q+1)^3$
Trefftz DG	$2q+1$	$(q+1)^2$

Number of Degrees of Freedom

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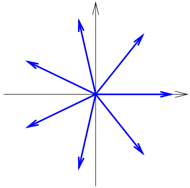
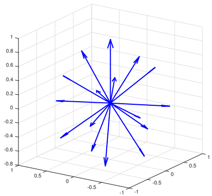
Number of Degrees of Freedom

Direction Vectors ($q = 3$):

2D



3D



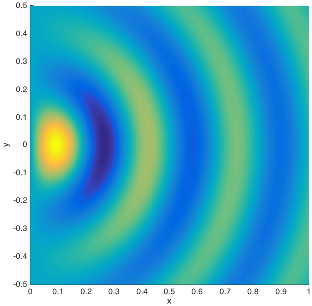
[Sloan & Womersley, 2004]

TDG/DG FEM Comparison

Consider the smooth (analytic) solution (for Acoustic Wave Propagation)

$$u(r,\theta) = \mathcal{J}_1(kr)\cos(\theta)$$

for $k = 20$ on the domain $\Omega = (0,1) \times (-1/2, 1/2)$.



Analytical Solution
(Real Part)

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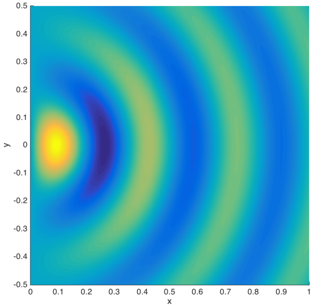
We solve using both a DGFEM (solid line) and Trefftz DGFEM (dashed).



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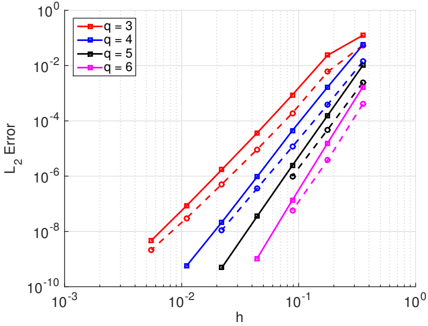

$\|u - u_{hp}\|_{L^2(\Omega)}$ vs. h
(h-refinement)

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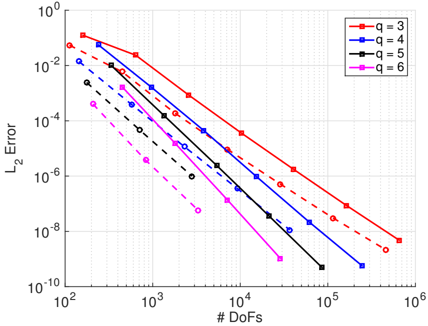

$\|u - u_{hp}\|_{L^2(\Omega)}$ vs. Degrees of Freedom
(h-refinement)

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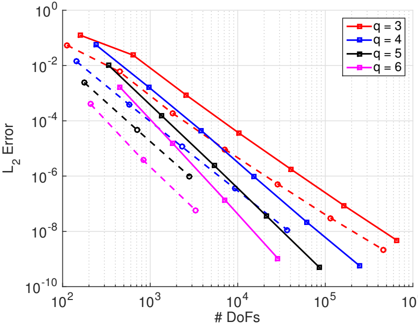
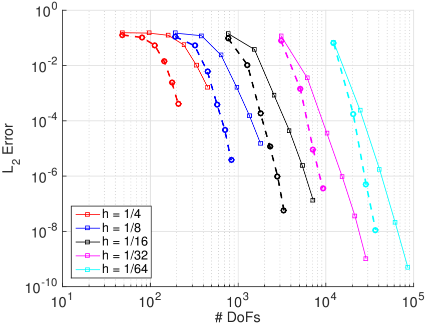


$\|u - u_{hp}\|_{L^2(\Omega)}$ vs. Degrees of Freedom (h-refinement)



$\|u - u_{hp}\|_{L^2(\Omega)}$ vs. Degrees of Freedom (p-refinement)

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Trefftz Discontinuous Galerkin FEM for Helmholtz

Find $u_{hp} \in V_p(\mathcal{T}_h)$ such that,

$$\mathcal{A}_h(u_{hp}, v_{hp}) = \ell_h(v_{hp}),$$

for all $v_{hp} \in V_p(\mathcal{T}_h)$, where

$$\begin{aligned}
\mathcal{A}_h(u,v) = &\int_{\mathcal{F}_h^I \cup \mathcal{F}_h^N} \{\!\{u\}\!\} [\![\nabla_h \bar{v}]\!] \, ds - \int_{\mathcal{F}_h^I \cup \mathcal{F}_h^N} \beta (ik)^{-1} [\![\nabla_h u]\!][\![\nabla_h \bar{v}]\!] \, ds \\
&- \int_{\mathcal{F}_h^I \cup \mathcal{F}_h^D} \{\!\{\nabla_h u\}\!\} \cdot [\![\bar{v}]\!] \, ds + \int_{\mathcal{F}_h^I \cup \mathcal{F}_h^D} \alpha i k [\![u]\!] \cdot [\![\bar{v}]\!] \, ds \\
&+ \int_{\mathcal{F}_h^R} (1-\delta) u \nabla_h \bar{v} \cdot \boldsymbol{n} \, ds - \int_{\mathcal{F}_h^R} \delta (ik\vartheta)^{-1} (\nabla_h u \cdot \boldsymbol{n})(\nabla_h \bar{v} \cdot \boldsymbol{n}) \, ds \\
&- \int_{\mathcal{F}_h^R} \delta \nabla_h u \cdot \boldsymbol{n} \bar{v} \, ds + \int_{\mathcal{F}_h^R} (1-\delta) ik\vartheta u \bar{v} \, ds, \\
\ell_h(v) = &- \int_{\mathcal{F}_h^R} \delta (ik\vartheta)^{-1} g_R \nabla_h \bar{v} \cdot \boldsymbol{n} \, ds + \int_{\mathcal{F}_h^R} (1-\delta) g_R \bar{v} \, ds.
\end{aligned}$$

Flux Parameters

Penalty Type	α	β	δ
DG-type Gittelson, Hiptmair & Perugia, 2009	$\mathtt{a}q_K^2/kh_K$	$\mathtt{b}kh_K/q_K$	$\mathtt{d}kh_K/q_K$
Constant Hiptmair, Moiola & Perugia, 2011	$\mathtt{a}$	$\mathtt{b}$	$\mathtt{d}$
UWVF Cessenat & Després, 1998	$1/2$	$1/2$	$1/2$
Non-Uniform Mesh Hiptmair, Moiola & Perugia, 2014	$\mathtt{a}h_{\max}/h_K$	$\mathtt{b}h_{\max}/h_K$	$\mathtt{d}h_{\max}/h_K$

For the rest of this talk we ignore Neumann boundary conditions.

Comparison of Flux Parameters (2D)

Consider the smooth (analytic) solution (for Acoustic Wave Propagation)

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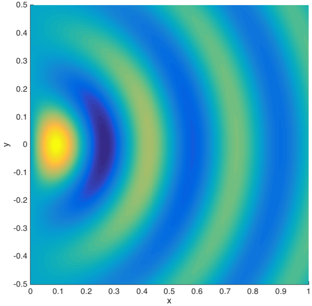

Re(Anal. Soln.) (k=20)

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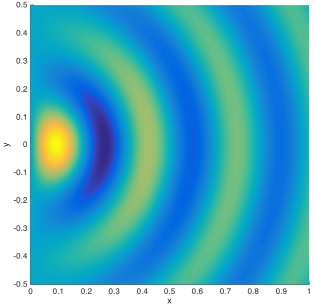
We solve using constant (solid line) and DG-type parameters (dashed).



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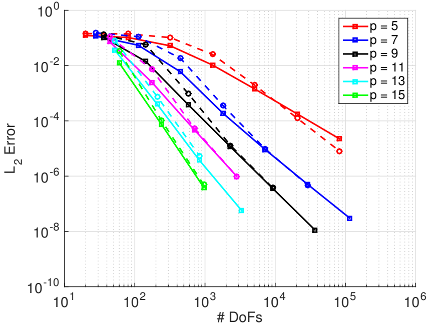
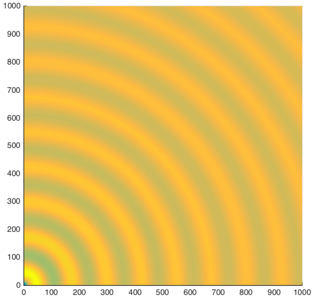


$k = 20$

To test the non-uniform parameters, we consider the solution

$$u(x, y) = \mathcal{H}_0^{(1)}(k\sqrt{x^2 + y^2}),$$

with $k = 50$, on the domain $\Omega = (0, 1)^2$, where $\mathcal{H}_0^{(1)}$ represents the Hankel function of the first kind of order 0.



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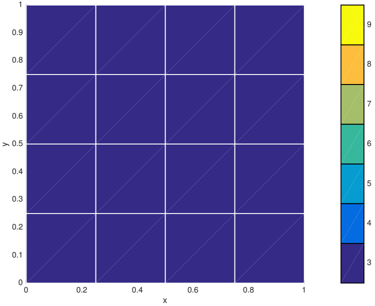

Mesh 1

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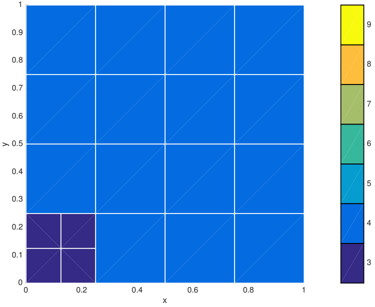

Mesh 2

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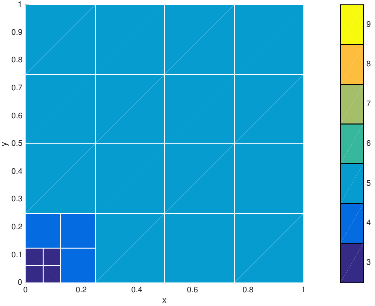

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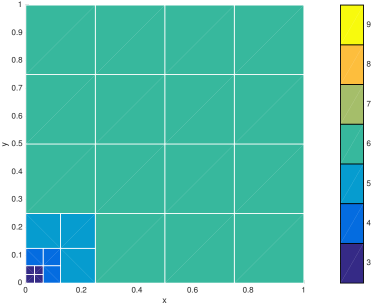

Mesh 4

Comparison of Flux Parameters (Non-Unif.)

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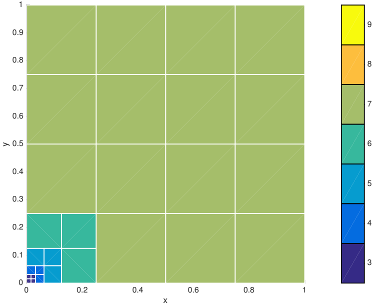

Mesh 5

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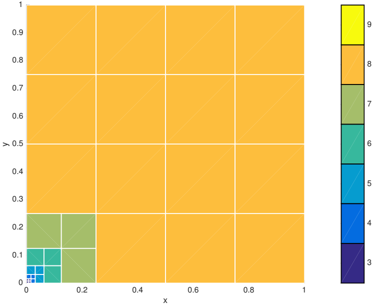

Mesh 6

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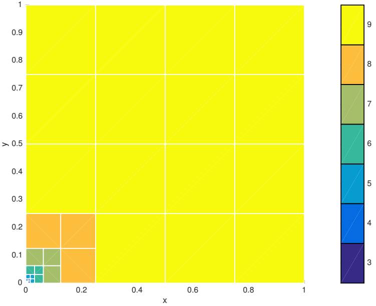

Mesh 7

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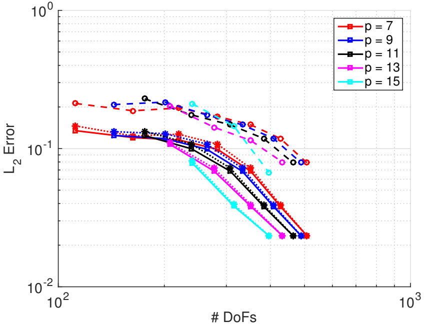

Constant (solid line), DG-type (dashed) & non-uniform (dotted) parameters

Section 2

Adaptive Refinement

Plane Wave Direction Refinement

Consider a plane wave analytical solution (for Acoustic Wave Propagation)

$$u(\boldsymbol{x}) = \mathrm{e}^{ik\boldsymbol{d}\cdot\boldsymbol{x}}$$

for $k = 20$ on the domain $\Omega = (0,1)^2$, where $\boldsymbol{d} = (1/\sqrt{2}, 1/\sqrt{2})$.

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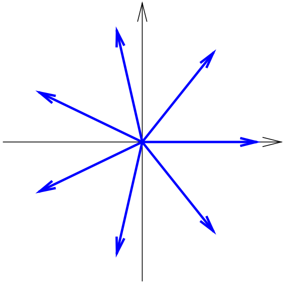
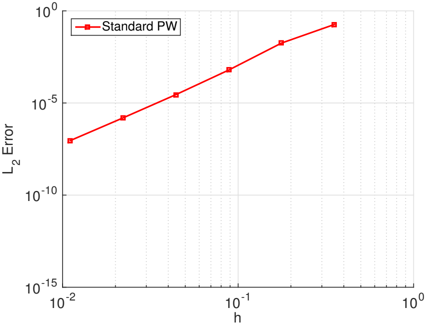
We evenly distribute directions $\boldsymbol{d}_\ell$, starting from $\boldsymbol{d}_1 = (1,0)$.



Plane Wave Directions ($q = 3$)



$\|u - u_{hp}\|_{L^2(\Omega)}$ vs. h

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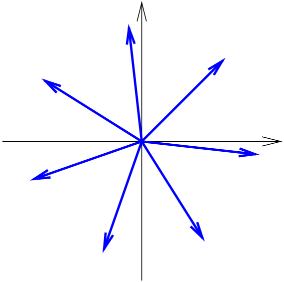
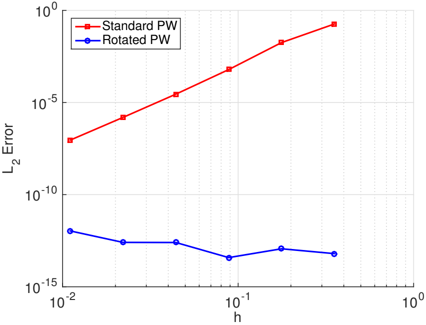
We evenly distribute directions $\boldsymbol{d}_\ell$, starting from $\boldsymbol{d}_1 = (1,0)$.
Rotating directions so that $\boldsymbol{d}_1 = \boldsymbol{d}$ gives (almost) the analytical solution.



Rotated Directions ($q = 3$)



$\|u - u_{hp}\|_{L^2(\Omega)}$ vs. h

Plane Wave Direction Refinement

Even for non-plane wave analytical solutions picking the correct main direction reduces the error.

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We need a way calculate/adapt the directions without the analytical solution. Several existing approaches exist:

- Ray-tracing — requires a source term. [Betcke & Phillips, 2012]
- Approximate

$$\frac{\nabla e(\boldsymbol{x}_0)}{ike(\boldsymbol{x}_0)},$$

  where e is the error. [Gittelson, 2008 (Master's Thesis)]
- Adding an extra unknown (the optimal angle of rotation) to the basis functions. [Amara, Chaudhry, Diaz, Djellouli & Fiedler, 2014]

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The eigenvector of the Hessian matching the largest eigenvalue should be the direction to use as the main direction, assuming the matching eigenvalue is significantly larger.

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Plane Wave Refinement Algorithm (2D)

Let $(\lambda_1, \boldsymbol{v}_1), (\lambda_2, \boldsymbol{v}_2)$ be the eigenpairs of $\mathbf{H}(\mathrm{Re}(u_h(\boldsymbol{x}_K)))$, and $(\mu_1, \boldsymbol{w}_1), (\mu_2, \boldsymbol{w}_2)$ the eigenpairs of $\mathbf{H}(\mathrm{Im}(u_h(\boldsymbol{x}_K)))$ s.t. $\lambda_1 \geq \lambda_2$, $\mu_1 \geq \mu_2$; then, for constant $C > 1$, we can select the first plane wave direction as follows:

$\lambda_1 \geq C\lambda_2$	$\mu_1 \geq C\mu_2$	$\lambda_1 \geq C\mu_1$	$\mu_1 \geq C\lambda_1$	First PW Direction
✓	✓	✓	✗	$\boldsymbol{v}_1$
✓	✓	✗	✓	$\boldsymbol{w}_1$
✓	✓	✗	✗	$(\boldsymbol{v}_1+\boldsymbol{w}_1)/2$
✓	✗	✓	✗	$\boldsymbol{v}_1$
✓	✗	✗	–	–
✗	✓	✗	✓	$\boldsymbol{w}_1$
✗	✓	–	✗	–
✗	✗	–	–	–

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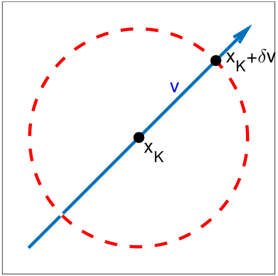

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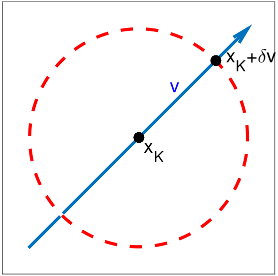

Evaluating at $\boldsymbol{x}_K + \delta\boldsymbol{v}$ we note that the normal is $\boldsymbol{v}$, so we can calculate

$$\frac{\nabla u_h(\boldsymbol{x}_K)\cdot\boldsymbol{v} + iku_h(\boldsymbol{x}_K)}{iku_h(\boldsymbol{x}_K)}.$$

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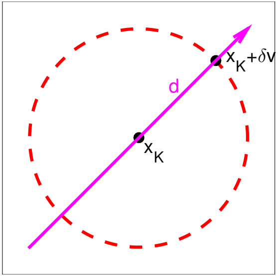

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We can compare this to the impedance for the u:

$$\frac{\nabla u(\boldsymbol{x}_K)\cdot\boldsymbol{v} + iku(\boldsymbol{x}_K)}{iku(\boldsymbol{x}_K)} = \begin{cases} 2, & \text{if } \boldsymbol{d} = \boldsymbol{v}, \\ & \end{cases}$$

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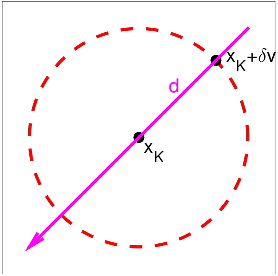

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To test the direction refinement, we consider the solution

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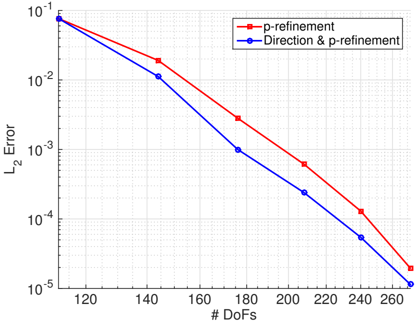
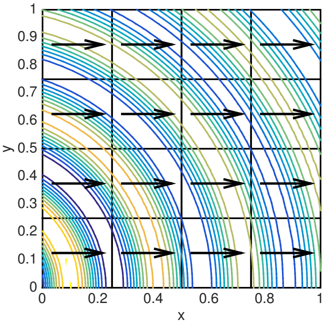
with $k=20$, on the domain $\Omega=(0,1)^2$.



$\|u-u_{hp}\|_{L^2(\Omega)}$ vs. DoF



First PW Direction ($p=3$)

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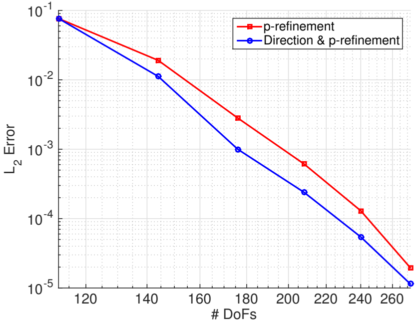

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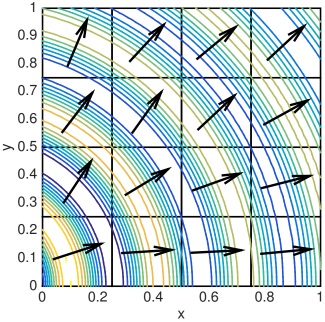

First PW Direction ($p=4$)

To test the direction refinement, we consider the solution

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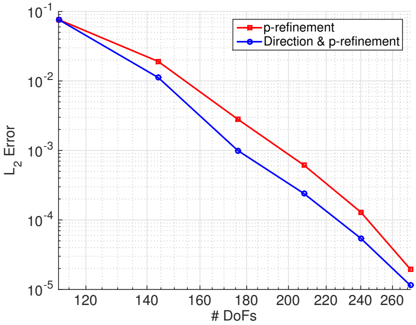
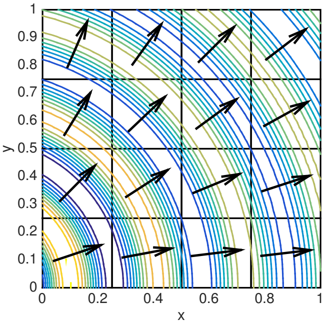


$\|u - u_{hp}\|_{L^2(\Omega)}$ vs. *DoF*

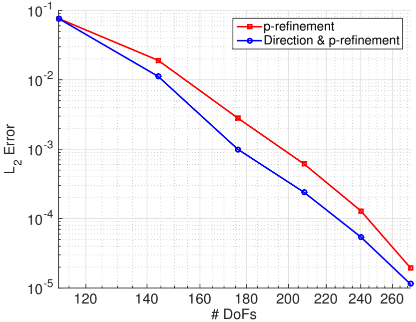


First PW Direction ($p = 5$)

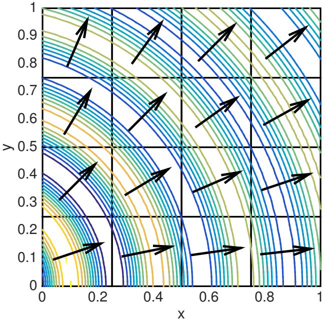
To test the direction refinement, we consider the solution

$$u(x,y) = \mathcal{H}_0^{(1)}\left(k\sqrt{(x+1/4)^2+y^2}\right),$$

with $k = 20$, on the domain $\Omega = (0,1)^2$.



$\|u - u_{hp}\|_{L^2(\Omega)}$ vs. DoF



First PW Direction ($p = 6$)

To test the direction refinement, we consider the solution

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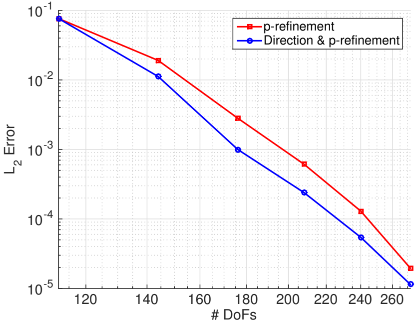
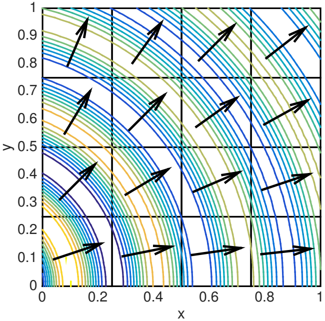


$\|u - u_{hp}\|_{L^2(\Omega)}$ vs. *DoF*



First PW Direction ($p = 7$)

Plane Wave Direction Refinement

To test the direction refinement, we consider the solution

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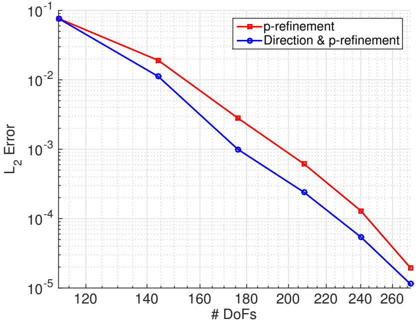
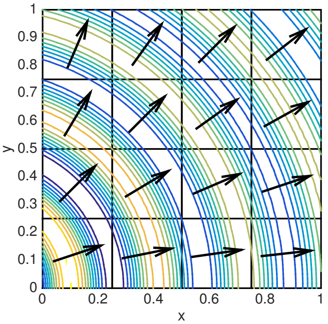


$\|u - u_{hp}\|_{L^2(\Omega)}$ vs. DoF



First PW Direction ($p = 8$)

An *a posteriori* error bounds exists for the *h*-version of the method in $\mathbb{R}_2$ (ignoring Neumann boundary conditions).

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A posteriori Error Bound — *h*-version Only

For the TDGFEM, with the constant flux parameters, the following error bound holds:

$$\|u-u_h\|^2_{L^2(\Omega)} \leq C(k,d_\Omega)\Bigg\{\left\|\alpha^{1/2}h_F^s[\![u_h]\!]\right\|^2_{L^2(\mathcal{F}_h^I\cup\mathcal{F}_h^D)} + \frac{1}{k^2}\|\beta^{\frac{1}{2}}h_F^s[\![\nabla u_h]\!]\|^2_{L^2(\mathcal{F}_h^I)} + \frac{1}{k^2}\left\|\delta^{1/2}h_F^s\left(g_R - \nabla u_h\cdot\boldsymbol{n}_F + ik\vartheta u_h\right)\right\|^2_{L^2(\mathcal{F}_h^R)}\Bigg\}$$

where s depends on the regularity of the solution to the adjoint problem ($z \in H^{3/2+s}(\Omega)$).

[Kapita, Monk & Warburton, 2015]

A posteriori Error Bound — *hp*-version

We propose the following potential *a posteriori* error bound for the *hp*-version with constant flux parameters:

$$\|u - u_{hp}\|^2_{L^2(\Omega)} \leq C \Bigg\{ k \left\| \alpha^{1/2} C_F(h,k,p) [\![u_{hp}]\!] \right\|^2_{L^2(\mathcal{F}_h^I \cup \mathcal{F}_h^D)} + \frac{1}{k} \| \beta^{\frac{1}{2}} C_F(h,k,p) [\![\nabla u_{hp}]\!] \|^2_{L^2(\mathcal{F}_h^I)} + \frac{1}{k} \left\| \delta^{1/2} C_F(h,k,p) \left(g_R - \nabla u_{hp} \cdot \boldsymbol{n}_F + iku_{hp} \right) \right\|^2_{L^2(\mathcal{F}_h^R)} \Bigg\}$$

where

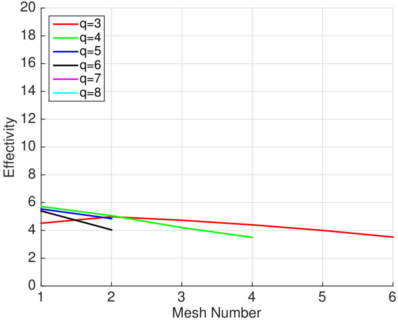
$$C_F(h,k,p)^2 = (h_F k)^s k^{-t} q_F^{-z}, \qquad \text{for } F \in \mathcal{F}_h.$$

Throughout the following we use $s = 3.5$, $t = 1$ and $z = 3$.

Consider the smooth (analytic) solution (for Acoustic Wave Propagation)

$$u(x,y) = \mathcal{H}_0^{(1)}(k\sqrt{(x+1/4)^2+y^2}),$$

on the domain $\Omega = (0,1)^2$.
Consider uniform h-refinement for $k = 10, 20, 30, 40, 50$.



Effectivity ($k = 10$)

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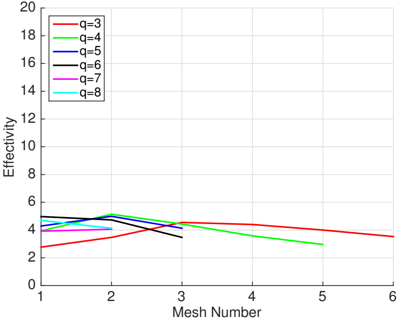

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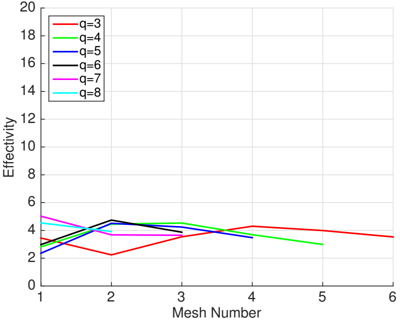

Effectivity ($k = 30$)

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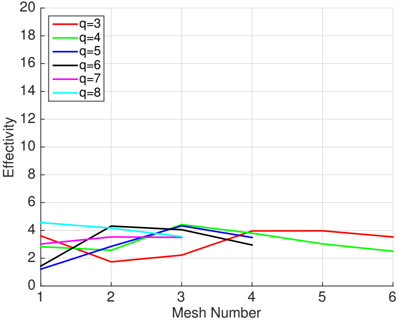

Effectivity ($k = 40$)

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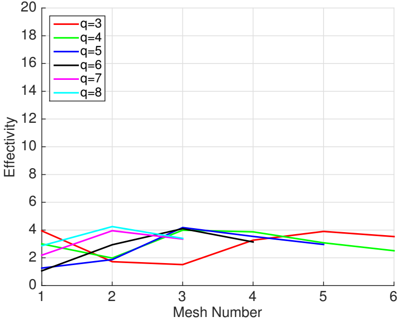

Effectivity ($k = 50$)

hp-adaptive Refinement

In order to select whether to perform *h*- or *p*-refinement at each refinement step usually involves estimates of the smoothness of the solution — several existing algorithms exist. [Mitchell, 2011]

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Instead choose to assume *p*-refinement is the best refinement at the first step for any element, then at further refinements decide whether to perform *h*- or *p*-refinement based on whether the expected error reduction is achieved by the previous refinement. [Melenk & Wohlmuth, 2001]

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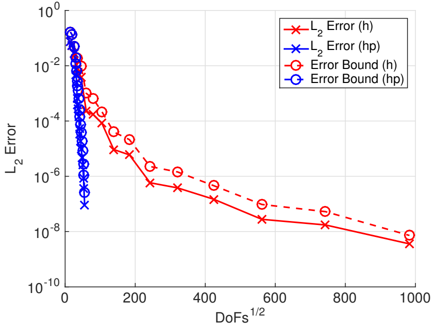
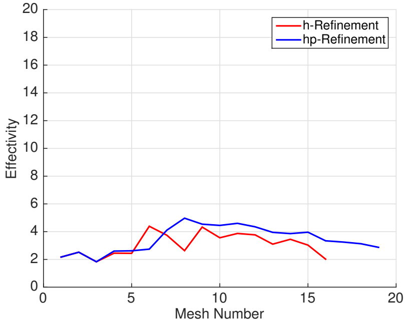
Consider h- and hp-refinement for $k = 20$.



L^2-Error & Error Bound



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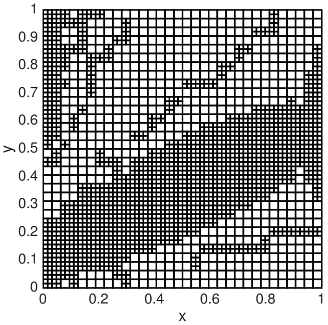
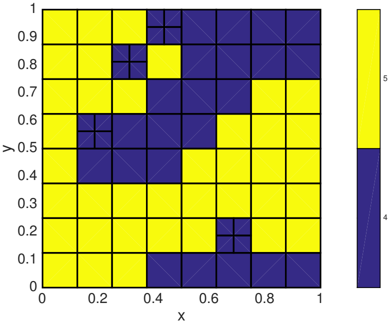


Mesh after 9 h-refinements

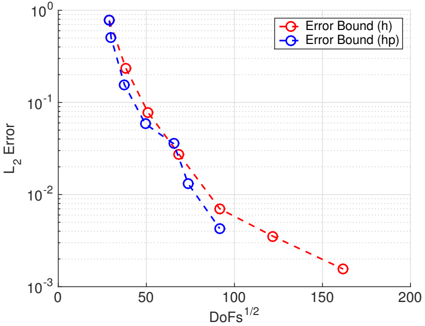


Mesh after 9 hp-refinements

Consider *h*- and *hp*-refinement for the sound-soft scatter problem with incidence field

$$u(x, y) = \exp(ik(x\cos(\pi/4) + y\sin(\pi/4)))$$

on the domain $\Omega = (0, 1)^2 \setminus (0.4, 0.6)^2$, with zero Dirichlet boundary on the obstacle and $k = 20$.



Error Bound

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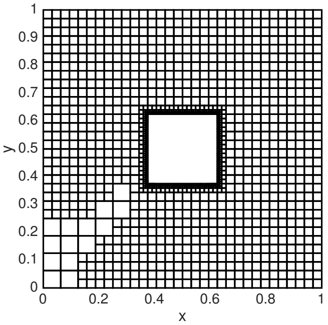
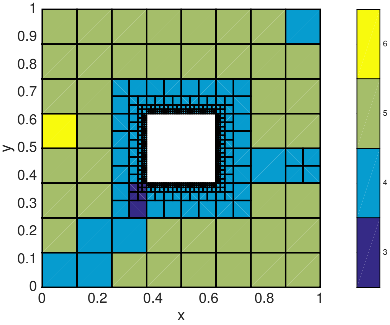


Mesh after 9 *h*-refinements



Mesh after 9 *hp*-refinements

Conclusion

Summary:

- *A priori* results exist for Trefftz DG, with Robin and Dirichlet BCs
- Various choice of flux parameters is required for the existing analysis
- The choice of flux parameters tends to make no difference on smooth solutions, although DG-style parameters appear poor for non-uniform refinement/singular problems.
- With plane wave basis functions it is possible to refine the wave directions.
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- *hp*-adaptive refinement.

Future Aims:

- Develop an algorithm for deciding on whether to perform *h* or *p* refinement based on only the numerical solution at the current step (rather than based estimates on expected convergence).
- Use the eigenvalues/eigenvectors to develop anisotropic *p*-refinement (unevenly spaced plane waves).